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A METHOD FOR ACHIEVING STABLE DISTRIBUTIONS OF WIRELESS MOBILE LOCATION IN MOTION SIMULATIONS

Tony Dean

Network Solutions Sector
1501 West Shure Drive
Motorola, Inc.
Arlington Heights, IL 60004, U.S.A.

ABSTRACT

A cellular engineer typically estimates system performance via simulation. Most cellular operations software provides data from which one can infer the average, busy hour, subscriber location distribution, which becomes an input to the simulation. When the simulation does not include mobility, as is typical with Monte Carlo simulations, modeling this distribution is a straight-forward task. However, when the simulation models mobility, it must do so in such a way that the subscriber location distribution is stable. We introduce a stochastic mobility model for the purpose of achieving and stabilizing *a priori* subscriber location distributions.

1 INTRODUCTION

A dynamic simulation of a cellular system requires a model of mobility. Simplistic mobility models choose a random velocity vector for a *particle* (mobile subscriber, automobile, etc.) and compute its destination based on physical laws of motion and the time period, or *epoch* between simulation steps. The primary disadvantage with this approach is that it is does not control the distribution of particle location. More sophisticated approaches model particle motion stochastically, such as (Rose and Yates 1997), (Liu, Bahl, and Chlamtac 1998), (Massey and Whitt 1993), and (Jabbari, Zhou, and Hillier 1998).

In most of these approaches, the object is to assign *a priori* behavior to the particles in motion, derive the resulting location distribution, and estimate its effects on the systems under study. Our objective is somewhat the opposite. Cellular operators typically have information on the location distribution of their subscribers. A cellular planning tool which incorporates dynamic simulation of subscriber mobility must be able to incorporate this *a priori* location distribution information.

The challenge in this effort is to find a stochastic process that achieves this distribution and satisfies other requirements of the simulation. In many respects, this is the reversal of the classic stochastic process problem: instead of beginning with the state transition probabilities and solving for the steady-state distribution, one begins with the steady-state distribution and solves for the state transition probabilities. As there is more leeway in this direction, there is an element of design involved.

We limit ourselves to discrete time, finite state Markov processes, so the location distribution data is assumed to be a finite, probability mass function (p.m.f.) on a two dimensional grid representing the location space. For convenience, we further assume that all p.m.f.'s have all positive values. (This condition can easily be relaxed with advantageous results.) We first develop the ideas in one dimension and then extend them to two. Finally, we confine ourselves to the process exhibited by a single particle. The aggregate location distribution produced by multiple particles with identical processes is identical.

We first develop our ideas in one dimension, where we define a class of Markov chains which captures both direction and mean speed of the particle. We then show how to extend such chains to two dimensions. Finally, we present results which demonstrate the advantages of simulating mobility with this class of Markov chain over using more simplistic mobility models.

1.1 Birth/Death Models

We assume that the discrete states of the particle's chain $\mathbf{X}$ consists of the integers 0,1,...,n-1 for some positive integer n and has the steady state distribution π_i, i = 0,...,n-1. We assume $\mathbf{X}$ is time homogeneous and define

$$p_{i,j} = \Pr\left[\mathbf{X}_{k+1} = j \middle| \mathbf{X}_k = i\right]$$

for any k and any states i and j. If we assume that **X** is a birth/death process, then the state transition probabilities satisfy the detailed balance equations

$$\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i} \tag{1}$$

for $i = 0,...,n\text{-}2$. See Figure 1 for an illustration of the state space. In this and other illustrations, we omit depiction of the same state transitions. Also, we break with tradition and depict the sates as adjoining "bins" instead of separated bubbles, as this more readily conveys the sense of the states as discrete locations. The states $i = 0,...,n\text{-}1$ are indicated in the lower left hand corner of each rectangle or bin.

A shortcoming of birth/death models of mobility is that the inter-bin probabilities are constrained by the inequality

$$p_{k,k+1} + p_{k,k-1} \leq 1\,,$$

due to the fact that the sum of all transition probabilities from a fixed state to other states must be one. This means that one or the other probabilities must be less than one half. One criterion of a mobility model is that particles move far enough in one direction to cause hand-offs. If the inter-bin probability is limited, this is not likely to happen.

Our solution, which the next section addresses, is to capture knowledge of the previous heading of the particle in the current state of the particle. We accomplish this by breaking the bins into substates, each of which indicates one of the two directions from which the particle entered the bin. This permits de-coupling of the left and right inter-bin transition probabilities, so that we may assign them higher values.

1.2 Bi-Directional Models

A *bi-directional* model of mobility is a Markov chain **X** with state transition probabilities $p_{i,j}$ for $i,j = 0,...,2n\text{-}1$ such that $p_{i,j} = 0$ if i is even and $i \leq j \leq i+2$ or if i is odd and $i\text{-}2 \leq j \leq i$. Intuitively, the bi-directional model breaks up the state space of the previous section into "eastbound" and "westbound" lanes. In any state, the process makes transitions to either the next bin in the direction of the current lane, or to the opposite bound lane in the same bin (a "U-turn", if you will), or remains in the same state. The state space and transitions are illustrated in Figure 2, where we include *intra*-bin transitions for only one of the bins. Each "bin" in Figure 1 has been divided into two states in Figure 2. Bin 0 becomes states 0 and 1, bin 2 becomes states 2 and 3, etc.

Just as the detailed balance equations (1) characterize a Birth/Death Markov chain, there is a version of the detailed balance equations that characterize a bi-directional Markov chain. These equations are presented in the next two propositions.

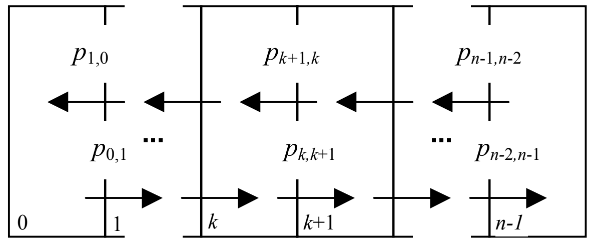

Figure 1: Birth/Death Model of Mobility

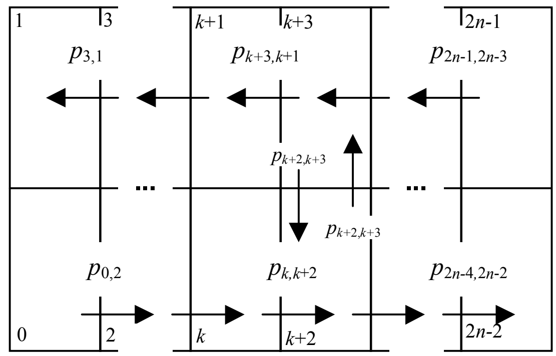

Figure 2: Bi-Directional Model of Mobility

Proposition 1

If $\pi = \langle \pi_i \mid i = 0,...,2n\text{-}1 \rangle$ is a positive-definite, p.m.f. and $p = \langle p_{i,j} \mid i,j = 0,...,2n\text{-}1 \rangle$ is a matrix with unit row sums and such that $p_{i,j} = 0$ unless i is even and $i \leq j \leq i+2$ or i is odd and $i\text{-}2 \leq j \leq i$, then p satisfies the Chapman-Kolmogorof (CK) equations if and only if

$$\begin{aligned} &\pi_k p_{k,k+2} = \pi_{k+3} p_{k+3,k+1},\ k = 0,2,...,2n-4 \\ &\pi_k \left(1 - p_{k,k}\right) = \pi_{k+1}\left(1 - p_{k+1,k+1}\right),\ k = 0,2,...,2n-2 \end{aligned} \tag{2}$$

<u>proof</u>: First, assume the CK equations. They imply that the flow into any state is equal to the flow out of that state. Thus,

$$\pi_0 p_{0,1} + \pi_3 p_{3,1} = \pi_1 p_{1,0}$$
$$\pi_1 p_{1,0} = \pi_0 p_{0,1} + \pi_0 p_{0,2}$$

Thus, $\pi_0 p_{0,2} = \pi_3 p_{3,1}$; i.e., the basis step of an inductive proof of the first equation of (2). Assume that $\pi_k p_{k,k+2} = \pi_{k+3} p_{k+3,k+1}$ for some even k. If $k \leq 2n\text{-}6$, the CK equations imply that the inflow to state k+3 equals the outflow. Similarly for state k+2 . Thus,

$$\pi_{k+2} p_{k+2,k+3} + \pi_{k+5} p_{k+5,k+3} = \pi_{k+3} p_{k+3,k+2} + \pi_{k+3} p_{k+3,k+1}$$
$$\pi_k p_{k,k+2} + \pi_{k+3} p_{k+3,k+2} = \pi_{k+2} p_{k+2,k+3} + \pi_{k+2} p_{k+2,k+4}$$

From the inductive assumption, we conclude $\pi_{k+2}p_{k+2,k+4} = \pi_{k+5}p_{k+5,k+3}$. This completes the induction step and establishes the identity $\pi_k p_{k,k+2} = \pi_{k+3}p_{k+3,k+1}$ for $k = 0,2,...,2n-6$. For $k = 2n-4$, the identity follows much as in the case for $k = 0$. This establishes the first equation of (2). The second identity follows from the first after applying the CK equation for state $k+1$.

Conversely, assume (2). If we replace $p_{k,k+2}$ by $1-p_{k,k}-p_{k,k+1}$ in the first equation of (2), then substitute for $p_{k,k}$ from the second equation, we get the CK equations for the odd states $k+1$, for $k = 0,2,...,2n-4$. The CK equation for state $2n-1$ follows from the second equation of (2) upon substitution of $p_{2n-2,2n-1}$ for $1-p_{2n-2,2n-2}$. The equations of (2) are entirely symmetric in even and odd indices, so we can adapt the preceding arguments to derive the CK equations for the even states k, for $k = 0,2,...,2n-4$. Therefore, (2) is equivalent to the CK equations. This completes the proof.

Because of the *detailed balance equations* (2), the probabilities of transition between all *even numbered* states of a bi-directional chain determine the entire set of state transition probabilities. So, to design a bi-directional chain which has a given stationary p.m.f., one might choose any set of values for the even state transition probabilities and use the detailed balance equations (together with the fact that p must have unit row sums) to assign values to the odd state transition probabilities. However, one wouldn't be assured that the values so assigned would lie in the unit interval, as is required for probabilities. The next proposition presents the requirements on the even state transition probabilities which insure that all values so defined are probabilities.

Proposition 2

Let $\pi=\langle \pi_i \mid i = 0,...,2n-1\rangle$ denote a positive valued, p.m.f. and $p = \langle p_{i,j} \mid i,j = 0,...,2n-1\rangle$ a matrix of real numbers with unit row sums such that $p_{i,j} = 0$ unless i is even and $i \le j \le i+2$ or i is odd and $i-2 \le j \le i$. Then π is the stationary distribution of a bi-directional chain with state transition probability matrix p if and only if p satisfies the detailed balance equations and

$$\left.\begin{array}{l} p_{k,k+2} \in [0,1], k = 0,2,...,2n-4 \\ p_{k,k} \in [0,1], k = 0,2,...,2n-2 \end{array}\right\} \tag{3}$$

$$p_{k,k+2} \le \frac{\pi_{k+1}}{\pi_k}, \frac{\pi_{k+2}}{\pi_k}, \frac{\pi_{k+3}}{\pi_k}, k = 0,2,...,2n-4 \tag{4}$$

$$1-\frac{\pi_{k+1}}{\pi_k} \le p_{k,k} \le 1-p_{k,k+2}, 1-\frac{\pi_{k-2}}{\pi_k}p_{k-2,k} \quad k = 2,4,...,2n-4 \tag{5}$$

$$1-\frac{\pi_1}{\pi_0} \le p_{0,0} \le 1-p_{0,2} \tag{6}$$

$$1-\frac{\pi_{2n-1}}{\pi_{2n-2}} \le p_{2n-2,2n-2} \le 1-\frac{\pi_{2n-4}}{\pi_{2n-2}}p_{2n-4,2n-2} \tag{7}$$

<u>proof</u>: Assume p is the matrix of state transition probabilities of a bi-directional chain with stationary distribution π. Equations (3) follow from the fact the p is a matrix of probabilities and the detailed balance equations follow from Proposition 1.

The third bound of (4) follows directly from the first detailed balance equation. The second bound follows from solving for $p_{k,k+2}$ in the CK equation for state $k+2$ and dropping negative terms. Using the first detailed balance equation again, we can replace $\pi_{k+3}p_{k+3,k+1}$ in the CK equation for state $k+1$ with $\pi_k p_{k,k+2}$. Solving for $p_{k,k+2}$ and dropping negative terms yields the first bound. This establishes (4).

The lower bound for $p_{k,k}$ in (5) follows by solving the second detailed balance equation for $p_{k,k}$ and dropping the term containing $p_{k+1,k+1}$. The first upper bound follows from the fact that $p_{k,k}$ and $p_{k,k+2}$ are transition probabilities out of the same state. The second upper bound follows from solving for $p_{k,k}$ in the CK equation for state k, and dropping the term containing $p_{k+1,k}$. This establishes (5). Inequalities (6) and (7) follow similarly. This establishes the left-right implication of Proposition 2.

Now assume the detailed balance equations and (3)-(7). Given Proposition 1 and the fact that p satisfies the detailed balance equations, we are done if we show that all of the elements of p lie in [0,1].

Property (3) tells us that all elements of p with even subscripts are probabilities. From (5) and (6) we infer that $p_{k,k}+p_{k,k+2}$ lies in [0,1] for $k = 0,...,2n-4$, so that $p_{k,k+1} = 1-(p_{k,k}+p_{k,k+2})$ lies in [0,1] for $k = 0,...,2n-4$, as well. Thus, all elements with even *first* subscripts are probabilities.

From the lower bounds in (5) and (6), we derive the inequalities

$$1-p_{k,k} \le \frac{\pi_{k+1}}{\pi_k}, \; k = 0,\ldots,2n-2$$

Multiplying through by π_k and applying the detailed balance equations implies $1-p_{k+1,k+1} \le 1$. This implies that the $p_{k+1,k+1} \ge 0$ for $k = 0,...,2n-2$. The second detailed balance equation implies that $1-p_{k+1,k+1}$ is nonnegative, and thus $p_{k+1,k+1} \le 1$, for $k = 0,...,2n-2$. Hence the $p_{k+1,k+1}$ are probabilities for $k = 0,...,2n-2$. By the first detailed balance equation, $p_{k+3,k+1} \ge 0$ for $k = 0,...,2n-2$. Using the third bound in (4) together with the first detailed balance equation yields $p_{k+3,k+1} \le 0$ for $k = 0,...,2n-2$ as well. Hence, the $p_{k+3,k+1}$ probabilities for $k = 0,...,2n-2$. Finally, applying both detailed balance equations yields

$$\pi_{k+1}\left(1-p_{k+1,k+1}-p_{k+1,k-1}\right) = \pi_k\left(1-p_{k,k}\right) - \pi_{k-2}p_{k-2,k}$$

for k = 2,...,2n-2. But (5) implies $1-p_{k,k} \geq (\pi_{k-2}/\pi_k)p_{k-2,k}$, so $1-p_{k+1,k+1}-p_{k+1,k-1} \geq 0$. Thus, $p_{k+1,k} = 1-p_{k+1,k+1}-p_{k+1,k-1} \geq 0$. Moreover, since we have already shown $p_{k+1,k+1} \geq 0$ and $p_{k+3,k+1} \geq 0$, we conclude $p_{k+1,k} \leq 1$ as well. So the $p_{k+1,k}$ are also probabilities. This completes the proof.

Typically, location distribution data does not include directional information. Consequently, in a bi-directional model, there is no reason to suppose that a particle is more likely to occupy the Eastbound substate of a bin than it is to occupy the Westbound substate. We say that a p.m.f. (and any bi-directional chain with such a p.m.f.) is *symmetric* if $\pi_k = \pi_{k+1}$ for k = 0,2,...,2n-2. This leads to the following corollary to Proposition 2, which we state without proof:

Proposition 3

Let $\pi = \langle \pi_i \mid i = 0,...,2n-1 \rangle$ denote a symmetric, positive-definite, p.m.f. and $p = \langle p_{i,j} \mid i,j = 0,...,2n-1 \rangle$ a matrix of real numbers with unit row sums such that $p_{i,j} = 0$ unless i is even and $i \leq j \leq i+2$ or i is odd and $i-2 \leq j \leq i$. Then π is the stationary distribution of a bi-directional chain with state transition probability matrix p if and only if p satisfies the detailed balance equations and

$$p_{k,k+2} \in [0,1], k = 0,2,...,2n-4$$

$$p_{k,k} \in [0,1], k = 0,2,...,2n-2$$

$$p_{k,k+2} \leq \frac{\pi_{k+2}}{\pi_k}, \quad k = 0,2,...,2n-4 \tag{8}$$

$$p_{k,k} \leq 1 - p_{k,k+2}, 1 - \frac{\pi_{k-2}}{\pi_k} p_{k-2,k} \quad k = 2,4,...,2n-4 \tag{9}$$

$$p_{0,0} \leq 1 - p_{0,2}$$

$$p_{2n-2,2n-2} \leq 1 - \frac{\pi_{2n-4}}{\pi_{2n-2}} p_{2n-4,2n-2}$$

1.3 The Canonical Bi-Directional Chain

With the aid of Proposition 3, we are able to produce a family of concrete examples of a symmetric, bi-directional process with p.m.f. π. Given $\mu \in (0,1]$, let

$$\left.\begin{aligned} p_{k,k+2} &= \mu \min\left(\frac{\pi_{k+2}}{\pi_k}, 1\right) \\ p_{k,k+1} &= 1 - p_{k,k+2} \\ p_{k,k} &= 0 \end{aligned}\right\} k = 0,2,\ldots,2n-4 \tag{10}$$

and let $p_{2n-2,2n-1} = 1$, $p_{2n-2,2n-2} = 0$. We then define $p_{k,k}$ and $p_{k,k-2}$ using the detailed balance equations and $p_{k,k-1} = 1-p_{k,k-2}$, for k = 3,5,...,2n-1. Finally, we define $p_{1,1} = 0$ and $p_{1,0} = 1$. With these definitions, p satisfies the conditions of Proposition 3, so it is the matrix of state transition probabilities of a symmetric, bi-directional chain. It turns out that

$$\left.\begin{aligned} p_{k,k-2} &= \mu \min\left(\frac{\pi_{k-2}}{\pi_k}, 1\right) \\ p_{k,k} &= 1 - p_{k,k-2} \\ p_{k,k-1} &= 0 \end{aligned}\right\} k = 3,5,\ldots,2n-1$$

The factor μ measures a kind of "persistence" in the chain: if μ = 1, an *inter*-bin transition is guaranteed if the probability of the target bin is greater than that of the source bin, otherwise it is the ratio of the target to source probabilities. For $\mu<1$, transitions from less probable to more probable bins are not guaranteed, but occur with probability μ.. We call this chain the *canonical* chain with *persistence* μ.

1.4 Modeling Velocity

Moving particles have velocity. A particle moving according to a bi-directional chain **X** has a zero mean velocity vector, because the motion is random. We can, however, calculate the mean, absolute distance D(**X**) the particle moves in bins (not counting substates) per transition. We call this the *mean transition displacement* or just *mean displacement*. This quantity, together with the time period simulated by the epoch, yields the mean speed of the particle. For any bi-directional chain, this quantity is given by

$$\mathrm{D}(\mathbf{X}) = \sum_{k=0,2,\ldots}^{2n-4} \pi_k p_{k,k+2} + \sum_{k=3,5,\ldots}^{2n-1} \pi_k p_{k,k-2} = 2 \sum_{k=0,2,\ldots}^{2n-4} \pi_k p_{k,k+2} \ ,$$

the simplification due to an application of the detailed balance equations.

Given a bi-directional chain **X** with D(**X**) = Δs and an epoch duration of $\Delta\tau$, we can simulate particle motion with mean speed equal to $n\Delta s/\Delta\tau$, for any positive integer n, simply by "executing" n transitions of **X** per epoch. However, to simulate speeds with any finer resolution than this, we need to be able to modify Δs.

It is easy to see from (10), that the mean displacement of the canonical chain is proportional to its persistence, μ. However, smaller values of μ lead to greater values of $p_{k,k+1}$. This means that the particle will change direction more often, potentially leading to fewer handoffs in a cellular simulation.

A better way to modify the mean displacement is to use a *scaling factor*: Let **X** be any bi-directional chain with state transition probability matrix p, and let $\lambda\in(0,1]$. Define p' as follows

$$p'_{i,j} = \lambda p_{i,j},\ i \neq j$$
$$p'_{i,i} = 1-\lambda\left(1-p_{i,i}\right),\ i = 0,1,\ldots,2n-1$$

It is straightforward to see that the new matrix p' satisfies the conditions of Proposition 2, so it is the matrix of state transition probabilities of a bi-directional chain **X′**. Because the inter-bin transition probabilities are λ times those of **X**, D(**X′**) = λD(**X**). Moreover, since $p'_{k,k\pm1} = \lambda p_{k,k\pm1}$, the tendency to change directions has even decreased. We call **X′** a *scaled*, canonical, bi-directional chain.

So, to simulate the motion of a particle with mean speed v, we do the following: Choose a canonical, bi-directional chain **X** with a desired persistence μ and let Δs = D(**X**). Let

$$n = \left\lceil \frac{v}{\Delta s/\Delta\tau} \right\rceil, \qquad \lambda = \frac{v}{n\Delta s/\Delta\tau}$$

Then, if **X′** is the chain obtained from **X** by applying the scaling factor λ, simulating n transitions of **X′** per epoch achieves mean speed v.

2 TWO DIMENSIONAL MOBILITY MODELS

Serious mobility modeling of cellular systems requires at least two dimensions. In this section, we extend the ideas of the previous section to two-dimensions by pasting together mobility models built on one dimensional cross sections of a two dimensional state space.

2.1 Extension of One Dimensional Models

Let S and T be finite state spaces and let π be a p.m.f. on $S\times T$. For each x in S, let $V(x) = \{x\}\times T$, which we call the *vertical cross section* through $S\times T$ at x. Similarly, for y in T, $L(y) = S\times\{y\}$ is the *lateral* cross section at y. We assume that π is non-zero on any lateral or vertical cross section. For each x in S and y in T, we define p.m.f.'s on T and S:

$$\pi_{V(x)}\left(y\right) = \frac{\pi\left(x,y\right)}{\sum_{t\in T}\pi\left(x,t\right)},\ \pi_{L(y)}\left(x\right) = \frac{\pi\left(x,y\right)}{\sum_{s\in S}\pi\left(s,y\right)} \qquad (11)$$

(In this section, we use functional notation for the arguments of p.m.f's and transition probabilities instead of subscript notation.)

Next, suppose that for each cross section C, there is a transition probability matrix p_C with π_C as its stationary distribution. Let α and β be numbers in (0,1) such that $\alpha+\beta = 1$. We define a transition probability matrix p on $S\times T$ as follows: For $\langle x,y\rangle$ and $\langle x',y'\rangle$ in $S\times T$, let

$$p\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} = \Pr\left[\mathbf{X}_{k+1} = \langle x',y'\rangle \middle| \mathbf{X}_k = \langle x,y\rangle\right]$$

for any k. Then, we define

$$\left.\begin{aligned} p\begin{pmatrix} x & x' \\ y & y \end{pmatrix} &= \alpha\, p_{L(y)}\left(x,x'\right),\ x\in S,\ y\in T,\ x'\neq x \\ p\begin{pmatrix} x & x \\ y & y' \end{pmatrix} &= \beta\, p_{V(x)}\left(y,y'\right),\ x\in S,\ y\in T,\ y'\neq y \\ p\begin{pmatrix} x & x \\ y & y \end{pmatrix} &= \alpha\, p_{L(y)}\left(x,x\right) + \beta\, p_{V(x)}\left(y,y\right) \end{aligned}\right\} (12)$$

with all other transition probabilities assigned zero values. Then, we show

Proposition 4

The matrix p is a transition probability matrix with stationary distribution π.

proof: We first show that p has unit row sums. For $z = \langle x,y\rangle$ in $S\times T$,

$$\sum_{z'\in S\times T} p\left(z,z'\right) =$$
$$p\begin{pmatrix} x & x \\ y & y \end{pmatrix} + \sum_{x'\in S\setminus\{x\}} p\begin{pmatrix} x & x' \\ y & y \end{pmatrix} + \sum_{y'\in T\setminus\{y\}} p\begin{pmatrix} x & x \\ y & y' \end{pmatrix}$$

By substituting the definitions (12), we obtain the expression

$$\alpha\sum_{x'\in S} p_{L(y)}\left(x,x'\right) + \beta\sum_{y'\in T} p_{V(x)}\left(y,y'\right)$$

which equals one because $p_{L(y)}$ and $p_{V(x)}$ have unit row sums and $\alpha+\beta = 1$.

Next, we show that π is the stationary distribution for p. For $z = \langle x,y\rangle$, $z' = \langle x',y'\rangle$ in $S\times T$,

$$\sum_{z\in S\times T}\pi\left(z\right)p\left(z,z'\right) = \pi\left(x',y'\right)p\begin{pmatrix} x' & x' \\ y' & y' \end{pmatrix} +$$
$$\sum_{x\in S\setminus\{x'\}}\pi\left(x,y'\right)p\begin{pmatrix} x & x' \\ y' & y' \end{pmatrix} + \sum_{y\in T\setminus\{y'\}}\pi\left(x',y\right)p\begin{pmatrix} x' & x' \\ y & y' \end{pmatrix}$$

Again, substituting the definitions (12), and rearranging terms yields

$$\alpha\sum_{x\in S}\pi(x,y')p_{L(y')}(x,x')+\beta\sum_{y\in T}\pi(x',y)p_{V(x')}(y,y')$$

Substituting (11), and using the fact that $\pi_{L(y')}$ and $\pi_{V(x')}$ are the stationary distributions for $p_{L(y')}$ and $p_{V(x')}$, we arrive at

$$\alpha\pi_{L(y')}(x')\sum_{x\in S}\pi(x,y')+\beta\pi_{V(x')}(y')\sum_{y\in T}\pi(x',y)$$

Applying (11) once more yields

$$\alpha\pi(x',y')+\beta\pi(x',y')=\pi(x',y')$$

which completes the proof.

Typically, one would choose $\alpha,\beta = ½$. Otherwise, the pattern of motion would have an elliptical bias. However, as will be shown later, there is a specific application for unequal assignment.

2.2 The Quad-Directional Chain

We use Proposition 4 to construct a mobility model for a two dimensional grid. Suppose we are given an $m\times n$ bin grid with p.m.f. $\pi_0 = \langle\pi_0(i,j) \mid i = 0,...,m\text{-}1, j = 0,...,n\text{-}1\rangle$. Let $S = \{0,...,2m\text{-}1\}$ and $T = \{0,...,2n\text{-}1\}$ be the coordinate spaces of a new state space $S\times T$, and let

$$\pi(x,y)=\pi_0(\lfloor x/2\rfloor,\lfloor y/2\rfloor)$$

for $\langle x,y\rangle$ in $S\times T$. In the new state space, each bin now has four substates $\langle x,y\rangle$, $\langle x+1,y\rangle$, $\langle x,y+1\rangle$, and $\langle x+1,y+1\rangle$ for some pair of even numbers x in S and y in T. Let α and β be chosen as in Proposition 4. Note that for each $x = 0,2,...,2m\text{-}2$, $\pi_{V(x)}$ and $\pi_{V(x+1)}$ are identical. Define $p_{V(x)}$ and $p_{V(x+1)}$ as the transition probability matrices for a bi-directional chain with p.m.f. $\pi_{V(x)}$. Define $p_{L(y)}$ similarly for $y = 0,2,...,2n\text{-}2$. Then, the Markov chain with transition probability matrix p, as defined for Proposition 4, is called a *quad-directional* chain. If all the cross sectional chains are scaled, canonical, bi-directional chains with the same persistence and scale factor, the quad-directional chain is said to be a scaled, canonical, quad-directional chain.

Figure 3 illustrates the state space and allowed transitions for a quad-directional chain. The diagram shows one complete bin and parts of others, bounded by thick lines. The substates in each bin are bounded from one another by thin lines. In the illustration, x and y are both even, so inter-bin transitions are allowed in positive directions. Intra-bin transitions are allowed to adjacent substates, $\langle x+1,y\rangle$, $\langle x,y+1\rangle$, and $\langle x+1,y+1\rangle$.

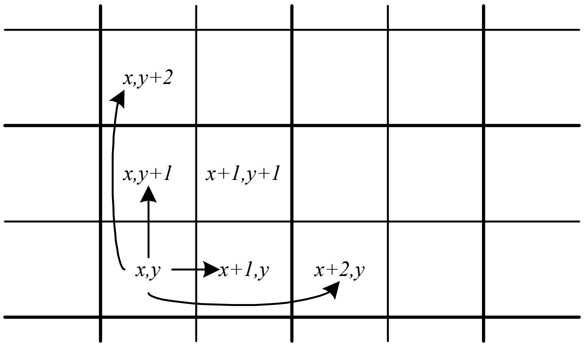

Figure 3: Quad-Directional Model of Mobility

2.3 Modeling Velocity

We compute the *mean transition displacement* D(**Z**) of a two dimensional chain **Z** just as for a one dimensional chain: as the mean number of bins moved in a single transition. We then simulate the velocity of a particle exactly as in section 1.4.

In the two dimensional case, however, we proceed more formally. Let $\mathbf{D}_L$ denote the random variable that assumes the value 1 if **Z** makes an inter-bin transition in the lateral direction and 0 if not. We define $\mathbf{D}_V$ similarly, $\mathbf{D} = \mathbf{D}_L + \mathbf{D}_V$, and D(**Z**) = **E**(**D**), where **E** is the expectation operator.

For a quad-directional chain with transition probability matrix p,

$$\mathbf{E}(\mathbf{D}_L)=\sum_{y=0}^{2n-1}\left\{\sum_{x=0,2,...}^{2m-4}\pi(x,y)\,p\begin{pmatrix}x & x+2\\ y & y\end{pmatrix}+\sum_{x=1,3,...}^{2m-3}\pi(x,y)\,p\begin{pmatrix}x+2 & x\\ y & y\end{pmatrix}\right\}$$

We can then substitute the definitions (11) and (12) and apply the detailed balance equations, arriving at

$$\mathbf{E}(\mathbf{D}_L)=2\alpha\sum_{y=0}^{2n-1}\left\{\sum_{x=0}^{2m-1}\pi(x,y)\sum_{x=0,2,...}^{2m-4}\pi_{L(y)}(x)\,p_{L(y)}(x,x+2)\right\}$$

Similarly,

$$\mathbf{E}(\mathbf{D}_V)=2\beta\sum_{x=0}^{2m-1}\left\{\sum_{y=0}^{2n-1}\pi(x,y)\sum_{y=0,2,...}^{2n-4}\pi_{V(x)}(y)\,p_{V(x)}(y,y+2)\right\}$$

2.4 Interpretation of Unequal α and β

Unequal values for α and β are useful in those cases for which the $m\times n$ grid consists of oblong, rather than square, bins. This occurs when location distribution data is collected by latitude and longitude. Suppose the local, average bin dimensions are Δx along latitudes and Δy along longitudes. Then, the mean transition *physical* displacement of **Z** is

$$\hat{D}(\mathbf{Z}) = \Delta x \mathbf{E}(\mathbf{D}_L) + \Delta y \mathbf{E}(\mathbf{D}_V) \tag{13}$$

The correct choice for α and β should be independent of π and p. If we choose π to be uniform, and the cross sectional chains to be canonical with unit persistence,

$$\mathbf{E}(\mathbf{D}_L) = \alpha \frac{m-1}{m}, \quad \mathbf{E}(\mathbf{D}_V) = \beta \frac{n-1}{n}$$

For large m and n, we may assume $\mathbf{E}(\mathbf{D}_L) = \alpha$ and $\mathbf{E}(\mathbf{D}_V) = \beta$. Moreover we expect that, in this model, the mean *physical* transition displacements in both directions are equal. Thus, by (13), $\alpha\Delta x = \beta\Delta y$. Since $\alpha+\beta = 1$, we conclude

$$\alpha = \frac{\Delta y}{\Delta x + \Delta y}, \quad \beta = \frac{\Delta x}{\Delta x + \Delta y}$$

3 SIMULATION RESULTS

The following example illustrates the advantages of using a quad-directional chain to model mobility. Particles arrive at a 100×100 grid of bins at a rate of 120 per epoch from a Poisson process. They take up locations in the grid according to a symmetric, bivariate, Gaussian distribution with mean at the grid center, and with standard deviation of the marginal distribution equal to 0.2. Particles move about the grid with exponentially distributed speeds with mean = 5 bins/epoch, and depart after an exponentially distributed time with mean equal to 3 ⅓ epochs. Final particle locations are collected from each of 100 independent replications of a 100 epoch simulation. In the plots, the histogram represents the fraction of particles falling in each of 50 equal-width annuli concentric with the center of the grid. The solid line represents the results that would be expected from a perfect Gaussian distribution.

Figure 4 is a histogram of particle location using a simplistic motion model as discussed in the introduction. At every epoch, each particle is moved an exponentially distributed number (mean = 5) of bin widths in a uniformly distributed direction. (If necessary, this step is reversed and repeated until the new particle position lies within the system boundaries.) Note that the distribution is significantly distorted from the true Gaussian, allowing more particles to occupy positions far from the center.

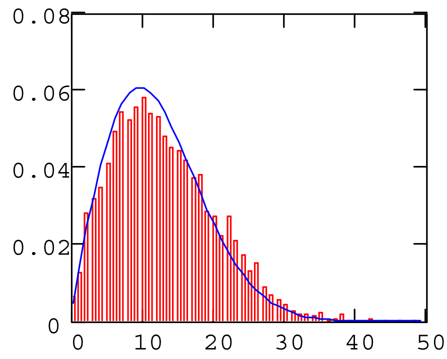

Figure 4: Simplistic Motion

Figure 5 is a histogram of particle location using quad-directional chains. Each chain makes a Poisson number of transitions per epoch, where the mean of the distribution is equal to the mean particle speed. Note that the histogram closely matches the ideal distribution.

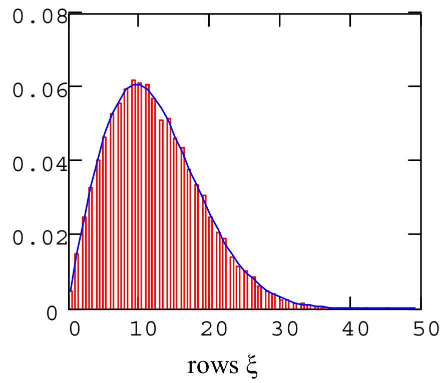

Figure 5: Quad-Directional Motion

4 CONCLUSION

The finite state Markov chain is an effective tool when mobility modeling must exhibit an *a priori* location distribution. More simplistic models can lead to distortion in the location distribution. Such distortion can have serious effects in modeling cellular systems. Examples include exclusion zones and roadways. Exclusion zones are areas in which the operator intends no coverage, such as over non-navigable terrain. A capacity simulation which allowed such locations to be occupied would report pessimistic results. In contrast, roadways are narrow bands of concentrated traffic where coverage is often highly focused. Allowing traffic to "leak" out of such areas could lead to optimistic capacity estimates on the roadway with pessimistic estimates just beyond.

The basic element of the Markov chain toolkit is the one dimensional, bi-directional chain. A specific example of such a chain is the scaled, canonical chain. The scaled, canonical chain is specified by its *persistence*, which is a measure of how likely the particle continues in its current direction, and its *scale factor*, which relates to the speed of the particle. By modifying the scale factor and the number of transitions per epoch, any particle speed may be simulated. One dimensional chains may then be "pasted" together to construct two dimensional chains. When the one dimensional chains are bi-directional, the two dimensional chain is called a *quad-directional* chain. Particle speed may be simulated in an entirely analogous manner as for one dimensional models.

The current work assumes that each particle has a fixed, mean speed. In many applications it would be useful to assign mean speeds as a function of location. In principle, this could be done with quad-directional chains. One might first extend the current work to non-rectangular simulation spaces. Then, one could partition a rectangular simulation space into a small number of areas over which the mean particle speed is fixed and construct quad-directional chains over each area. Particles would not be able to move from one area to another, but there is probably a simple modification that would allow this. Ultimately, one would like to assign a speed or even a velocity distribution to each bin, allowing a "continuous" change in speed/velocity from bin to bin.

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AUTHOR BIOGRAPHY

TONY DEAN is a principal staff engineer at Motorola, Inc., where he conducts performance analysis and modeling of cellular infrastructure systems. His research interests lie in the discrete event simulation of such systems. Prior to joining Motorola, he contributed in the development of a patented token- ring based voice and data communications system, engaged in software development for factory data collection terminals, and supported software products at Digital Equipment Corp. and Burroughs Corp. (now Unisys). He holds a Ph.D. degree in mathematical logic from Indiana University. His email address is <Anthony.Dean@motorola.com>.