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GENERATING DAILY CHANGES IN MARKET VARIABLES USING A MULTIVARIATE MIXTURE OF NORMAL DISTRIBUTIONS

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ABSTRACT

The mixture of normal distributions provides a useful extension of the normal distribution for modeling of daily changes in market variables with fatter-than-normal tails and skewness. An efficient analytical Monte Carlo method is proposed for generating daily changes using a multivariate mixture of normal distributions with arbitrary covariance matrix. The main purpose of this method is to transform (linearly) a multivariate normal with an input covariance matrix into the desired multivariate mixture of normal distributions. This input covariance matrix can be derived analytically. Any linear combination of mixtures of normal distributions can be shown to be a mixture of normal distributions.

1 INTRODUCTION

The normal distribution is the most commonly used model of daily changes in market variables. Many studies (Wilson [1993, 1998], Zangari [1996], Venkataraman [1997], Duffie and Pan [1997] and Hull and White [1998]) show that the distributions of daily changes, such as returns in equity, foreign exchanges, and commodity markets, are frequently asymmetric with fat tails. The assumption of normality is far from perfect and often inappropriate. Mixture of normals is a more general and flexible distribution for fitting the market data of daily changes. It fully takes into account the kurtosis and skewness in market variables. In addition, the normal distribution is a special case of the mixture of normal distributions.

Mixture of normal distributions has been successfully applied in many fields including economics, marketing, and finance (Clark [1973], Zangari [1996], Venkataraman [1997], Duffie and Pan [1997], Hull and White [1998], and Wang [2000]). Recently the mixture of normal distributions has become a popular model for the distribution of daily changes in market variables with fat tails. Another widely used model of fat-tailed distributions is the multivariate t distribution (Zangari [1996], Wilson [1998], and Glasserman, Heidelberger, and Shahabuddin [2000]). Pointed out by Glasserman, Heidelberger, and Shahabuddin [2000], a shortcoming of the multivariate t distribution is that all daily changes in market variables have equally fat tails (sharing the same degrees of freedom). They use a *copula* to extend this model to allow multiple degrees of freedom. However, non-linear inverse transformations are needed for this extension.

The commonly used method for generating a multivariate distribution with arbitrary marginal distributions and a covariance matrix is the three-step method (Schmeiser [1991], Cario and Nelson [1997], and Hull and White [1998]). The early work on this topic can be found in Mardia [1970], and Li and Hammond [1975]. Step one is to generate a multivariate normal with an input covariance matrix. Step two is to transform this multivariate normal into a multivariate uniform distribution on (0,1). Step three is to transform this multivariate uniform distribution on (0,1) into the desired multivariate distribution via inverse functions. Both the transformations in steps one and two are nonlinear. There is no general, analytical way, to determine the input covariance matrix for step one. We have to solve nonlinear equations to derive a numerical approximation of the input covariance matrix. The most difficult part is to find the inverse marginal distributions. It is often time consuming, but sometimes there is no other alternative.

We propose an efficient Monte Carlo method for generating daily changes in market variables using a multivariate mixture of normal distributions with an arbitrary covariance matrix. It is different from the three-step method. We transform (linearly) a multivariate normal with an input covariance matrix into the desired multivariate mixture of normal distributions. This input covariance matrix can be derived analytically. The marginal distributions are mixtures of normals and may have a different number of components. In other words, daily changes may have different degrees of fat tails and skewness.

In calculation portfolio value-at-risk (VaR), we provide an efficient analytic computation method. We prove that a linear combination of mixtures of normals is a mixture of normals under our setting. Therefore under the assumption of the multivariate mixture of normals, the total portfolio return is a mixture of normals.

Our work is related to the work of calculating value-at-risk by Zangari [1996], Venkataraman [1997], Hull and White [1998], and Li [1999]. Zangari [1996] proposes a RiskMetrics^{TM} [1995] method that allows for a more realistic model of the financial return tail distribution. The current RiskMetrics^{TM} assumption is that returns follow a conditional normal distribution. They used a mixture of two distributions to model fat tails. Gibbs sampler is the tool of estimating this return distribution. Venkataraman [1997] used the same model of mixture of two normals, but the estimation technique was the quasi-Bayesian maximum likelihood approach, which was first proposed by Hamilton [1991]. Hull and White [1998] proposed a maximum likelihood method to estimate parameters of the mixture of two normal distributions. Market data was divided by standard deviation into four categories, then they estimated parameters by fitting the quantiles of the distribution. In addition, they described a method of generating a multivariate distributions with arbitrary marginal distributions and a covariance matrix. Their idea is basically the same as the three-step method. Li [1999] uses the theory of estimating functions to construct an approximate confidence interval for calculation of value at risk. Kurtosis and skewness are explicitly used in his study.

Most of work in the literature are focused on estimating the mixture of normal distributions. There is less work on generating daily changes using a multivariate mixture of normal distributions. The difficulty is in how to handle the correlation matrix. The main contribution of this paper is to provide an efficient Monte Carlo method for generating a multivariate mixture of normal distributions with an arbitrary covariance matrix. The correlation matrix can be handled efficiently and analytically.

This paper proceeds as follows. Section 2 discusses fat tails and skewness in daily change distributions. We derive the general results of the kurtosis and skewness for mixture of normal distributions. In an example, we compare the normal distribution with the mixture of two normals. Our figure shows the significant difference between the two densities in the sense of kurtosis and skewness even though they have the same mean and variance. Section 3 discusses how to generate a multivariate mixture of normals with an arbitrary covariance matrix. The marginal distribution of each variables is a mixture of normals, but the number of components in the mixture may be different for each variable. In the univariate case, we introduce a method demonstrating how to generate the univariate mixture of normals. In the multivariate case, we derive an efficient method to generate a multivariate mixture of normals. A detailed ready-to-use algorithm is given. Section 4 provides an efficient analytic method for calculating portfolio VaR. Under the assumption of the multivariate mixture of normals, calculating portfolio VaR is just as simple as under the normality assumption. Section 5 provides a summary and conclusions.

2 FAT TAILS AND SKEWNESS IN DAILY CHANGE DISTRIBUTIONS

Many studies (Duffie and Pan [1997] and Hull and White [1998]) show that daily changes in many variables, such as S&P 500, NASDAQ, NYSE All Share, and particularly exchange rates, exhibit significant amounts of positive kurtosis and negative skewness. Distributions of daily changes in these variables have fat tails and are typically skewed to the left. This implies that extremely large market moves and particularly negative returns are more likely than a normal distribution would predict when taking into account the kurtosis and skewness. In this section, first of all, we describe the definition of kurtosis and skewness. Secondly, we introduce mixture of normal distribution. Finally, we provide an example of the mixture of two normal distributions and compare this with a normal distribution. A figure demonstrates the significant difference of kurtosis and skewness between the two densities even though they have the same mean and variance.

2.1 Kurtosis and Skewness

Kurtosis is a measure of how fat the tails of a distribution are, which is found from the fourth central moment of daily change. It is very sensitive to extremely large market moves. Skewness is a measure of asymmetry, which is found from the third central moment of daily change. It measures the degree of difference between positive deviations from the mean and negative deviation from the mean. In general, the normalized skewness and kurtosis of a random variable X are defined as follows, respectively (See Casella and Berger [1990]):

$$\alpha_3 = \frac{\mathrm{E}(X-\mu)^3}{\sigma^3} \quad \text{and} \quad \alpha_4 = \frac{\mathrm{E}(X-\mu)^4}{\sigma^4}. \tag{1}$$

where $\mu = \mathrm{E}(X)$ and $\sigma^2 = \mathrm{E}(X-\mu)^2$. The standardized kurtosis is defined as

$$\alpha_4' = \frac{\mathrm{E}(X-\mu)^4}{\sigma^4} - 3,$$

which is a relative measure used for comparison with the normal density. Any normally distributed random variable has kurtosis of three and skewness of zero. As discussed in Duffie and Pan [1997], for many markets, returns have fatter than normal tails with negative skewness. The S&P 500 daily returns from 1986 to 1996, for example, have an extremely high sample kurtosis of 111 and negative skewness of -4.81. Obviously, the normal distribution used as a model to these market data is quite inappropriate.

2.2 Mixture of Normal Distributions

In this subsection, we describe the univariate mixture of k normal distributions and derive its basic properties. The mixture of two normal distributions, as a simple example, is discussed. Comparing with the standard normal distribution, it has the same mean and variance, but different kurtosis and skewness. Figure 1 shows the significant difference between the two densities.

In general, the cumulative distribution function (cdf) of a mixture of k normal random variable X is defined by

$$F(x) = \sum_{j=1}^{k} p_j \Phi\left(\frac{x-\mu_j}{\sigma_j}\right), \tag{2}$$

where Φ is the cdf of $N(0,1)$. Therefore its probability density function (pdf) is

$$f(x) = \sum_{j=1}^{k} p_j \phi_j(x;\mu_j,\sigma_j^2), \tag{3}$$

where, for $j = 1, \cdots, k$,

$$\phi_j(x;\mu_j,\sigma_j^2) = \frac{1}{\sqrt{2\pi}\ \sigma_j} e^{-\frac{(x-\mu_j)^2}{2\sigma_j^2}},$$

$$0 \le p_j \le 1, \quad \sum_{j=1}^{k} p_j = 1.$$

After a direct calculation by using the definition of (1), we derive the following basic properties.

Proposition 2.2.1. *If X is a mixture of k normals with pdf (3), then its mean, variance, skewness, and kurtosis are,*

$$\begin{aligned}
\mu &= \sum_{j=1}^{k} p_j\mu_j, \\
\sigma^2 &= \sum_{j=1}^{k} p_j(\sigma_j^2+\mu_j^2) - \mu^2 \\
\alpha_3 &= \frac{1}{\sigma^3}\sum_{j=1}^{k} p_j(\mu_j-\mu)[3\sigma_j^3 + (\mu_j-\mu)^2] \text{ and} \\
\alpha_4 &= \frac{1}{\sigma^4}\sum_{j=1}^{k} p_j[3\sigma_j^4 + 6(\mu_j-\mu)^2\sigma_j^2 + (\mu_j-\mu)^4].
\end{aligned}$$

In the following example, we show that the model of a normal distribution is quite inappropriate for fitting market data, since its density does not take into accounts the fat tails and skewness.

Example 2.1. *A Mixture of Two Normal Distributions.*

We consider a mixture of two normal distributions (3) with the following parameters

$$p = 0.5,\ \mu_1 = 0.5,\ \mu_2 = 0.5,\ \sigma_1 = 0.5,\ \text{and } \sigma_2 = 1.32.$$

We use Proposition 2.2.1 to compute its skewness and kurtosis. The results, compared to the standard normal distribution are summarized in Table 1. We compare the

Table 1: Standard Normal versus Mixture of Normals

Distribution	Standard Normal	Mixture of Normals
Mean	0	0
Variance	1	1
Skewness	0	-0.75
Kurtosis	3	6.06

two densities in Figure 1. The density of the standard normal is symmetric with skewness of 0 and kurtosis of 3 while the density of the mixture of two normals is asymmetric with skewness of -0.75 and kurtosis of 6.06, even though they have the same mean and variance. This indicates that the density of the mixture of normals is skewed to the left with fat tails. Duffie and Pan [1997] show that the S&P 500 daily returns from 1986 to 1996 has an extremely high sample kurtosis and negative skewness. Our Figure 1 is similar to their Figure 3. Obviously, compared to the normal distribution, the mixture of normals is a more general and flexible model of fitting market data of daily changes. It, of course, takes account the skewness and kurtosis.

3 GENERATING MIXTURES OF NORMAL VARIATES

In this section, we propose a new method for generating a multivariate mixture of normal distributions with arbitrary covariance matrix, which is different from the general three-step idea. Our method contains two steps. In step one, we generate a multivariate normal with an input covariance matrix. Later on we see how this covariance matrix will be derived analytically. In step two, we transform this multivariate normal into the desired multivariate mixture of normals linearly with any given arbitrary covariance matrix. Our discussion starts with the univariate case, then the multivariate case, follows.

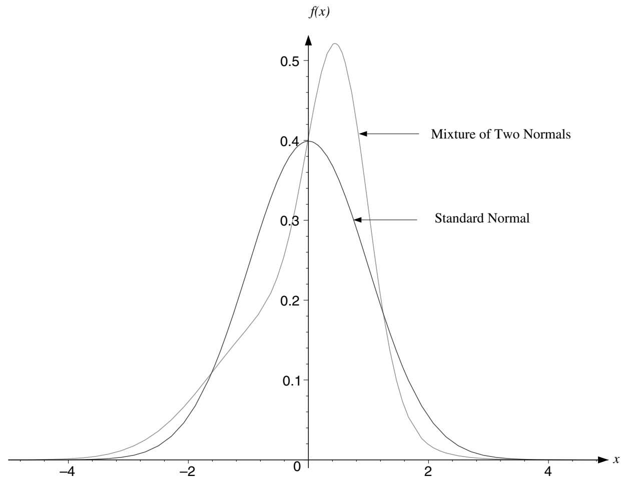

Figure 1: Probability Density Functions of Standard Normal and Mixture of Two Normals with the same mean and variance, but different kurtosis and skewness.

3.1 The Univariate Case

We discuss how to generate a single daily change using a mixture of k univariate normal distributions. The following result provides a feasible procedure of implementation. Detailed proof can be found in Wang [2001].

Proposition 3.1.1. *If $Y \sim N(0,1)$, $U \sim U(0,1)$, and X and U are independent each other, then*

$$X = \sum_{j=1}^{k} (\sigma_j Y + \mu_j) I_{\left\{\sum_{l=1}^{j-1} p_l \le U < \sum_{l=1}^{j} p_l\right\}}$$

is a mixture of k normals with cdf of (2), where $I_{\{\cdot\}}$ is the indicate function, and $\sum_{l=1}^{0} p_l = 0$.

As a direct application of Proposition 3.1.1, generating a mixture of normals with cdf (2) can be easily accomplished as follows:

Algorithm 3.1.

1. *Generate Y from $N(0,1)$.*
2. *Generate U from $U(0,1)$.*
3. *Return*

$$X = \sum_{j=1}^{k} (\sigma_j Y + \mu_j) I_{\left\{\sum_{l=1}^{j-1} p_l \le U < \sum_{l=1}^{j} p_l\right\}},$$

where $\sum_{l=1}^{0} p_l = 0$.

3.2 The Multivariate Case

We assume that $\mathbf{X} = (X_1, \cdots, X_n)'$ is a random vector of daily changes in market variables. The marginal distribution of each component X_i is a univariate mixture of k_i normals with pdf:

$$f_{X_i}(x) = \sum_{h=1}^{k_i} p_{i_h} \frac{1}{\sqrt{2\pi}\ \sigma_{i_h}} e^{-\frac{(x-\mu_{i_h})^2}{2\sigma_{i_h}^2}}, \qquad (4)$$

where

$$0 \le p_{i_h} \le 1, \ \ h = 1, \cdots, k_i, \ \ \sum_{h=1}^{k_i} p_{i_h} = 1, \ \ i = 1, \cdots, n.$$

The covariance matrix of $\mathbf{X}$ is

$$\Sigma_X = [\sigma_{ij}(X)], \qquad (5)$$

where $\sigma_{ij}(X) = \mathrm{Cov}(X_i, X_j), \quad i, j = 1, \cdots, n.$ Note that our definition of the multivariate mixture of normal distributions is different from Johnson [1987]. He mimics the univariate case to yield the multivariate case (Johnson [1987], page 56):

$$pN_m(\mu_1, \Sigma_1) + (1-p)N_m(\mu_2, \Sigma_2),$$

which is the cdf of a mixture of two multivariate normal distributions. Following his definition, all the marginal distributions must have the same number of components. In general, there are many multivariate distributions having mixtures of normal distributions as their marginal distributions. Our definition is one of them for a multivariate mixture of normal distributions. Under our definition, each marginal distribution may have a different number of components.

In the univariate case, we know how to generate each single component X_i from Proposition 3.1.1. In the multivariate case, the most difficult part is how to handle the covariance matrix Σ_X. The following result provides an efficient and easy-to-implement procedure to handle the covariance matrix. See Wang [2000] for the detailed proof.

Proposition 3.2.1. *Let $\mathbf{U} = (U_1, \cdots, U_n)'$, where U_is are independent $U(0,1)$ random variables, and $\mathbf{Y} = (Y_1, \cdots, Y_n)'$, where the Y_is are $N(0,1)$ random variables with covariance matrix*

$$\Sigma_Y = [\sigma_{ij}(Y)],$$

where

$$\sigma_{ij}(Y) = \mathrm{Cov}(Y_i, Y_j), \quad i, j = 1, \cdots, n.$$

Define

$$X_i = \sum_{h=1}^{k_i} (\sigma_{i_h} Y_i + \mu_{i_h}) I_{\left\{\sum_{l=1}^{h-1} p_{i_l} \le U_i < \sum_{l=1}^{h} p_{i_l}\right\}},$$

where

$$\sum_{l=1}^{0} p_{i_l} = 0, \quad i = 1, \cdots, n.$$

If $\mathbf{U}$ *and* $\mathbf{Y}$ *are independent, then* $\mathbf{X} = (X_1, \cdots, X_n)'$ *is a multivariate mixture of normals with mean*

$$\mu_X = (\mu_1, \cdots, \mu_n)',$$

where

$$\mu_i = \sum_{h=1}^{k_i} p_{i_h} \mu_{i_h}, \ i = 1, \cdots, n,$$

and covariance matrix

$$\Sigma_X = [\sigma_{ij}(X)],$$

where $\sigma_{ij}(X) =$

$$\begin{cases} \sigma_{ij}(Y) \sum_{h=1}^{k_i} \sum_{l=1}^{k_j} p_{i_h} p_{j_l} \sigma_{i_h} \sigma_{j_l} \\ + \sum_{h=1}^{k_i} \sum_{l=1}^{k_j} p_{i_h} p_{j_l} (\mu_{i_h} - \mu_i)(\mu_{j_l} - \mu_j)], \\ \qquad for \quad i \neq j, \quad and \quad i, j = 1, \cdots, n, \\ \sum_{h=1}^{k_i} p_{i_h} (\sigma_{i_h}^2 + \mu_{i_h}^2) - \left(\sum_{h=1}^{k_i} p_{i_h} \mu_{i_h}\right)^2, \\ \qquad for \quad i = j, \quad and \quad i = 1, \cdots, n. \end{cases} \tag{6}$$

Furthermore, for any given desired covariance matrix Σ_X, we can calculate the analytical input matrix Σ_Y which is required for generating the multivariate mixture of normals X. As a direct result of Proposition 3.2.1, we have the following main result.

Proposition 3.2.2. *Under all assumptions of Proposition 3.2.1,* $\sum_Y$ *can be derived in terms of* $\sum_X$ *where* $\sigma_{ij}(Y) =$

$$\begin{cases} \dfrac{\sigma_{ij}(X) - \sum_{h=1}^{k_i} \sum_{l=1}^{k_j} p_{i_h} p_{j_l} (\mu_{i_h} - \mu_i)(\mu_{j_l} - \mu_j)}{\sum_{h=1}^{k_i} \sum_{l=1}^{k_j} p_{i_h} p_{j_l} \sigma_{i_h} \sigma_{j_l}}, \\ \qquad for \quad i \neq j, \quad and \quad i, j = 1, \cdots, n \\ 1, \qquad for \quad i = j, \quad and \quad i = 1, \cdots, n. \end{cases}$$

Cholesky Decomposition is a commonly used method for generating the multivariate normal with an arbitrary covariance matrix (Johnson [1987], and Law and Kelton [2000]). Algorithms of the Cholesky method can found in Fishman [1973]. We consider the Cholesky method.

Proposition 3.2.3. (Cholesky Decomposition) *For any given covariance matrix* $\Sigma_Y = [\sigma_{ij}(Y)]$*, if* $\mathbf{Z} = (Z_1, \cdots, Z_n)'$*, where the* Z_i*s are independent* $N(0, 1)$ *random variables, there exists a unique lower triangular matrix C, such that*

$$\Sigma_Y = CC',$$

and $C\mathbf{Z}$ *is a multivariate normal with covariance matrix* Σ_Y*. Furthermore, matrix C can be found from the following recursive formula:*

$$c_{ij}(Y) = \frac{\sigma_{ij}(Y) - \sum_{k=1}^{j-1} c_{ik}(Y) c_{jk}(Y)}{\sqrt{\sigma_{jj}(Y) - \sum_{k=1}^{j-1} c_{jk}^2(Y)}}$$

where

$$\sum_{k=1}^{0} c_{ik}(Y) c_{jk}(Y) = 0, \quad 1 \le j \le i \le n.$$

Based on the results of Propositions 3.2.2 and 3.2.3, generating a multivariate mixture of normals with the marginal pdfs of (4) and covariance matrix of $\Sigma_X = [\sigma_{ij}]$ can be accomplished as follows:

Algorithm 3.2.

1. *Calculate* $\sum_Y$*, where* $\sigma_{ij}(Y) =$

$$\begin{cases} \dfrac{\sigma_{ij}(X) - \sum_{h=1}^{k_i} \sum_{l=1}^{k_j} p_{i_h} p_{j_l} (\mu_{i_h} - \mu_i)(\mu_{j_l} - \mu_j)}{\sum_{l=1}^{k_j} p_{i_h} p_{j_l} \sigma_{i_h} \sigma_{j_l}}, \\ \qquad for \quad i \neq j, \quad and \quad i, j = 1, \cdots, n \\ 1, \qquad for \quad i = j, \quad and \quad i = 1, \cdots, n. \end{cases}$$

2. *Calculate C, where*

$$c_{ij}(Y) = \frac{\sigma_{ij}(Y) - \sum_{k=1}^{j-1} c_{ik}(Y) c_{jk}(Y)}{\sqrt{\sigma_{jj}(Y) - \sum_{k=1}^{j-1} c_{jk}^2(Y)}}$$

   and

$$\sum_{k=1}^{0} c_{ik}(Y) c_{jk}(Y) = 0, \quad 1 \le j \le i \le n.$$

3. *Generate* $Z = (Z_1, \cdots, Z_n)'$*, where the* Z_i*s are from* $N(0, 1)$.
4. *Generate* $U = (U_1, \cdots, U_n)'$*, where the* U_i*s are from* $U(0, 1)$.
5. *Calculate* $Y = (Y_1, \cdots, Y_n)'$*, where* $Y_i = \sum_{k=1}^{i} c_{ik}(Y) Z_i$.

6. *Return* $X = (X_1, \cdots, X_n)'$, *where* $X_i = \sum_{h=1}^{k_i} (\sigma_{i_h} Y_i + \mu_{i_h}) I_{\left\{\sum_{l=1}^{h-1} p_{i_l} \le U_i < \sum_{l=1}^{h} p_{i_l}\right\}}$, *and* $\sum_{l=1}^{0} p_{i_l} = 0$.

4 PORTFOLIO VaR

In recent years, VaR has become a new benchmark for managing and controlling risk (RiskMetrics™ [1995], Jorion [1997], and Dowd [1998]). The VaR measures the worst expected loss over a given time interval under normal market conditions at a given confidence level, and provides users with a summary measure of market risk. Suppose that $\mathbf{X} = (X_1, \cdots, X_n)'$ is a random vector of portfolio returns. For any portfolio $w = (w_1, \cdots, w_n)'$ (where w_i is the weight on asset i), the total return of portfolio w is $R_w = \sum_{i=1}^{n} w_i X_i$. Precisely q_w, the VaR at the $100(1-\alpha)\%$ confidence level of a portfolio w for a specific time period, is the solution to

$$\mathrm{P}\,(R_w \le -q_w) = \alpha. \tag{7}$$

In this section, we derive an important property of the mixture of normal distributions. Also, a portfolio VaR can be calculated efficiently without using Monte Carlo simulation.

Under our setting (4), if the marginal distributions of portfolio returns X_i $(i = 1, \cdots, n)$ are the mixture of normals, then the total return of a portfolio w $R_w = \sum_{i=1}^{n} w_i X_i$ is a (univariate) mixture of normal distributions. See Wang [2000] for the detailed proof.

Proposition 4.0.4. *We assume that* $\mathbf{X} = (X_1, \cdots, X_n)'$ *is a random vector of market daily returns and the marginal distribution of each component* X_i *is a univariate mixture of* k_i *normals with pdf (4) and covariance matrix* $\sum_X$. *Then, for any (linear) portfolio* $w = (w_1, \cdots, w_n)'$, *the portfolio return* $R_w = \sum_{i=1}^{n} w_i X_i$ *is a univariate mixture of normal distributions. The cdf is given by*

$$\mathrm{P}(R_w \le x) = \sum_{h_1=1}^{k_1} \cdots \sum_{h_n=1}^{k_n} p_{1_{h_1}} \cdots p_{n_{h_n}} \Phi\left(\frac{x - \mu_{h_1 \cdots h_n}}{\sigma_{h_1 \cdots h_n}}\right),$$

where Φ *is the cdf of* $N(0, 1)$,

$$\mu_{h_1 \cdots h_n} = \sum_{i=1}^{n} w_i \mu_{i_{h_i}},$$

and

$$\sigma_{h_1 \cdots h_n} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{i_{h_i}} \sigma_{j_{h_j}} \sigma_{ij}(Y).$$

Proposition 4.0.4 provides an efficient way to calculate portfolio VaR. The degree of difficulty for calculating the portfolio VaR is equivalent to find the quantile of a (univariate) mixture of normal distributions.

Proposition 4.0.5. *Under all assumptions of Proposition 4.0.4, the VaR* q_w *of a given portfolio* w *is the solution to:*

$$\sum_{h_1=1}^{k_1} \cdots \sum_{h_n=1}^{k_n} p_{1_{h_1}} \cdots p_{n_{h_n}} \Phi\left(\frac{-q_w - \mu_{h_1 \cdots h_n}}{\sigma_{h_1 \cdots h_n}}\right) = \alpha. \tag{8}$$

The right hand side of equation (8) is a monotone decreasing function in q_w. Therefore the VaR q_w can be calculated numerically to any desired degree of accuracy. The computation is just as easy as calculating a few quantiles of normal distributions. There is no need of the Monte Carlo simulation.

Note that there is no general, analytical way, to calculate portfolio VaR when the profit-and-loss function is a nonlinear function of portfolio returns. Under the model of the multivariate mixture of normal distributions, our Monte Carlo method in Section 3 provides an efficient way to calculate a portfolio VaR. Comparing this with the model of a multivariate normal distribution, there is not much extra work for the Monte Carlo simulation. The main part of extra work is to generate the uniform (0,1) vector, which is quite cheap and fast.

5 CONCLUSIONS

Many studies shown that frequently distributions of daily changes have fat tails and skew to the left. The mixture of normal distributions provides a useful extension of the normal for the modeling of daily changes with fatter-than-normal tails and skewness. The main result of this paper is to provide an efficient analytical Monte Carlo method for generating daily changes using a multivariate mixture of normal distributions with arbitrary covariance matrix. An easy-to-implement and ready-to-use algorithm is described. In general, the most difficult part of generating multivariate distributions is how to handle the covariance matrix. We derive the input covariance matrix analytically without using any numerical approximations or inverse transformations. In addition, the commonly used multivariate normal model is a special case of the multivariate mixture of normals. We also provide an efficient analytical method for calculating portfolio VaR without using Monte Carlo simulation. Under the assumption of multivariate mixture of normals, calculating a portfolio VaR is just as simple as under the normality assumption, since any linear combination of mixtures of normal distributions is a mixture of normal distributions. Overall our method is more general, appropriate, and efficient.

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