Proceedings of the 2004 Winter Simulation Conference
R. G. Ingalls, M. D. Rossetti, J. S. Smith, and B. A. Peters, eds.

OVERLAPPING VARIANCE ESTIMATORS FOR SIMULATIONS

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ABSTRACT

We examine properties of overlapped versions of the standardized time series area and Cramér–von Mises estimators for the variance parameter of a stationary stochastic process, e.g., a steady-state simulation output process. We find that the overlapping estimators have the same bias properties as, but lower variance than, their nonoverlapping counterparts; the new estimators also perform well against the benchmark batch means estimator. We illustrate our findings with analytical and Monte Carlo examples.

1 INTRODUCTION

This paper discusses new "overlapping" variance estimators that can be used in the analysis of steady-state simulations. Such an analysis might start off with, at the very least, an estimate of the unknown mean μ of a steady-state output process, $\{Y_i, i \geq 1\}$ — the sample mean $\bar{Y}_n$ being the obvious candidate. Since $\bar{Y}_n$ is a random variable, the experimenter ought to estimate its variability as well. One such measure is the *variance parameter*, $\sigma^2 \equiv \lim_{n\to\infty} n\,\mathsf{Var}(\bar{Y}_n)$.

There are many techniques in the literature concerning the estimation of σ^2. In particular, the well-known methods of (nonoverlapping) batch means (NBM), overlapping batch means (OBM), spectral analysis, regeneration, autoregressive modeling, and standardized time series (STS) are discussed in simulation texts such as Law and Kelton (2000).

A common strategy used by NBM, OBM, and STS employs *batching* of the observations — instead of considering all of the observations at once, break them up into smaller batches (sometimes disjoint, sometimes not, depending on the analysis method) and use an appropriate trick. For instance, NBM splits the observations into adjacent, but disjoint batches, assumes that the resulting sample (batch) means from each batch are approximately independent and identically distributed (i.i.d.) normal random variables (r.v.'s), and then applies "standard" variance estimation techniques to the batch means (as explained in §3.4). STS batched estimators also use adjacent, but disjoint batches. The idea is to form a separate STS estimator from each batch, assume they are i.i.d., and then average them (see §§3.2 and 3.3). OBM forms overlapping batches, with the full realization that the resulting overlapping batch means are *not* independent (though they are identically distributed and asymptotically normal). Then OBM applies the "standard" sample-variance estimator, appropriately scaled, to these highly correlated overlapping batch means. This technique uses results from spectral theory to obtain an estimator that is provably superior to NBM, at least asymptotically for certain serially correlated time series.

Concerning measures of the performance of estimators for σ^2, we most often care about the bias and variance, as well as the resulting mean squared error (MSE). Batching typically increases bias, but decreases variance; its effect on MSE takes a bit more work to analyze. What is nice about the OBM estimator is that it has the same bias as, but smaller variance than, the corresponding performance measures for NBM. Thus, OBM gives better (asymptotic) performance than NBM "for free".

So what happens if we try to apply overlapping batching techniques to the STS estimators? The current article does just this, and the news is almost all good: The resulting overlapped STS estimators have about the same bias, but substantially smaller variance than their nonoverlapped counterparts.

There has already been a great deal of progress on the study of overlapping estimators. Meketon and Schmeiser (1984) introduced the OBM methodology. Welch (1987) relates OBM to certain spectral estimators, and looks into the effects of partial overlapping. Goldsman and Meketon (1986), Song (1988), and Song and Schmeiser (1993) derive bias and variance properties of OBM estimators. Additional early work on the subject is undertaken by Pedrosa and Schmeiser (1993, 1994), who establish covariance properties between OBM estimators and subsequently propose a batch-size determination algorithm. Damerdji (1991, 1994, 1995) derives consistency results (both in the strong and mean-square senses) for a variety of variance estimators, including OBM and an overlapping version of a certain STS estimator. He also establishes a formal link between the spectral and overlapping methodologies.

This paper is organized as follows. We present background material in §2, where we introduce a number of benchmark estimators. §3 reviews the effects of batching the observations into nonoverlapping batches, then forming an STS estimator from each batch. §4 does the same, except that the batches are now allowed to overlap. That section also examines the asymptotic performance characteristics of the new overlapping estimators. We find that the overlapping estimators almost always outperform their nonoverlapped counterparts in terms of MSE. We give some exact and empirical examples and comparisons in §5, while §6 summarizes our results and provides conclusions. Most of the proofs and some additional details relevant to the current paper are deferred to Alexopoulos et al. (2004).

2 BACKGROUND

We consider a stationary stochastic process $\{Y_i, i \ge 1\}$, which we assume satisfies a Functional Central Limit Theorem (FCLT). This assumption applies to a broad class of processes, and will allow us to determine the limiting properties of the various variance estimators under consideration in this paper.

Assumption FCLT There exist constants μ and positive σ such that as $n \to \infty$, $X_n \Rightarrow \sigma\mathcal{W}$, where $\mathcal{W}$ is a standard Brownian motion process, "$\Rightarrow$" denotes weak convergence as $n \to \infty$, and

$$X_n(t) \equiv \frac{\lfloor nt \rfloor(\bar{Y}_{\lfloor nt \rfloor} - \mu)}{\sqrt{n}} \quad \text{for} \quad t \ge 0,$$

where $\bar{Y}_j \equiv \sum_{k=1}^{j} Y_k/j$, $j = 1, 2, \ldots$, and $\lfloor\cdot\rfloor$ is the greatest integer function.

Glynn and Iglehart (1990) list several different sets of sufficient conditions — usually in the form of moment and mixing conditions — for Assumption FCLT to hold. The constants μ and σ^2 in the assumption can be identified with the process mean and variance parameter, respectively.

The *standardized time series* of the stationary Y_i's is (cf. Schruben 1983)

$$T_n(t) \equiv \frac{\lfloor nt \rfloor(\bar{Y}_n - \bar{Y}_{\lfloor nt \rfloor})}{\sigma\sqrt{n}} \quad \text{for} \quad 0 \le t \le 1.$$

Under Assumption FCLT, it can be shown that

$$(\sqrt{n}(\bar{Y}_n - \mu), \sigma T_n) \Rightarrow (\sigma\mathcal{W}(1), \sigma\mathcal{B}), \tag{1}$$

where $\mathcal{B}$ is a standard Brownian bridge process on [0, 1].

2.1 The Weighted Area Estimator

This subsection deals with the weighted area estimator for σ^2 (cf. Goldsman, Meketon, and Schruben 1990; Goldsman and Schruben 1990). We define the square of the weighted area under the standardized time series and its limiting functional as

$$A(f; n) \equiv \left[\frac{1}{n}\sum_{k=1}^{n} f(k/n)\sigma T_n(k/n)\right]^2$$

and

$$A(f) \equiv \left[\int_0^1 f(t)\sigma\mathcal{B}(t)\,dt\right]^2,$$

respectively, where $f(t)$ is continuous on the interval [0, 1] and normalized so that $\mathsf{Var}(\int_0^1 f(t)\mathcal{B}(t)\,dt) = 1$. Then $\int_0^1 f(t)\sigma\mathcal{B}(t)\,dt \sim \sigma\mathsf{Nor}(0, 1)$, and under mild conditions, the continuous mapping theorem (cf. Billingsley 1968, Theorem 5.1) implies that $A(f; n) \xrightarrow{\mathcal{D}} A(f) \sim \sigma^2\chi_1^2$, where "$\xrightarrow{\mathcal{D}}$" denotes convergence in distribution as $n \to \infty$. For this reason, we call $A(f; n)$ the *weighted area* estimator for σ^2.

Denote the covariance function $R_k \equiv \mathsf{Cov}(Y_1, Y_{1+k})$, $k = 0, \pm1, \pm2, \ldots$, and the associated quantity $\gamma \equiv -2\sum_{k=1}^{\infty} kR_k$ (cf. Song and Schmeiser 1995). Further, the notation $p(n) = o(q(n))$ means that $p(n)/q(n) \to 0$ as $n \to \infty$. The next theorem gives expressions for the expected value and variance of the weighted area estimator.

Theorem 1 (see, e.g., Foley and Goldsman 2000) Suppose that $\{Y_i, i \ge 1\}$ is a stationary process for which Assumption FCLT holds, $\sum_{k=1}^{\infty} k^2|R_k| < \infty$, and $\sum_{k=-\infty}^{\infty} R_k = \sigma^2 > 0$. Further, suppose that $A^2(f; n)$ is uniformly integrable (cf. Billingsley 1968). Then

$$\mathsf{E}[A(f; n)] = \sigma^2 + \frac{[(F(1) - \bar{F}(1))^2 + \bar{F}^2(1)]\gamma}{2n} + o(1/n)$$

and

$$\text{Var}(A(f;n)) \rightarrow \text{Var}(A(f)) = \text{Var}(\sigma^2\chi_1^2) = 2\sigma^4$$

as $n \to \infty$, where the functions $F(s) \equiv \int_0^s f(t)\,dt$, $0 \le s \le 1$, and $\bar{F}(u) \equiv \int_0^u F(s)\,ds$, $0 \le u \le 1$. Note that this limiting variance does not depend on the form of the weighting function.

Example 1 Schruben (1983) studied the area estimator with constant weighting function $f_0(t) \equiv \sqrt{12}$, for all $t \in [0,1]$; in this case, Theorem 1 implies that $\mathsf{E}[A(f_0;n)] = \sigma^2 + 3\gamma/n + o(1/n)$. If one chooses weights having $F(1) = \bar{F}(1) = 0$, the resulting estimator is *first-order unbiased* for σ^2, i.e., its bias is $o(1/n)$. Examples of such weighting functions are the quadratic $f_2(t) \equiv \sqrt{840}(3t^2 - 3t + 1/2)$ (Goldsman, Meketon, and Schruben 1990; Goldsman and Schruben 1990) and the "orthonormal" sequence of weights $f_{\cos,j}(t) = \sqrt{8}\pi j \cos(2\pi j t)$, $j = 1, 2, \ldots$ (Foley and Goldsman 2000). It can be shown that the orthonormal estimators' limiting functionals $A(f_{\cos,1}), A(f_{\cos,2}), \ldots$ are i.i.d. $\sigma^2\chi_1^2$.

2.2 The Weighted Cramér–von Mises Estimator

We give an overview of the weighted CvM estimator for σ^2 (cf. Goldsman, Kang, and Seila 1999). We begin by defining some notation for the weighted area under the square of the STS and its limiting functional,

$$C(g;n) \equiv \frac{1}{n}\sum_{k=1}^{n} g(k/n)\,[\sigma T_n(k/n)]^2$$

and

$$C(g) \equiv \int_0^1 g(t)(\sigma\mathcal{B}(t))^2\,dt,$$

respectively, where $g(t)$ is a weighting function normalized so that $\mathsf{E}[C(g)] = \sigma^2$ and $\frac{d^2}{dt^2}g(t)$ is continuous and bounded on [0, 1]. Under mild assumptions, the continuous mapping theorem implies that $C(g;n) \xrightarrow{\mathcal{D}} C(g)$, and we call $C(g;n)$ the *weighted CvM* estimator for σ^2. Theorem 2 gives results on the expected value and variance of the weighted CvM estimator.

Theorem 2 (cf. Goldsman, Kang, and Seila 1999) Under conditions similar to those of Theorem 1,

$$\mathsf{E}[C(g;n)] = \sigma^2 + \frac{\gamma}{n}(G-1) + o(1/n)$$

and

$$\begin{aligned}\text{Var}(C(g;n)) &\rightarrow \text{Var}(C(g)) \\ &= 4\sigma^4\int_0^1 g(t)(1-t)^2\int_0^t g(s)s^2\,ds\,dt \qquad (2)\end{aligned}$$

as $n \to \infty$, where $G \equiv \int_0^1 g(t)\,dt$.

Example 2 The CvM estimator with constant weighting function $g_0(t) \equiv 6$ has $\mathsf{E}[C(g_0;n)] = \sigma^2 + 5\gamma/n + o(1/n)$. If one chooses weights having $G = 1$ (in addition to the normalizing and second derivative constraints), Theorem 2 implies that the CvM estimator $C(g;n)$ has bias $o(1/n)$. An example of such a first-order unbiased weighting function is the quadratic $g_{2;c}(t) \equiv 51 - c/2 + ct - 150t^2$, where $t \in [0,1]$ and c is real.

The choice of weighting function $g(t)$ affects the variances of $C(g;n)$ and $C(g)$. (The weighting function $f(t)$ of §2.1 affects the variance of $A(f;n)$, but *not* that of $A(f)$, which is always $\text{Var}(A(f)) = 2\sigma^4$.)

Example 3 Theorem 2 implies that $\text{Var}(C(g_0)) = 4\sigma^4/5$. Similarly, $\text{Var}(C(g_{2;c})) = (c^2 - 300c + 26856)\sigma^4/2520$; this variance is minimized by $g_2^\star(t) \equiv g_{2;150}(t)$, in which case $\text{Var}(C(g_2^\star)) = 121\sigma^4/70$. Although $\text{Var}(C(g_2^\star)) > \text{Var}(C(g_0))$, the estimator $C(g_2^\star;n)$ is first-order unbiased for σ^2, while $C(g_0;n)$ is not.

3 ESTIMATORS FROM NONOVERLAPPING BATCHES

This section examines what happens if we divide the run into contiguous, nonoverlapping *batches*, form an STS estimator from each batch, and take the average of these estimators.

3.1 Batching Basics

We will work with b contiguous, nonoverlapping batches of observations, each of length m, from the simulation output, $Y_1, Y_2, \ldots, Y_n$ (where we assume that $n = bm$). Thus, the observations $Y_{(i-1)m+1}, Y_{(i-1)m+2}, \ldots, Y_{im}$ constitute batch i, $1 \le i \le b$.

To parallel the discussion in §2, the standardized time series from batch i is

$$T_{i,m}(t) \equiv \frac{\lfloor mt\rfloor(\bar{Y}_{i,m} - \bar{Y}_{i,\lfloor mt\rfloor})}{\sigma\sqrt{m}},$$

for $0 \le t \le 1$ and $1 \le i \le b$, where

$$\bar{Y}_{i,j} \equiv \frac{1}{j}\sum_{k=1}^{j} Y_{(i-1)m+k},$$

for $1 \le j \le m$ and $1 \le i \le b$. If we define $Z_i(m) \equiv \sqrt{m}(\bar{Y}_{i,m} - \mu)$, $1 \le i \le b$, then under the same mild conditions as before,

$$(Z_1(m), Z_2(m), \ldots, Z_b(m); \sigma T_{1,m}, \sigma T_{2,m}, \ldots, \sigma T_{b,m}) \Rightarrow (\sigma Z_1, \sigma Z_2, \ldots, \sigma Z_b; \sigma \mathcal{B}_0, \sigma \mathcal{B}_1, \ldots, \sigma \mathcal{B}_{b-1}), \quad (3)$$

where the Z_i's are i.i.d. standard normal random variables, and $\mathcal{B}_s$ denotes a standard Brownian bridge process on $[s, s+1]$, for $s \in [0, b-1]$, i.e.,

$$\mathcal{B}_s(t) = \mathcal{W}(s+t) - \mathcal{W}(s) - t[\mathcal{W}(s+1) - \mathcal{W}(s)],$$

for $t \in [0, 1]$. It is easy to see that $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{b-1}$ are independent Brownian bridges.

3.2 Batched Area Estimator

We define the area estimator formed exclusively from batch i as

$$A_i(f; m) \equiv \left[\frac{1}{m}\sum_{k=1}^{m} f(k/m)\sigma T_{i,m}(k/m)\right]^2,$$

$1 \le i \le b$. The *batched area* estimator for σ^2 is

$$\mathcal{A}(f; b, m) \equiv \frac{1}{b}\sum_{i=1}^{b} A_i(f; m).$$

Since the $T_{i,m}$'s, $1 \le i \le b$, converge to independent Brownian bridge processes as m becomes large (with fixed b), we shall assume that the $A_i(f; m)$'s, $1 \le i \le b$, are asymptotically independent as $m \to \infty$. Then by the remarks in §2.1, we have $\mathcal{A}(f; b, m) \xrightarrow{\mathcal{D}} \sigma^2 \chi_b^2 / b$.

Theorem 1 implies

$$\mathsf{E}[\mathcal{A}(f; b, m)] = \sigma^2 + \frac{[(F(1) - \bar{F}(1))^2 + \bar{F}^2(1)]\gamma}{2m} + o(1/m). \quad (4)$$

Further, if assume the uniform integrability of $\mathcal{A}^2(f; b, m)$, we have

$$\lim_{m\to\infty} b\,\mathsf{Var}(\mathcal{A}(f; b, m)) = \mathsf{Var}(A(f)) = 2\sigma^4. \quad (5)$$

3.3 Batched CvM Estimator

Similarly, the CvM estimator formed exclusively from batch i is

$$C_i(g; m) \equiv \frac{1}{m}\sum_{k=1}^{m} g(k/m)\left[\sigma T_{i,m}(k/m)\right]^2,$$

$1 \le i \le b$. The *batched CvM* estimator for σ^2 is

$$\mathcal{C}(g; b, m) \equiv \frac{1}{b}\sum_{i=1}^{b} C_i(g; m).$$

Theorem 2 implies

$$\mathsf{E}[\mathcal{C}(g; b, m)] = \sigma^2 + \frac{\gamma}{m}(G - 1) + o(1/m). \quad (6)$$

As before, for fixed b,

$$\lim_{m\to\infty} b\,\mathsf{Var}(\mathcal{C}(g; b, m)) = \mathsf{Var}(C(g)), \quad (7)$$

where $\mathsf{Var}(C(g))$ is given by (2).

3.4 NBM Estimator

The quantities $\bar{Y}_{i,m}$, $1 \le i \le b$, are referred to as the *batch means* of the process $Y_1, Y_2, \ldots, Y_n$, and are often assumed to be i.i.d. normal random variables, at least for large enough batch size m. This assumption immediately suggests the NBM estimator for σ^2,

$$\mathcal{N}(b, m) \equiv \frac{m}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m} - \bar{Y}_n)^2 \xrightarrow{\mathcal{D}} \frac{\sigma^2\chi_{b-1}^2}{b-1},$$

as $m \to \infty$ with b fixed (cf. Glynn and Whitt 1991; Schmeiser 1982; and Steiger and Wilson 2001). The NBM estimator is one of the most popular for σ^2, and serves as a benchmark for comparison.

Under mild conditions, it is well known that (see Chien, Goldsman, and Melamed 1997; Goldsman and Meketon 1986; Song and Schmeiser 1995; among others)

$$\mathsf{E}[\mathcal{N}(b, m)] = \sigma^2 + \frac{\gamma(b+1)}{bm} + o(1/m). \quad (8)$$

Further, for fixed b,

$$\lim_{m\to\infty} (b-1)\,\mathsf{Var}(\mathcal{N}(b, m)) = 2\sigma^4.$$

3.5 Recapitulation

So we see that as $m \to \infty$, the batched area, batched CvM, and NBM estimators are all asymptotically unbiased for σ^2. In addition, the variances of these estimators are all inversely proportional to the number of batches (at least for sufficiently large batch size). Of course, for fixed m and b and, hence, for fixed sample size $n = mb$, some estimators will tend to perform better than others. Certainly, we want to use estimators with low bias and variance; but for fixed n, decreasing one usually comes at the expense of increasing the other — the well-known trade-off that we have already mentioned.

One could argue that NBM, as the benchmark method, has moderate bias and variance. The good news is that STS area and CvM estimators with certain weighting functions can outperform NBM in terms of large-sample bias, and in the case of CvM, in terms of variance as well. In the sequel, we will show that the use of overlapping batches with respect to any particular estimator preserves its expected value, while reducing its variance — a free bonus.

4 ESTIMATORS FROM OVERLAPPING BATCHES

Here we consider estimators based on *overlapping* batches, à la Meketon and Schmeiser (1984). More specifically, we construct a number of overlapping batches, form an STS estimator from each batch, and take the average of these estimators.

4.1 Overlapping Batching Basics

We assume that there are n observations $Y_1, Y_2, \ldots, Y_n$ on hand. We form $n-m+1$ overlapping batches, each of size m. In particular, the observations $Y_i, Y_{i+1}, \ldots, Y_{i+m-1}$ constitute batch i, $1 \le i \le n-m+1$. We will continue to denote $b \equiv n/m$ as before; but obviously, when speaking in the context of overlapping batches, b can no longer be interpreted as "the number of batches".

To parallel the discussion in §3.1, the standardized time series from overlapping batch i is

$$T^O_{i,m}(t) \equiv \frac{\lfloor mt \rfloor (\bar{Y}^O_{i,m} - \bar{Y}^O_{i,\lfloor mt \rfloor})}{\sigma\sqrt{m}},$$

for $0 \le t \le 1$ and $1 \le i \le n-m+1$, where

$$\bar{Y}^O_{i,j} \equiv \frac{1}{j}\sum_{k=0}^{j-1} Y_{i+k},$$

for $1 \le i \le n-m+1$ and $1 \le j \le m$. Under the same mild conditions as before,

$$\sigma T^O_{sm,m} \Rightarrow \sigma \mathcal{B}_s, \quad 0 \le s \le b-1, \quad s \text{ fixed}.$$

4.2 Overlapping Area Estimator

We define the overlapping area estimator from overlapping batch i by

$$A^O_i(f;m) \equiv \left[\frac{1}{m}\sum_{k=1}^{m} f(k/m)\sigma T^O_{i,m}(k/m)\right]^2,$$

$1 \le i \le n-m+1$. The *overlapping area* estimator for σ^2 is

$$\mathcal{A}^O(f;b,m) \equiv \frac{1}{n-m+1}\sum_{i=1}^{n-m+1} A^O_i(f;m).$$

Then the continuous mapping theorem implies that, as $m \to \infty$,

$$\begin{aligned}\mathcal{A}^O(f;b,m) &\xrightarrow{\mathcal{D}} \mathcal{A}^O(f;b) \\ &\equiv \frac{1}{b-1}\int_0^{b-1}\left(\sigma\int_0^1 f(u)\mathcal{B}_s(u)\,du\right)^2 ds. \qquad (9)\end{aligned}$$

Meanwhile, Theorem 1 gives

$$\begin{aligned}&\mathsf{E}[\mathcal{A}^O(f;b,m)] \qquad (10)\\ &= \sigma^2 + \frac{[(F(1)-\bar{F}(1))^2 + \bar{F}^2(1)]\gamma}{2m} + o(1/m).\end{aligned}$$

Note that the expected value of the overlapping area estimator matches that of the batched area estimator.

The next series of results, proven in Alexopoulos et al. (2004), concern the asymptotic covariance between two overlapping area estimators, formed from two weighting functions, say $f_k(t)$ and $f_\ell(t)$. First of all, it can be shown that under mild conditions, as $m \to \infty$, we have

$$\begin{aligned}&\mathsf{Cov}(\mathcal{A}^O(f_k;b,m), \mathcal{A}^O(f_\ell;b,m)) \\ &\quad \to \mathsf{Cov}(\mathcal{A}^O(f_k;b), \mathcal{A}^O(f_\ell;b)).\end{aligned}$$

But how do we calculate this asymptotic covariance? A couple of lemmas point the way.

Lemma 1 Define

$$p_{k\ell}(s,r) \equiv \int_0^1\int_0^1 f_k(u)f_\ell(v)\mathsf{Cov}(\mathcal{B}_s(u),\mathcal{B}_r(v))\,du\,dv.$$

Then for $0 \le y \le 1$,

$$\begin{aligned}&p_{k\ell}(0,y) \qquad (11)\\ &= \bar{F}_k(1)[F_\ell(1-y) - \bar{F}_\ell(1-y) - \bar{F}_\ell(1)y] \\ &\quad + \bar{F}_k(y)\bar{F}_\ell(1) - \int_0^{1-y} f_\ell(u)\bar{F}_k(y+u)\,du,\end{aligned}$$

where $F_k(x)$, $F_\ell(x)$, $\bar{F}_k(x)$, and $\bar{F}_\ell(x)$ are analogous to the functions defined in §2.1.

If we happen to be dealing with weighting functions $f_k(t)$ and $f_\ell(t)$ such that $\bar{F}_k(1) = \bar{F}_\ell(1) = 0$, then most of the terms in (11) disappear. In this case, some easy calculus yields the equivalent expression,

$$p_{k\ell}(0, y) \;=\; \int_0^{1-y} F_\ell(u) F_k(y+u)\, du. \qquad (12)$$

The first-order unbiased weighting functions $f_2(t)$ and $f_{\cos,j}(t)$ from Example 1 all satisfy the condition $\bar{F}(1) = 0$, making the calculation of $\mathsf{Var}(\mathcal{A}^O(f;b))$ particularly easy for these cases.

Lemma 2 Define

$$q_{k\ell}(y) \;\equiv\; p^2_{k\ell}(0, y) + p^2_{\ell k}(0, y).$$

Then for $b > 1$, we have

$$\begin{aligned} &\mathsf{Cov}\left(\mathcal{A}^O(f_k; b), \mathcal{A}^O(f_\ell; b)\right) \\ &\quad = \frac{2\sigma^4}{(b-1)^2} \int_0^1 (b-1-y) q_{k\ell}(y)\, dy. \qquad (13) \end{aligned}$$

So in order to calculate $\mathsf{Cov}(\mathcal{A}^O(f_k; b), \mathcal{A}^O(f_\ell; b))$, we apply (11) (or (12) if appropriate) to the definition of $q_{k\ell}(y)$ and plug into (13). We take $f(t) = f_k(t) = f_\ell(t)$ when we want to calculate $\mathsf{Var}(\mathcal{A}^O(f;b))$.

Contrary to our findings in §§2.1 and 3.2, where we did not use overlapping, some examples illustrate the fact that the variance of the overlapping area estimator *does* depend on the choice of weighting function.

Example 4 For the overlapping constant-weighted area estimator from Example 1, we have from Equations (11) and (13),

$$\begin{aligned} &\mathsf{Var}(\mathcal{A}^O(f_0; b, m)) \\ &\quad \to \mathsf{Var}(\mathcal{A}^O(f_0; b)) \;=\; \frac{24b - 31}{35(b-1)^2}\sigma^4 \;\doteq\; \frac{24}{35b}\sigma^4. \end{aligned}$$

This compares favorably to the batched constant-weighted area estimator's asymptotic ($m \to \infty$) variance, $\mathsf{Var}(\mathcal{A}(f_0; b)) = 2\sigma^4/b$ (see Equation (5)).

Example 5 For the overlapping area estimator with first-order unbiased quadratic weighting function $f_2(t)$ from Example 1, Equations (12) and (13) imply

$$\mathsf{Var}(\mathcal{A}^O(f_2; b)) \;=\; \frac{3514b - 4359}{4290(b-1)^2}\sigma^4.$$

This compares well to the analogous batched quadratically weighted area estimator's variance, $\mathsf{Var}(\mathcal{A}(f_2; b)) = 2\sigma^4/b$.

Example 6 For the overlapping area estimators from the family of orthonormal first-order unbiased weights $f_{\cos,i}(t)$, $i = 1, 2, \ldots$, we have from Example 1 and Equations (12) and (13),

$$\mathsf{Var}(\mathcal{A}^O(f_{\cos,j}; b)) \;\doteq\; \frac{8\pi^2 j^2 + 15}{12\pi^2 j^2 b}\sigma^4. \qquad (14)$$

Again, the analogous batched weighted area estimator has $\mathsf{Var}(\mathcal{A}(f_{\cos,j}; b)) = 2\sigma^4/b$.

Remark 1 One can average the orthonormal estimators $\mathcal{A}^O(f_{\cos,j}; b)$, $j \geq 1$, and use Equation (13) to obtain estimators with smaller variance (cf. Alexopoulos et al. 2004).

4.3 Overlapping CvM Estimator

We define the overlapping CvM estimator from overlapping batch i by

$$C_i^O(g; m) \;\equiv\; \frac{1}{m}\sum_{k=1}^{m} g(k/m)\left[\sigma T^O_{i,m}(k/m)\right]^2,$$

$1 \leq i \leq n - m + 1$. The *overlapping CvM* estimator for σ^2 is

$$\mathcal{C}^O(g; b, m) \;\equiv\; \frac{1}{n-m+1}\sum_{i=1}^{n-m+1} C_i^O(g; m).$$

Then by the continuous mapping theorem, as $m \to \infty$,

$$\begin{aligned} &\mathcal{C}^O(g; b, m) \qquad (15) \\ &\quad \xrightarrow{\mathcal{D}} \;\; \mathcal{C}^O(g; b) \;\equiv\; \frac{1}{b-1}\int_0^{b-1}\int_0^1 g(u)\sigma\mathcal{B}_s^2(u)\, du\, ds. \end{aligned}$$

Meanwhile, Theorem 2 implies

$$\mathsf{E}[\mathcal{C}^O(g; b, m)] \;=\; \sigma^2 + \frac{\gamma(G-1)}{m} + o(1/m). \qquad (16)$$

So the expected value of the overlapping CvM estimator is the same as that of the batched CvM estimator.

Alexopoulos et al. (2004) give mild conditions for which

$$\mathsf{Var}(\mathcal{C}^O(g; b, m)) \;\to\; \mathsf{Var}(\mathcal{C}^O(g; b)), \text{ as } m \to \infty.$$

We next give a preliminary result from Alexopoulos et al. (2004) that will help to simplify the subsequent calculations for $\mathsf{Var}(\mathcal{C}^O(g; b))$.

Lemma 3

$$\mathsf{Var}(\mathcal{C}^O(g; b)) \;=\; \frac{4\sigma^4}{(b-1)^2}\int_0^1 (b-1-y) q(y)\, dy, \qquad (17)$$

where

$$\begin{aligned} q(y) & \\ \equiv\ & \int_0^1 \int_0^1 g(u)g(v)\mathsf{Cov}^2(\mathcal{B}_0(u), \mathcal{B}_y(v))\, du\, dv \\ =\ & y^2 \int_0^{1-y} g(v)v^2 \int_0^y g(u)u^2\, du\, dv \\ & +(1-y)^2 \int_{1-y}^1 g(v)(1-v)^2 \int_0^y g(u)u^2\, du\, dv \\ & +\int_0^{1-y} g(v) \int_y^{y+v} g(u)(u-y+vy-uv-uvy)^2\, du\, dv \\ & +y^2 \int_{1-y}^1 g(v)(1-v)^2 \int_y^1 g(u)(1-u)^2\, du\, dv \\ & +(1+y)^2 \int_0^{1-y} g(v)v^2 \int_{y+v}^1 g(u)(1-u)^2\, du\, dv. \end{aligned}$$

Carrying out the tedious algebra allows us to calculate the desired variance for a particular weighting function.

Example 7 For the overlapping constant-weighted CvM estimator from Example 2, we have, as $m \to \infty$,

$$\begin{aligned} & \mathsf{Var}(\mathcal{C}^O(g_0; b, m)) \\ & \to\ \mathsf{Var}(\mathcal{C}^O(g_0; b)) = \frac{88b - 115}{210(b-1)^2}\sigma^4 \doteq \frac{44}{105b}\sigma^4. \end{aligned}$$

This compares favorably to the batched constant-weighted CvM estimator's asymptotic variance, $\mathsf{Var}(\mathcal{C}(g_0; b)) = 4\sigma^4/5b$.

Example 8 For the overlapping CvM estimator with quadratic weight $g_{2;c}(t)$ from Example 2, we have

$$\begin{aligned} & \mathsf{Var}(\mathcal{C}^O(g_{2;c}; b)) \\ & = \frac{3876480b + 187c^2 - 56100c - 690300}{4989600(b-1)^2}\sigma^4. \end{aligned}$$

This quantity is minimized with respect to c by the choice $c = 150$ (uniformly in b), i.e., with weighting function $g_2^\star(t)$ as in Example 2. Then

$$\mathsf{Var}(\mathcal{C}^O(g_2^\star; b)) = \frac{10768b - 13605}{13860(b-1)^2}\sigma^4 \doteq \frac{0.777}{b}\sigma^4.$$

This compares well to the batched quadratic CvM estimator's asymptotic variance, $\mathsf{Var}(\mathcal{C}(g_2^\star; b)) = 121\sigma^4/70b$.

4.4 OBM Estimator

Using the notation from §4.1, we define the ith overlapping batch mean as $\bar{Y}^O_{i,m} \equiv \sum_{k=0}^{m-1} Y_{i+k}/m$, for $1 \le i \le n - m + 1$. The OBM estimator for σ^2, originally studied by Meketon and Schmeiser (1984) (with a slightly different scaling constant), is

$$\mathcal{O}(b, m) \equiv \frac{nm}{(n-m+1)(n-m)} \sum_{i=1}^{n-m+1} (\bar{Y}^O_{i,m} - \bar{Y}_n)^2.$$

Under mild conditions, Goldsman and Meketon (1986) and Song and Schmeiser (1995) show that, for large b,

$$\mathsf{E}[\mathcal{O}(b, m)] = \sigma^2 + \frac{\gamma}{m} + o(1/m), \tag{18}$$

a result that certainly makes sense in light of Equation (8). As for the OBM estimator's variance, Alexopoulos et al. (2004), Damerdji (1995), and Meketon and Schmeiser (1984) find that

$$\begin{aligned} & \lim_{m\to\infty} \mathsf{Var}(\mathcal{O}(b, m)) \\ & \to\ \frac{(4b^3 - 11b^2 + 4b + 6)\sigma^4}{3(b-1)^4} \doteq \frac{4\sigma^4}{3b}, \end{aligned} \tag{19}$$

with the approximate result holding for large b.

4.5 Recapitulation

Parallelling the discussion in §3.5, we see that as $m \to \infty$, the overlapping area, overlapping CvM, and OBM estimators are all asymptotically unbiased for σ^2. In addition, the variances of these estimators are all inversely proportional to the ratio $b = n/m$ (for sufficiently large batch size). We have yet to prove that the estimators are consistent in mean square as both m and $b \to \infty$, though we believe this to be true (cf. Damerdji 1995).

All of the overlapping estimators preserve the bias properties of their nonoverlapping counterparts. Thus, we found that the overlapping area and overlapping CvM estimators with certain "unbiased" weighting functions can beat OBM in terms of large-sample bias. An added feature is that the overlapping STS estimators also defeat their nonoverlapped counterparts as well as OBM in terms of variance, sometimes by quite a bit.

5 EMPIRICAL EXAMPLE

In this section, we illustrate via Monte Carlo simulation the performance of the various estimators on a simple first-order autoregressive process [AR(1)]. This stationary process is defined as $Y_i = \phi Y_{i-1} + \epsilon_i$, $i \ge 1$, where the ϵ_i's are i.i.d. $\mathsf{Nor}(0, 1-\phi^2)$ r.v.'s, and Y_0 is a $\mathsf{Nor}(0, 1)$ r.v. initialized independently of the other observations. The AR(1) process has covariance function $R_k = \phi^{|k|}$ for all k, so that $\sigma^2 = (1+\phi)/(1-\phi)$ and $\gamma = -2\phi/(1-\phi)^2$.

In the current example, we set $\phi = 0.9$, corresponding to a highly positive autocorrelation structure and variance parameter $\sigma^2 = 19$. Based on 10,000 replications, we estimated the expected values and variances of a variety of nonoverlapping and overlapping area, CvM, and batch means variance estimators. Representative results are given in Table 1, where $b = 20$ and $m = 1000$. The last column in Table 1 provides the asymptotic (as $m \to \infty$) variance of each variance estimator obtained analytically in previous sections.

Table 1: Empirical Performance of the Variance Estimators for $b = 20$ and $m = 1000$

Variance Estimator	Estimated Mean	Estimated Variance	Asymptotic Variance
$\mathcal{A}^O(f_0; b, m)$	18.51	12.89	12.83
$\mathcal{A}(f_0; b, m)$	18.47	33.84	36.10
$\mathcal{A}^O(f_2; b, m)$	18.98	15.34	15.37
$\mathcal{A}(f_2; b, m)$	18.91	35.52	36.10
$\mathcal{A}^O(f_{\cos,1}; b, m)$	18.98	14.85	14.89
$\mathcal{A}(f_{\cos,1}; b, m)$	18.91	35.60	36.10
$\mathcal{C}^O(g_0; b, m)$	18.15	7.79	7.83
$\mathcal{C}(g_0; b, m)$	18.14	13.61	14.44
$\mathcal{C}^O(g_2^\star; b, m)$	18.96	14.53	14.56
$\mathcal{C}(g_2^\star; b, m)$	18.91	29.96	31.20
$\mathcal{O}(b, m)$	18.74	25.86	25.56
$\mathcal{N}(b, m)$	18.82	37.11	38.00

From the last two columns of Table 1, one can see that the estimated variances of all variance estimators for the AR(1) process with $\phi = 0.9$ are consistent with their analytically obtained asymptotic counterparts. Moreover, the expected values of all variance estimators are similar and close to σ^2. Hence, this empirical study confirms our earlier conclusions based on our theoretical results. In particular, the overlapped versions of the various area, CvM, and batch means estimators have similar expected values but smaller variances than the nonoverlapped versions.

6 SUMMARY AND CONCLUSIONS

In this paper, we studied the overlapping versions of the standardized times series area and Cramér–von Mises variance estimators for steady-state simulations. We obtained asymptotic expressions for the expected values and variances of these overlapping variance estimators, and we compared them with the corresponding nonoverlapping estimators as well as with the overlapping and nonoverlapping batch means estimators. These expressions show that the overlapping versions of the standardized times series variance estimators provide similar bias but smaller variance than their nonoverlapping counterparts — sometimes by a large margin. We confirmed these asymptotic results by an empirical study that showed that the overlapping variance estimators perform as advertised.

Algorithms for computing the estimates as well as the related time-complexity issues are given in Alexopoulos et al. (2004).

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