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PORTFOLIO CREDIT RISK ANALYSIS INVOLVING CDO TRANCHES

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ABSTRACT

Credit risk analysis for portfolios containing CDO tranches is a challenging task for risk managers. We propose here a basis function approach for CDO tranche valuation and portfolio risk analysis at horizon, based on a multi-step Monte Carlo simulation model. The idea is to approximate the expected value of the tranche at horizon by a linear combination of basis functions, which are chosen to best characterize the current state of the associated CDO. It can be generalized for portfolio risk analysis involving any complex financial instruments.

1 INTRODUCTION

Collateralized debt obligation (CDO) is an asset-backed security whose underlying collateral is typically a portfolio of credit risky instruments such as bonds and loans. The different layers of CDO securities, which receive cash flows generated from the collateral portfolio according to a prioritized payment structure ('waterfall'), are called tranches.

With the rapid growth and development of CDOs and other credit derivatives in the recent years, portfolio credit risk management has become a more challenging job for risk managers. Some quantitative tools, such as Moody's KMV's Portfolio Manager™ and RiskMetrics Group's CreditManager®, have been widely used for portfolio credit risk analysis. However, they were initially designed for portfolio of standalone credit exposures, including bank loans, corporate bonds, CDS, revolvers, etc. For a portfolio that contains both CDO tranches and standalone exposures, there is no simple method to analyze the risk due to the difficulty of valuating tranches and addressing correlation between tranches and standalone exposures. User usually has to either treat each tranche as a bond with similar rating or rely on his own judgment.

The portfolio credit risk analysis involving CDO tranches has elicited interests from both academics and practitioners, as well as from the regulators due to the adaptation of practitioner models for Basel II (Gordy 2004).

The portfolio value distribution is important for portfolio managers to understand return and risk of the portfolio. The portfolio in consideration could be either a single tranche or a mixture of tranches and bonds/loans. In general the simulation can be performed under risk-neutral or physical probability measure. For a risk-neutral valuation, the cash flows are generated under risk-neutral measure and discounted by the risk-free rate; for a physical valuation, the cash flows are generated under physical measure and discounted by a user-specified spread added to the risk-free rate.

Pricing and hedging of financial instruments are usually performed on an as-of date basis, i.e., by discounting all the future cash flows back to the as-of date. While for the portfolio VaR analysis, economic capital allocation and risk contribution calculation, a future date or horizon analysis is needed. Intuitively, as-of date value has a wider distribution since it incorporates the uncertainty throughout the lifetime of the exposures while horizon value distribution only contains uncertainty from as-of date to the horizon.

Standard approach for deriving portfolio value at horizon is to simulate a sample path of the portfolio from as-of date to horizon and valuate individual exposures at horizon conditional on the realized state at horizon. This valuation is easy for standalone exposure, whose value solely depends on its credit state at horizon, more specifically its default probability over the remaining maturity. However, CDO tranches cannot be valuated the same way, because there are a lot of factors that affect the value of a tranche, such as the collateral credit quality, default correlation, tranche structure, credit enhancement policy, reinvestment strategy, etc. Also the tranche value is a path-dependent process, which cannot be simply determined by the variables measured at horizon.

There have been a number of efforts for accommodating CDO tranches into the portfolio credit risk analysis. One approach (Gordy 2004) is to jointly model rating changes for collateral pool obligors and portfolio obligors, conditional on outcomes at horizon, re-price CDO tranche using a simple valuation method, such as Duffie and Garleanu (2001). Peretyatkin and Perraudin (2003) proposed another ratings-based approach using multi-period Monte Carlo simulation.

2 MODEL DETAILS

We are here presenting a basis function approach for CDO tranche valuation and portfolio credit risk analysis at horizon, built upon a multi-step Monte Carlo simulation framework (Morokoff 2003).

2.1 Problem Setup

Consider a portfolio containing n_T CDO tranches and n_S standalone exposures, with initial notional amounts w_i^0, $i=1,\ldots,n_T+n_S$. The portfolio value at horizon is given by,

$$\Pi(t_H) = \sum_{i=1}^{n_T+n_S} w_i^0 FV_i(0,t_H) + \sum_{i=1}^{n_T+n_S} w_i^{t_H} V_i(t_H),$$

where t_H is the horizon time, $FV_i(0,t)$ denotes the time t value of all cash flows received before t for \$1 notional investment on exposure i at time 0, w_i^t is the notional amount at time t. $V_i(t)$ represents the value of exposure i at time t, defined as the expected discounted future cash flows for \$1 notional investment on exposure i at time t. They can be calculated as

$$FV_i(0,t_H) = \sum_{\{l:t_{i,l} \le t_H\}} \frac{c_{i,l}}{DF_{t_{i,l}}^{t_H}},$$

and

$$V_i(t_H) = E\left(\sum_{\{l:t_{i,l} > t_H\}} c_{i,l} DF_{t_H}^{t_{i,l}} \mid \mathbb{F}_{t_H} \right),$$

where DF_t^T denotes the discount factor for calculating time t value of cash flow at a future time T, $c_{i,l}$ is the l-th cash flow received for \$1 notional investment on exposure i at time $t_{i,l}$. $\mathbb{F}_t$ is the filtration at t, which can be understood as the collection of all random events that have occurred up to time t.

The portfolio value distribution can be translated into the loss distribution by specifying a zero loss point, then used to calculate other outputs like expected loss and VaR. The zero loss point is usually chosen as the risk-free return value of the portfolio.

For a standalone exposure, the value at horizon can be computed, given the realized credit state at horizon and remaining cash flows. As for a tranche exposure, however, we cannot valuate it in an easy way because there are many factors that affect it. One option is to launch a sub-simulation to estimate the tranche value conditional on the realized state at horizon, but the computational cost of this simulation within simulation would be prohibitive. Another option is to use the ratings-based method, which has the drawback of not taking specific deal structure and path-dependency of tranche value process into account.

2.2 Basis Function Approach

The idea of this approach is to approximate the tranche value at horizon by a linear combination of chosen basis functions and then derive the portfolio value distribution at horizon. This is similar to the least-squares method by Longstaff and Schwarz (2001) for valuing American options with Monte Carlo simulation. One difference is that here we only need to perform the regression at one time step—the horizon time.

Assume the tranche value can be approximated by,

$$V_i(t) \approx \sum_{k=1}^{K_i} \beta_{i,k} h_{i,k}(t),$$

where $h_{i,k}(t)$ are basis functions that contain the information of the CDO deal associated with tranche i up to time t, K_i is the number of basis functions selected for tranche i. The coefficients $\beta_{i,k}$ are estimated through the least-squares regression. The discounted value of future tranche cash flows from one simulation run is compared to the tranche expected value conditional on $\mathbb{F}_t$--given by the formula above, the coefficients are chosen to minimize the sum of squared differences over all the simulation runs. This implicitly assumes that the deviations of the realized sample value of the discounted future cash flows from its true mean are independent across the simulation runs. In the matrix format, we have

$$\underset{N\times 1}{\widetilde{V}_i} = \underset{N\times K_i}{\widetilde{H}_i} \cdot \underset{K_i\times 1}{\widehat{B}_i} + \underset{N\times 1}{\widetilde{\varepsilon}_i},$$

where $\widetilde{V}_i$ is the vector that contains the discounted values for tranche i from N simulation runs, $\widetilde{H}_i$ is the matrix of the basis function values evaluated at t_H, $\widehat{B}_i$ is the coefficient vector and $\widetilde{\varepsilon}_i$ is the residual.

The regression coefficients can be solved through Gauss-Jordan elimination or QR decomposition. When using QR decomposition, the result can be expressed as the solution to an upper triangular K_i x K_i system of equations. Both the upper triangular matrix and the right hand side of the linear system can be updated sequentially as new simulation runs are performed, so that it is unnecessary to save any giant matrices of basis function values to compute the coefficients. Therefore this calculation should not cost significantly more computation time than currently used in the multi-step simulation for portfolio of tranches.

Once the coefficients are determined, conditional expected tranche value at horizon may be estimated through the basis functions fitted by the regression coefficients, given a simulation realization at horizon. A portfolio value distribution at horizon can be derived by summing up the values of all the exposures contained in the portfolio at horizon during each simulation run.

The simulation procedure is as follows:

1. Simulate the portfolio from the as-of date till last exposure matures, under risk-neutral measure.
2. Discount the cash flows for each exposure to the as-of date using the risk-free rate.
3. For each tranche that is not completely amortized or liquidated by horizon time t_H, discount the corresponding cash flows after t_H to time t_H using the risk-free rate, then calculate the basis function values at t_H. Update the tranche discounted value matrix $\widetilde{V}_i$ and basis function matrix $\widetilde{H}_i$, $i=1,\ldots,n_T$.
4. Repeat step 1-3 for N_1 simulation runs, use QR decomposition to solve the regression coefficients $\widehat{B}_i$, $i=1,\ldots,n_T$.
5. Simulate the portfolio from the as-of date till t_H, under physical measure.
6. For each exposure, calculate the time t_H value of cash flows before t_H. Valuate the standalone exposure at t_H using its realized credit state and valuate the tranche by a linear combination of basis functions values at t_H.
7. Repeat step 5-6 for N_2 simulation runs, obtain the portfolio value distribution at horizon.
8. Determine the zero loss point, calculate the expected loss and unexpected loss for the entire portfolio as well as for each exposure, also calculate the correlations among exposure values.
9. Calculate the portfolio risk and return measures, including expected return, risk contribution, VaR and economic capital.

2.3 Choice of Basis Functions

The optimal choice of basis functions is a set of orthogonal functions that are most explanatory of the conditional expected value. With this in mind, we want to select the basis functions that best characterize the tranche status, and are not highly correlated with each other to avoid getting a singular matrix during matrix decomposition.

Further research will be required to determine a sufficient set of basis functions to adequately estimate the expected tranche value conditional on the state at horizon. Note that each tranche in the portfolio may have a different set of basis functions. Some possible choices of basis functions are suggested below, as well as the appropriate polynomials of them.

- Collateral default loss up to horizon;
- Amortization amount up to horizon;
- Weighted average default probabilities at horizon;
- Weighted average loss given default at horizon;
- Weighted average coupon/spread at horizon;
- Collateral portfolio value at horizon;
- Spot interest rate at horizon;
- Tranche notional at horizon.

The numerical results are not yet available at this moment and we will report them once they are ready.

3 KEY OUTPUTS

The following model outputs can be produced for the portfolio in consideration:

3.1 Portfolio Value Distribution

- As-of date value distribution
- Horizon value and loss distribution
- Expected loss and unexpected loss
- Expected return of the portfolio
- Correlation of exposure values

3.2 Risk Contribution

Risk contribution is defined as the marginal contribution of individual exposure to the volatility of the portfolio value. It is given by

$$RC^i = \frac{\partial UL_p}{\partial w_i} = \frac{\sum_j w_j \rho_{ij} UL_i UL_j}{UL_p} = \frac{\rho_{ip} UL_i UL_p}{UL_p} = \rho_{ip} UL_i$$

where ρ_{ip} is the correlation between the value of exposure i and the value of the portfolio, UL stands for the unexpected loss which is calculated as the standard deviation of the value distribution. Another output, tail risk contribution, measures the contribution to the risk of an extreme event.

3.3 VaR and Economic Capital

Portfolio VaR measures the portfolio extreme loss and economic capital is the amount of capital that banks set aside to buffer against this loss. They can both be calculated given the portfolio value distribution at horizon.

4 CONCLUSION

We propose a basis function approach for CDO tranche valuation and risk analysis at horizon for portfolios that contain CDO tranches. It accommodates the complicated deal structure and path-dependency of value process in the tranche valuation, and does not require much additional work to an as-of date Monte Carlo valuation of the portfolio. Portfolio value distribution at horizon, expected loss, unexpected loss, risk contribution, VaR and economic capital are the key outputs. Besides CDO tranches, this model can be easily adapted to portfolio risk analysis involving any complex financial instruments such as combo notes or tranches from a CDO of CDOs.

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