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SIMULATION OUTPUT ANALYSIS VIA DYNAMIC BATCH MEANS

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ABSTRACT

This paper is focused on estimating the quality of the sample mean from a steady-state simulation experiment with consideration of computational efficiency, memory requirement, and statistical efficiency. In addition, we seek methods that do not require knowing run length *a priori*. We develop an algorithm of nonoverlapping batch means that is implemented in fixed memory by dynamically changing both batch size and number of batches as the simulation runs. The algorithm, denoted by DBM for Dynamic Batch Means, requires computation time similar to other batch means data-collection methods, despite its fixed memory requirement. To achieve satisfactory statistical efficiency of DBM, we propose two associated estimators, $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$, of the variance of the sample mean and investigate their statistical properties. Our study shows that the estimator $\widehat{V}_{PBM}$ with parameter $w = 1$ is, as a practical matter, better than the other proposed estimators.

1 INTRODUCTION

Consider steady-state data $\{Y_1, Y_2, \ldots\}$ generated from a stochastic simulation experiment for estimating the only performance measure θ by $\widehat{\theta}$, where $\widehat{\theta}$ is a function of $\{Y_1, Y_2, \ldots, Y_n\}$ and n is the run length. We discuss the problem of determining the quality of $\widehat{\theta}$ from both the practitioner's and the researcher's point of view.

A fundamental problem that the practitioner faces is how to determine the quality of the point estimator $\widehat{\theta}$. In other words, what would happen if the simulation experiment were repeated with different random numbers. Usually the variability of $\widehat{\theta}$, the sampling error, is measured by the standard error of $\widehat{\theta}$, $\mathrm{ste}(\widehat{\theta})$, or by its square, $\mathrm{var}(\widehat{\theta})$. Once the standard error is estimated, it can be used to compute a confidence interval of θ or a tolerance interval of $\mathrm{F}_{\hat{\theta}}$, or to conclude how many digits of $\widehat{\theta}$ are meaningful (Song and Schmeiser 1994).

From the practitioner's point of view, a good solution to the problem should meet the following three criteria.

- Small storage requirements: Data storage either in random access memory (ram) or in disk space should require only $O(1)$ space compared with the sample size. Computation cost is free if the practitioner does not need the results from the simulation experiment immediately. Therefore, he or she could run the simulation experiment as long as he or she desires without any restriction on ram or disk space.
- Fast computation: The computation of estimators should involve no more than $O(n)$ computation time. Computationally intensive estimators, e.g., bootstrapping and jackknifing, are inappropriate to use in steady-state simulation because time spent in computing estimators could be used to generate more observations from the simulation experiment.
- Good statistical properties: An estimator, in terms of the mean-square-error (mse) criterion, should have small bias and variance. For the confidence interval of θ, the commonly used criteria are the probability of coverage, the expected length of the the confidence interval, and the standard deviation of length of the confidence interval (Schmeiser 1982).

Therefore, one of the primary tasks for a researcher is to develop procedures and estimators associated with the procedure to estimate $\mathrm{var}(\widehat{\theta})$ or $\mathrm{ste}(\widehat{\theta})$ that satisfy the three solution criteria discussed above.

Often the performance measure θ is a population mean $\mathrm{E}(Y)$ and its associated point estimator $\widehat{\theta} = \overline{\overline{Y}}_n = n^{-1}\sum_{i=1}^{n} Y_i$, the sample mean. We focus on estimating the variance of the sample mean, $\mathrm{var}(\overline{\overline{Y}}_n)$. Throughout this paper, we use the mse criterion to measure the quality of a variance estimator of the sample mean.

This paper consists of two investigations: one is to develop Dynamic Batch Means (DBM), an output-analysis procedure that meets the criteria discussed above; the other is to study two different types of variance estimators to estimate $\mathrm{var}(\widehat{\theta})$ via DBM.

The approach to develop DBM is by using Fishman's idea of doubling batch sizes for Nonoverlapping Batch Means (NBM), see Fishman (1978). NBM is conceptually straightforward, dividing the observations $Y_1, Y_2, \ldots, Y_n$ into b batches each of which is of size m. Therefore, the ith batch consists of the observations $Y_{(i-1)m+1}, Y_{(i-1)m+2}, \ldots, Y_{im}$ for $i = 1, 2, \ldots, b$. Both NBM and DBM transform the correlated observations into fewer batch means that are approximately normal distributed and uncorrelated. NBM and other batching methods as discussed in previous research literature, e.g., Pedrosa (1994), require $O(n)$ space for the data storage. DBM uses only finite memory, increases batch size dynamically as run length increases, and computes the variance of sample mean estimates according to the value of current batch size. Since DBM requires only $O(1)$ space for data storage and $O(n)$ computation time, practitioners can run their simulation experiments via DBM as long as they desire without any restriction on ram or disk space.

To estimate the variance of the sample mean via DBM, we may sometimes have the situation that the number of observations in the last batch is less than previous batches. For other output-analysis methods with $O(n)$ space for data storage, the partial batch becomes asymptotically insignificant as the run length n increases. For DBM, however, as n increases, the batch size increases as well since DBM uses only $O(1)$ space for data storage. The partial batch becomes important if the partial batch size is only a bit less than the size of previous batches and the number of batches is small. We consider two different types of variance estimators to estimate the variance of sample mean: one that truncates the partial batch, denoted by $\widehat{V}_{TBM}$, and one that considers the partial batch, denoted by $\widehat{V}_{PBM}$.

This papers is organized as follows. In Section 2 we review NBM and the quadratic-form coefficients of NBM variance estimator. In Section 3, we propose DBM. In Section 4 we develop and study two variance estimators associated with DBM. We evaluate the performance of these two estimators in Section 5.

2 BACKGROUND

We summarize background information about batching methods, especially for NBM and its quadratic-form estimator. General background can be found in Law and Kelton (1991).

2.1 Nonoverlapping Batch Means Methods

Batching is a classical methodology for estimating the sampling error of $\widehat{\theta}$. Several output-analysis methods based on batching have been proposed, e.g., NBM by Conway (1963), Overlapping Batch Means (OBM) by Meketon and Schmeiser (1984), and Standardized Time Series (STS) by Schruben (1983). NBM has some advantages over other output-analysis methods. In addition to being easy to understand and easy to implement, batch means can be extended by analogy to estimators other than the sample mean, for example the sample standard deviation (Schmeiser et al. 1990). We consider only $\widehat{\theta} = \overline{\overline{Y}}$, the sample mean of the simulation output data $Y_1, Y_2, \ldots, Y_n$.

As discussed in Section 1, NBM divides the observations $Y_1, Y_2, \ldots, Y_n$ into b nonoverlapping batches, each of size m (assuming for now that $n = bm$). The NBM estimator of $\mathrm{var}(\overline{\overline{Y}}_n)$ is defined as

$$\widehat{\mathrm{V}}^{(N)} = \frac{m}{n} \sum_{i=1}^{b} \frac{(\overline{Y}_{i,m} - \overline{\overline{Y}}_n)^2}{b-1}, \tag{1}$$

where the ith nonoverlapping batch mean is

$$\overline{Y}_{i,m} = \frac{1}{m} \sum_{k=1}^{m} Y_{(i-1)m+k}, \tag{2}$$

$i = 1, 2, \ldots, b$. Schmeiser (1982) suggests using ten to thirty NBM batches if only two criteria for the confidence interval procedure are considered: coverage probability and expected length. Song and Schmeiser (1995) and Pedrosa (1994) consider mse for batch size. Goldsman and Schmeiser (1997) discuss the computational efficiency NBM. The variance estimators of NBM requires $O(n)$ time for computation and $O(1)$ space for storage if the batch size m is known.

2.2 Quadratic-Form Estimators

Song and Schmeiser (1993) use the quadratic-form structure to study the properties of different variance estimators of the sample mean, e.g., NBM, OBM, and STS. To derive quadratic-form coefficients is straightforward but requires tedious algebra. This approach, however, provides us intuition into an estimator by visualizing it through a three-dimensional graph of the quadratic-form coefficients. As discussed in Song and Schmeiser (1993), for example, the quadratic-form argument is another way to show that the OBM estimator is asymptotically equivalent to the classical-spectral estimator with Bartlett's window (e.g., Priestley (1981), p. 437–443). Also, these quadratic-form coefficients can be used for computing the covariance of quadratic-form estimators numerically.

The NBM estimator of $\mathrm{var}(\overline{\overline{Y}}_n)$ defined in Equation (1) can be written as the quadratic form

$$\widehat{\mathrm{V}}^{(N)} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}^{(N)} Y_i Y_j, \tag{3}$$

for constant coefficients $q_{ij}^{(N)}$. Song and Schmeiser (1993) derive the quadratic-form coefficients for $\widehat{\mathrm{V}}^{(N)}$ as

$$q_{ij}^{(N)} = \begin{cases} n^{-2} & \text{if } i = 1, 2, \ldots, n; \\ & j = f(i), \ldots, l(i), \\ -n^{-2}(b-1)^{-1} & \text{otherwise,} \end{cases} \tag{4}$$

where $l(i) = \lceil i/m \rceil m$ denotes the subscript of the last observation in the batch that contains Y_i and $f(i) = l(i) - m + 1$ denotes the subscript of the first observation in the batch that contains Y_i. For instance, if $n = 48$ and $m = 12$, then the first observation of the second batch is Y_{13} and the last observation of the second batch is Y_{24}. Therefore, $f(i) = 13$ and $l(i) = 24$ for $i = 13, 14, \ldots, 24$.

Figure 1 shows the the cross-product coefficients $q_{ij}^{(N)}$ for $\widehat{\mathrm{V}}^{(N)}$. The horizontal plane consists of the subscripts i and j, where $1 \le i, j \le n$, and the surface represents the corresponding value of $q_{ij}^{(N)}$. Based on whether i and j are from the same batch or not, NBM assigns all cross products within each batch the same positive weight and all cross products outside the batches the same negative weight.

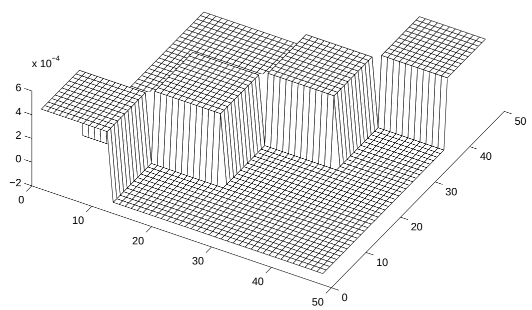

Figure 1: NBM Quadratic-From Coefficients $q_{i,j}^{(N)}$ for $n = 48$ and $m = 12$

3 DYNAMIC BATCH MEANS

In this section, we propose and discuss the Dynamic Batch Mean (DBM) method, which requires only finite memory, even when n is not known *a priori*. DBM is an updating procedure for the cumulative statistics to estimate $\mathrm{var}(\overline{\overline{Y}}_n)$.

3.1 The DBM Method

The key idea of developing DBM is collapsing a vector. In general-purpose computer languages, e.g. C, the observations from simulation are stored in a vector. For DBM, a practitioner needs to specify the size of the vector where the observations are stored: $2k$ where k is a positive integer. A nonoverlapping batch sum, by analogy, is a component of the vector or a cell. Instead of keeping each individual observation, DBM stores the sum of observations for each batch. DBM is called each time that a new observation y_n is generated from a simulation experiment. Whenever the vector is full DBM collapses these $2k$ cells into k cells. The idea of collapsing a vector in DBM is illustrated in Fig 2.

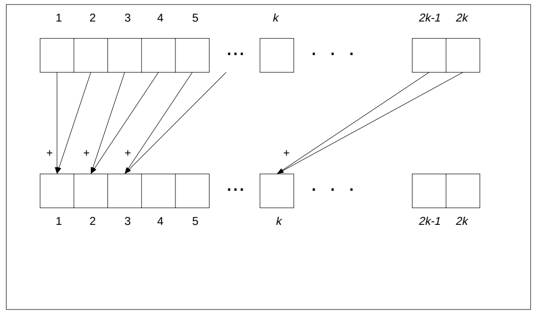

Figure 2: The Idea of Collapsing in DBM

Let $\underline{A}$ be the vector of size $2k$ where DBM stores batch sums. Let $\underline{A}[c]$ be the current cell where DBM stores the latest observation, where $c = 1, \ldots, 2k$, and let p be the number of observations stored in the c-th cell, where $p = 1, \ldots, m$. The **DBM** algorithm is

To update $\underline{A}$ with the new observation y_n, given k.

Initially, before y_1, $p = 1$, $m = 1$, $c = 0$.

Step 1 If the c-th cell has room ($p < m$), then $p \leftarrow p+1$ and go to Step 4.

Step 2 If the vector has room ($c < 2k$), then 2.a. Else 2.b.

- 2.a Increment the current cell, $c \leftarrow c + 1$.
- 2.b Collapse the vector. Initialize the current cell c and double the batch size m.
  - (1) For ($i = 1, \ldots, k$) $A[i] \leftarrow A[2i-1] + A[2i]$;
  - (2) $c \leftarrow k + 1$ and $m \leftarrow 2m$.

Step 3 Initialize the value of p and the sum stored in the current cell, $p \leftarrow 1$ and $A[c] = 0$.

Step 4 Add the new observation y_n in the current cell, $A[c] \leftarrow A[c] + y_n$.

Step 5 Return.

DBM adds a new observation in the current cell if the number of observations contained in the current cell, p, is less than the full-batch size m, indicating the vector is not full. As long as the c-th cell contains the same number of observations as the full-batch size and the vector is full, DBM collapses theses $2k$ cells into k cells. After collapsing the vector, DBM updates the full-batch size by doubling the previous value.

3.2 Discussion

Rather than obtaining the state variables, c, p, and m, directly from the program, DBM could compute these values by using n and k alone. Given the values of n and k, the values of m, c, and p are also determined. For m, its value is determined by

$$m = 2^{\lceil \log_2 \frac{n}{k} \rceil - 1}, \tag{5}$$

where $m = 1, 2, 4, 8, 16, \ldots$. Given the value of m from Equation (5), the current batch is

$$c = \lceil \frac{n}{m} \rceil, \tag{6}$$

where $c = 1, \ldots, 2k$. From Equations (5) and (6), the number of observations in the current batch is

$$p = n - (c-1)m, \tag{7}$$

where $p \in \{1, \ldots, m\}$. Conversely, the observation number can be computed using $n = (c-1)m + p$.

As mentioned previously, a practitioner needs to specify the size of the vector, $2k$, before using DBM. For DBM, once the number of observations n is determined, the number of DBM batches, c, is determined by Equations (5) and (6). The value for the number of DBM batches is between k and $2k$ if $n > k$. The number of batches for NBM Schmeiser citeschmeiser82 suggested can be applied for DBM since DBM is NBM in essence. He suggests using ten to thirty NBM batches for the reasons discussed in Section 2. The value of k we suggest for DBM is between ten and fifteen, so that the number of DBM batches is in the range from ten to thirty.

For a fixed value of k, therefore, DBM meets the first two criteria discussed in Section 1, i.e., $O(1)$ storage and $O(n)$ computation time.

The reason we use the idea of doubling batch size instead of tripling batch size or a higher rate is to reduce the potential wasting of degrees of freedom. Doubling batch size is more statistically efficient. In the case of $n = 12$ and $k = 3$, using the idea of doubling batch size creates six batches each of size two. On the other hand, using tripling batch size creates only four batches each of size three.

DBM can be applied for analyzing outputs from a simulation experiment with the initial-transient problem by combining existing methods for the initial-transient problem, e.g. initial-data truncation. As long as we decide the truncation point, we start to use DBM after the truncation point. DBM can be also applied for analyzing outputs without combining the methods for the initial-transient problem since we can run a simulation experiment via DBM as long as we desire. As simulation run length n increases, the batch size in DBM increases. Therefore, the initial bias becomes asymptotically negligible. Nevertheless, we consider only steady-state simulation throughout this paper because the statistical performances of variance estimators for DBM are more tractable.

4 ESTIMATING VARIANCE OF THE SAMPLE MEAN VIA DBM

Given the information about the vector $\underline{A}$ and its state variables, c, p, and m from DBM, the sample mean $\overline{\overline{Y}}_n$ is obtained by $\sum_{i=1}^{c} A[i]/n$. The i-th batch mean is obtained by $\overline{Y}_i = A[i]/m$ for $i = 1, 2, \ldots, c-1$ and the last-batch mean in DBM is obtained by $\overline{Y}_c = A[c]/p$. As n increases, the full-batch size m also increases. Practitioners may end up with the situation that the last-batch size p is relatively large. We propose and study two estimator $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ to estimate $\mathrm{var}(\overline{\overline{Y}}_n)$ when the number of observations in the last batch is not the same as previous full batches.

4.1 Definition of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$

The estimator $\widehat{V}_{TBM}$ of $\mathrm{var}(\overline{\overline{Y}}_n)$ is defined as

$$\widehat{V}_{TBM} = (\frac{bm}{n}) \frac{\sum_{i=1}^{b} (\overline{Y}_{i,m} - \overline{\overline{Y}}_{bm})^2}{b(b-1)}, \tag{8}$$

where b is the number of full batches, $b = c - 1 + \lfloor p/m \rfloor$, $\overline{Y}_{i,m}$ denotes the i-th nonoverlapping batch mean, and $\overline{\overline{Y}}_{bm}$ denotes the sample mean of $Y_1, \ldots, Y_{bm}$. The right fraction in Equation (8) is the classical NBM estimator of $\mathrm{var}(\overline{\overline{Y}}_{km})$, which does not consider the final partial batch. If $p = m$, Equation (8) is equivalent to the NBM estimator of $\mathrm{var}(\overline{\overline{Y}}_n)$. The left fraction in Equation (8) is a correction factor; asymptotically the variance of $\overline{\overline{Y}}_n$ over the variance of $\overline{\overline{Y}}_{bm}$ is $(bm)/n$. If the batch means are asymptotically uncorrelated then the correction factor bm/n is equal to the asymptotic ratio of the variance of $\overline{\overline{Y}}_n$ to the variance of $\overline{\overline{Y}}_{bm}$. In other words,

$$\lim_{n \to \infty} \frac{n}{bm} \frac{\mathrm{var}(\overline{\overline{Y}}_n)}{\mathrm{var}(\overline{\overline{Y}}_{bm})} = 1, \text{ where } bm \leq n. \tag{9}$$

The estimator $\widehat{V}_{PBM}$ of $\mathrm{var}(\overline{\overline{Y}}_n)$ is defined as

$$\widehat{V}_{PBM} = \frac{\sum_{i=1}^{c-1} (\overline{Y}_{i,m} - \overline{\overline{Y}}_n)^2 + (\frac{p}{m})^w (\overline{Y}_{c,p} - \overline{\overline{Y}}_n)^2}{(c-1)(c-2+(\frac{p}{m})^{w-1} + \frac{p}{m})}, \tag{10}$$

where $w \geq 0$, c is the number of batches stored in DBM, p is the partial-batch size, and m is the full-batch size. In Equation (10), we use the function $(p/m)^w$ to weight variance of the batch mean for the partial batch compared to variance of the batch mean for the full batch, which is weighted to one.

The estimator $\widehat{V}_{TBM}$ can be rewritten as the quadratic form

$$q_{ij}^{(T)} = \begin{cases} \dfrac{bm}{n^3} & \text{if } i = 1, 2, \ldots, bm \\ & \text{and } j = f(i), \ldots, l(i), \\ \dfrac{-bm}{n^3(b-1)} & \text{if } i = 1, 2, \ldots, bm-1 \\ & \text{and } j = l(i)+1, \ldots, bm, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

where $f(i)$ and $l(i)$ are defined in Section 2. These quadratic-form coefficients are similar to those quadratic-form coefficients of the classical NBM estimator in Equation (4). One can simply multiply the quadratic-form coefficients in Equation (4) by $(bm)/n$ to get the result in Equation (11) because $\widehat{V}_{TBM}$ considers the partial batch but truncates it. Therefore, $\widehat{V}_{TBM}$ weights zeros to those data in the final partial batch.

The estimator $\widehat{V}_{PBM}$ can be rewritten as the quadratic form

$$q_{ij}^{(P)} = \begin{cases} \dfrac{\dfrac{(c+p/m-2)^2+c-2}{(c-1+p/m)^2m^2+(p/w)^w}}{(c+(p/m)^{w-1}+p/m-2)(c-1)} \\ \text{if } i = 1, 2, \ldots, (c-1)m \\ \text{and } j = f(i), \ldots, l(i); \\ \dfrac{\dfrac{-c+1-2p/m+(p/w)^w}{(c-1+p/m)^2m^2}}{(c+(p/m)^{w-1}+p/m-2)(c-1)} \\ \text{if } i = 1, 2, \ldots, (c-1)m-1 \\ \text{and } j = l(i)+1, \ldots, (c-1)m; \\ \dfrac{\dfrac{(c-1)p^2+(c-1)^2(p/m)^w m^2}{(c-1+p/m)^2m^2p^2}}{(c+(p/m)^{w-1}+p/m-2)(c-1)} \\ \text{if } i = (c-1)m+1, \ldots, n \\ \text{and } j = (c-1)m+1, \ldots, n; \\ \dfrac{\dfrac{-(c-1)(p/m)^w m - p^2/m}{(c-1+p/m)^2m^2p}}{(c+(p/m)^{w-1}+p/m-2)(c-1)} \\ \text{if } \min(i,j) = 1, 2, \ldots, (c-1)m \\ \text{and } \max(i,j) = (c-1)m+1, \ldots, n, \end{cases} \tag{12}$$

where $f(i)$ and $l(i)$ are defined in Section 2. The first part of $q_{ij}^{(P)}$ is the quadratic coefficient when both the observations Y_i and Y_j are in the same full batch. The second part of $q_{ij}^{(P)}$ is the quadratic coefficient when both the observations Y_i and Y_j are in different full batches. The third part of $q_{ij}^{(P)}$ is the quadratic coefficient when the observations in which both Y_i and Y_j are in the partial batch. The fourth part of $q_{ij}^{(P)}$ is the quadratic coefficient when one of the observations is in a full batch and the other is in the partial batch.

To get some intuition for the estimators $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$, we can view them through three-dimensional graphs $q_{ij}^{(T)}$ or $q_{ij}^{(P)}$ versus i and j. Using Equations (11) and (12), we view $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ through three-dimensional graphs $q_{ij}^{(T)}$ and $q_{ij}^{(P)}$ versus i and j. From Figure 3 we see the missing partial batch in $\widehat{V}_{TBM}$. From Figures 4–5, we can see how the quadratic-form coefficients of $\widehat{V}_{PBM}$ change by setting different value of w. As w approaches infinity, the three-dimensional graphs of $q_{ij}^{(P)}$ become almost identical to the three-dimensional graph of $q_{ij}^{(T)}$ except for those $q_{ij}^{(T)}$'s within the partial batch. Comparing Figure 2 with Figure 3, we see how $\widehat{V}_N$ and $\widehat{V}_{TBM}$ weights differently on the edges of the three-dimensional graphs.

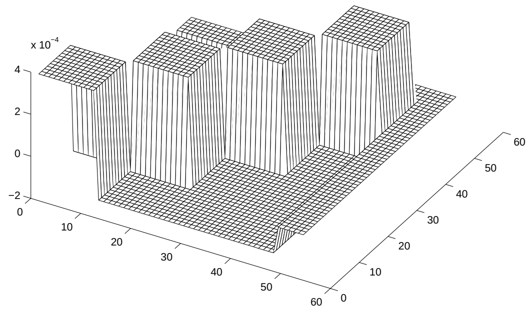

Figure 3: Graph of the Quadratic-Form Coefficients of $\widehat{V}_{TBM}$, $q_{i,j}^{(T)}$ with a Half Partial Batch: $n = 54$, $b = 4$, and $m = 12$

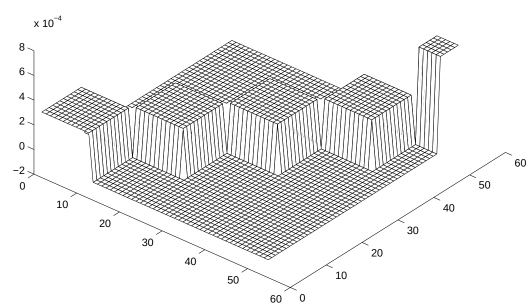

Figure 4: Graph of the Quadratic-Form Coefficients of $\widehat{V}_{PBM}$, $q_{i,j}^{(P)}$, with a Half Partial Batch and $w = 1$, $n = 54$, $c = 5$, and $m = 12$

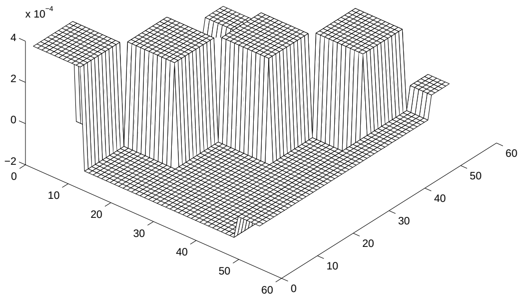

Figure 5: Graph of the Quadratic-Form Coefficients of $\widehat{V}_{PBM}$, $q_{i,j}^{(P)}$, with a Half Partial Batch and $w = 3$, $n = 54$, $c = 5$, and $m = 12$

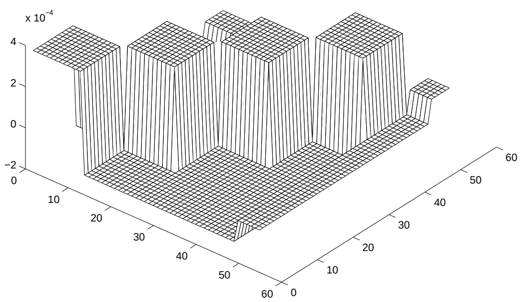

Figure 6: Graph of the Quadratic-Form Coefficients of $\widehat{V}_{PBM}$, $q_{i,j}^{(P)}$, with a Half Partial Batch and $w = 50$, $n = 54$, $c = 5$, and $m = 12$

4.2 Statistical Properties of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$

For IID data, $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ are unbiased (Yeh 1999). The estimator $\widehat{V}_{TBM}$ is biased in general, however, and the bias of $\widehat{V}_{PBM}$ is not tractable.

Result 4.1 *If full-batch means are uncorrelated, then*

$$\mathrm{var}(\widehat{V}_{TBM}) = \frac{m^2\sigma^4(m)}{n^2 b}(\alpha_4(m) - \frac{b-3}{b-1}), \tag{13}$$

where $\alpha_4(m)$ is the kurtosis of a full-batch mean and $\sigma^2(m)$ is the variance for the distribution of a full-batch mean, $\overline{Y}_i$, a function of m and in turn a function of n.

Proof: From Wilks (1962) p. 200, we have

$$\mathrm{var}(\frac{\sum_{i=1}^{b}(\overline{Y}_{i,m} - \overline{\overline{Y}}_{bm})^2}{b(b-1)}) = \frac{\sigma^4(m)}{b^3}(\alpha_4(m) - \frac{b-3}{b-1}).$$

Therefore,

$$\begin{aligned}\mathrm{var}(\widehat{V}_{TBM}) &= \frac{(bm)^2}{n^2}\mathrm{var}(\frac{\sum_{i=1}^{b}(\overline{Y}_{i,m} - \overline{\overline{Y}}_{bm})^2}{b(b-1)}) \\ &= \frac{b^2m^2\sigma^4(m)}{n^2b^3}(\alpha_4(m) - \frac{b-3}{b-1}) \\ &= \frac{m^2\sigma^4(m)}{n^2 b}(\alpha_4(m) - \frac{b-3}{b-1}).\end{aligned}$$

Pedrosa (1994) shows that under certain sufficient conditions the variance of the OBM estimator, $\widehat{V}^{(O)}$, can be approximated as

$$\mathrm{var}(\widehat{V}^{(O)}) \approx 2\ R_0^2\ {\gamma_0}^2 \left[\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij}^{(O)})^2\right], \tag{14}$$

where $R_0 = \mathrm{var}(Y_i)$ and $\gamma_0 = \sum_{h=-\infty}^{\infty}\rho_h$, and ρ_h is the lag-h autocorrelation. Similar results still hold for $\widehat{V}^{(N)}$. That is,

$$\mathrm{var}(\widehat{V}^{(N)}) \approx 2\ R_0^2\ {\gamma_0}^2 \left[\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij}^{(N)})^2\right]. \tag{15}$$

The variance of $\widehat{V}_{PBM}$ can be computed by Equation (15) since $\widehat{V}_{PBM}$ is just a variant of $\widehat{V}^{(N)}$. We apply golden section search to find that the variance of $\widehat{V}_{PBM}$ is minimal when $w = 1$ in several different cases. To verify this statement visually, we generate several graphs of the variance $\widehat{V}_{PBM}$ with different partial-batch sizes. From Figures 7–10, the variance of $\widehat{V}_{PBM}$ is minimized when $w = 1$. When $p = m$, the variance of $\widehat{V}_{PBM}$ is constant regardless of the value of w. For $p = m$, $(p/m)^w = 1$ and therefore the variance of $\widehat{V}_{PBM}$ is the same for all $w \geq 0$.

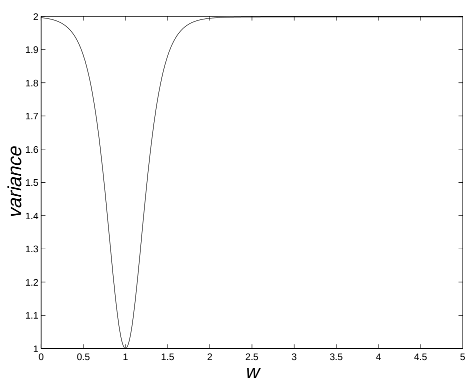

Figure 7: Variance of $\widehat{V}_{PBM}$ as a Function of w for n=2049, c=3, m=1024, p=1, R_0=1 and γ_0=1

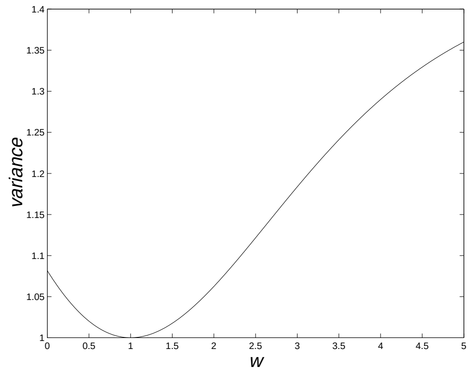

Figure 8: Variance of $\widehat{V}_{PBM}$ as a Function of w for n=2560, c=3, m=1024, p=512, R_0=1 and γ_0=1

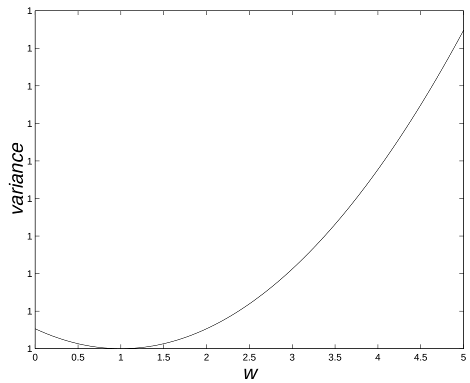

Figure 9: Variance of $\widehat{V}_{PBM}$ as a Function of w for n=3071, c=3, m=1024, p=1023, R_0=1 and γ_0=1

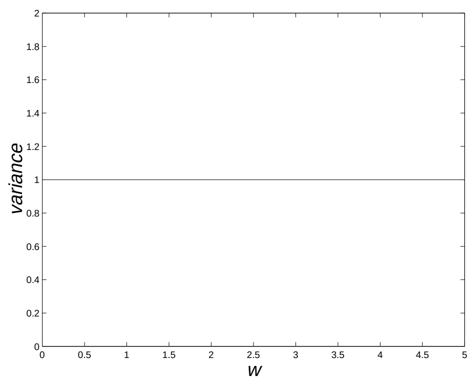

Figure 10: Variance of $\widehat{V}_{PBM}$ as a Function of w for n=3072, c=3, m=1024, p=1024, R_0=1 and γ_0=1

In the case of dependent data, however, the bias of $\widehat{V}_{PBM}$ is not easy to derive. We show empirically in the next section that $w = 1$ works well for dependent, as well as independent, data.

In DBM, the number of full batches b is bouncing between k and $2k$ as n increases. Therefore, the limiting value of $n^2 b\ \mathrm{var}(\widehat{V}_{TBM})$ does not exist. Equations (16) and (17), however, provide asymptotic upper bound and lower bounds on $n^2 b\ \mathrm{var}(\widehat{V}_{TBM})$.

Asymptotically $\alpha_4(m) \to 3$ and $m\sigma^2(m) \to R_0\gamma_0$. Because $k \le b \le 2k$, Equation (13) implies

$$\limsup_{n\to\infty} n^2 b\ \mathrm{var}(\widehat{V}_{TBM}) = {R_0}^2\gamma_0^2(3 - \frac{k-3}{k-1}) \qquad (16)$$

and

$$\liminf_{n\to\infty} n^2 b\ \mathrm{var}(\widehat{V}_{TBM}) = {R_0}^2\gamma_0^2(3 - \frac{2k-3}{2k-1}). \qquad (17)$$

Another interesting property of the asymptotic value of $n^2 b\ \mathrm{var}(\widehat{V}_{TBM})$ is that the fractions $3-(k-3)/(k-1)$ and $3-(2k-3)/(2k-1)$ in Equations (16) and (17) have some connections to the classical NBM estimator, $\widehat{V}^{(N)}$. Goldsman and Meketon (1986) show that the classical NBM estimator has following property:

$$\lim_{n\to\infty} n^2 b\ \mathrm{var}(\widehat{V}^{(N)}) = {R_0}^2\gamma_0^2 c_v \qquad (18)$$

if $b \to \infty$ as $n \to \infty$, where the variance constant $c_v = 2$. Comparing to the fractions $3-(k-3)/(k-1)$ and $3-(2k-3)/(2k-1)$ in Equations (16) and (17), the fractions approach 2 as k goes to infinity. The value of k in DBM, however, is fixed. Therefore, the variance constant for $\mathrm{var}(\widehat{V}_{TBM})$ is bouncing between $3-(k-3)/(k-1)$ and $3-(2k-3)/(2k-1)$. In other words, asymptotically $\widehat{V}_{TBM}$ in DBM is not as efficient as the classical NBM estimator $\widehat{V}^{(N)}$. The variance of $\widehat{V}_{PBM}$ exhibits similar behaviors.

5 COMPARISON OF $\widehat{V}_{TBM}$ AND $\widehat{V}_{PBM}$ FOR DEPENDENT AND NON-NORMAL DATA

In Section 4, we propose two variance estimators, $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$, and study some of their statistical properties. The performances of these estimators are, however, unknown in more realistic situations. The question of which estimator practitioners should use in practice is considered here.

To compare the DBM variance estimators $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ in more-general situations, we examine them using non-normal and autocorrelated data, AR(1) and EAR(1) (Schmeiser and Song 1989). We empirically compare the performances of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ and conclude that, as a practical matter, $\widehat{V}_{PBM}$ with $w = 1$ is the mse-optimal variance estimator. We discuss the design of the Monte Carlo experiments and present the results next.

5.1 The Monte Carlo Experiment

The purpose of this empirical study is to compare the performance of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ in a more realistic situation and to see how mse changes as w, k, ϕ and the marginal distributions change. The results for $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ are presented in terms of squared bias, variance and mse.

At each of the design points, two processes are simulated: AR(1) with $\phi = 0.85$ and EAR(1) with $\phi = 0.85$. These two processes for each design point are initialized from the appropriate steady-state distribution. The sample size is $n = 1153$. Six estimators, $\widehat{V}_{PBM}$ with $w = 0.5, 1, 2, 10, 50$ and $\widehat{V}_{TBM}$, are compared. The DBM parameter k we choose in the Monte Carlo experiments is between 2 and 80. Table 1 gives the associated k value with full-batch size m, number of full batches $c-1$, and partial-batch size p when $n = 1153$.

Table 1: The Relationship of Number of Full Batches $c-1$, Full-Batch Size m, and Partial-Batch Size p, for $k = 2$ to 80 with $n = 1153$

k	$c-1$	m	p
2	2	512	129
3 to 4	4	256	129
5 to 9	9	128	1
10 to 18	18	64	1
19 to 36	36	32	1
37 to 72	72	16	1
73 to 80	144	8	1

The results are based on 50 independent micro-replications within 60 independent macro-replications at each design point. Common random numbers are used. In all cases, $\mathrm{var}(\overline{\overline{Y}}_n) = 1$. Therefore, the degenerate estimator $\widehat{V} = 0$, which is dimensionless, has bias $= 1$, $\mathrm{var}(\widehat{V}) = 0$, and mse $= 1$.

5.2 MSE of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$

The numerical studies in Yeh (1999) demonstrate that the mse of $\widehat{V}_{PBM}$ with $w = 1$ is better than the mse of $\widehat{V}_{PBM}$ with w other than 1. Here, we use $w = 1$ for $\widehat{V}_{PBM}$ and compare it to $\widehat{V}_{TBM}$ in this section.

As a practical matter, $\widehat{V}_{PBM}$ with $w = 1$ is better than $\widehat{V}_{TBM}$. For k relatively small, the mse of $\widehat{V}_{PBM}$ with $w = 1$ is much smaller than the mse of $\widehat{V}_{TBM}$. Even a tiny partial batch helps to reduce more variance than squared bias it introduced. As k increases, mse of $\widehat{V}_{PBM}$ with $w = 1$ does not dominate $\widehat{V}_{TBM}$ but is close to the mse of $\widehat{V}_{TBM}$. As shown in Figures 11 and 12, the estimator $\widehat{V}_{PBM}$ with $w = 1$ is often best when k is small and is close to the best when k is large. The amount by which $\widehat{V}_{PBM}$ with $w = 1$ loses, in practice, is insignificant. Because we want to set k small, the estimator $\widehat{V}_{PBM}$ with $w = 1$ is the best estimator.

The results of AR(1) are similar to the results of EAR(1) even though the marginal distributions are different. By the central limit theorem, batch means converge to the normal distribution as batch size becomes large. Therefore, the similarity between the results is no surprise.

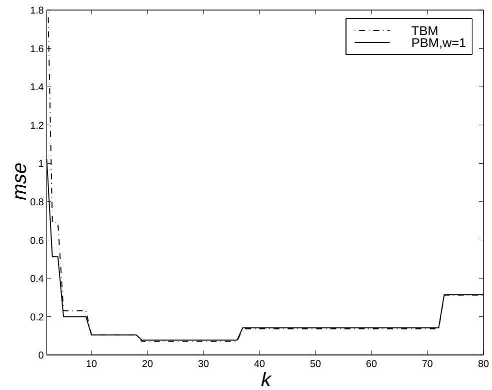

Figure 11: MSE of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ with $w = 1$ for AR(1) with $\phi = 0.85$

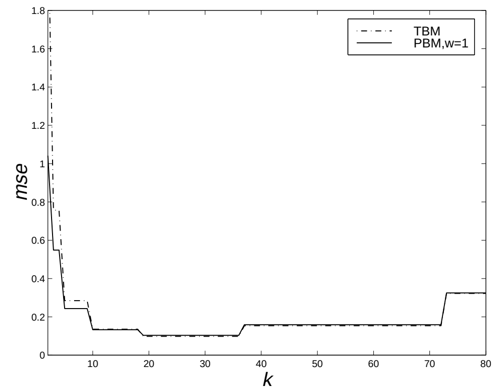

Figure 12: MSE of $\widehat{V}_{TBM}$ and $\widehat{V}_{PBM}$ with $w = 1$ for EAR(1) with $\phi = 0.85$

6 CONCLUSION

We conclude that DBM is a method that satisfies all three desired properties. For small storage requirement, most of the algorithms require $O(n)$ storage; DBM requires only finite $O(1)$ storage and does not require knowing run length *a priori*. For fast computation, in respect to computational efficiency, no algorithm can estimate the variance of the sample mean in less than $O(n)$ time; DBM requires $O(n)$ computation time. For good statistical properties, we conclude that the variance estimator $\widehat{V}_{PBM}$ with $w = 1$, as a practical matter, is the best estimator for use with DBM. Based on analytical and experimental comparisons with the power w of $\widehat{V}_{PBM}$ and with $\widehat{V}_{TBM}$, the estimator $\widehat{V}_{PBM}$ with $w = 1$ provides a good mse performance in general.

The estimator $\widehat{V}_{PBM}$ with $w = 1$ could be applied in NBM as well whenever there exists a partial batch. Including the information contained in the partial batch improves statistical efficiency.

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