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APPROXIMATE/PERFECT SAMPLERS FOR CLOSED JACKSON NETWORKS

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ABSTRACT

In this paper, we propose two samplers for the product-form solution of basic queueing networks, closed Jackson networks with multiple servers. Our approach is sampling via Markov chain, but it is NOT a simulation of behavior of customers in queueing networks. We propose two of new ergodic Markov chains both of which have a unique stationary distribution that is the product form solution of closed Jackson networks. One of them is for approximate sampling, and we show it mixes *rapidly*. To our knowledge, this is the first approximate polynomial-time sampler for closed Jackson networks with multiple servers. The other is for perfect sampling based on monotone CFTP (coupling from the past) algorithm proposed by Propp and Wilson,and we show the *monotonicity* of the chain.

1 INTRODUCTION

A Jackson network is one of the basic and significant models in queueing network theory. In the model, customers receive service at nodes each of which has multiple exponential servers on first-come-first-served (FCFS) basis and move probabilistically to a next node when service is completed. Jackson (1957) showed that a Jackson network has a product-form solution as the steady-state distribution of customers in the network (Jackson 1963, Gordon and Newell 1967). By computing the normalizing constant of the product-form solution, we can obtain important performance measures like as throughput, rates of utilization of stations, and so on.

There is well-known Buzen's algorithm (Buzen 1973), which computes the normalizing constant of the product-form solution. However, the running time of Buzen's algorithm is pseudo-polynomial time depending on the number of customers in a closed network. Chen and O'Cinneide (1998) proposed a randomized algorithm based on Markov chain Monte Carlo (MCMC), but it is weakly polynomial-time algorithm in some very special cases. Ozawa (2004) proposed a perfect sampler for closed Jackson networks with single servers, however his chain mixes in pseudo-polynomial time.

In this paper, we are concerned with sampling from the product-form solution of closed Jackson networks with multiple servers. Thus, we assume that a given network is strongly connected, a class of customers is unique, no customer leaves or enters the network, and each node has multiple servers. Then, we propose two ergodic Markov chains both of which have a unique stationary distribution that is the product-form solution of a closed Jackson network. Here we note that they are NOT a simulation of networks, but just have a unique stationary distribution which is the same as a product-form solution of a network.

Mainly, we discuss the convergence of the chains. We show that the *mixing time* of the chain for approximate sampling is bounded by $n(n-1)\ln(K\varepsilon^{-1})/2$ for an arbitrary positive $\varepsilon < 1$, where n is the number of nodes and K is the number of customers. To our knowledge, this is the first approximate polynomial-time sampler for closed Jackson networks with multiple servers. A key idea which derives polynomiality is not to simulate behavior of customers in a network, while both algorithms of Chen and O'Cinneide (1998) and Ozawa (2004) simulate behavior of customers. We show the mixing time by using a technique of *path coupling* introduces by Bubley and Dyer (1997). On the other hand, we show that the other chain is *monotone*, and design a perfect sampler based on monotone CFTP (coupling from the past) algorithm proposed by Propp and Wilson (1996).

There are two benefits at least, if we have a fast sampler. One is that we may design a fast randomized algorithms for computing normalizing constant, and so for throughput. Actually, we can design a polynomial-time randomized approximation scheme, though we will not deal with it in this paper (see Section 5 for more detail). The other is that a fast sampler finds a state with respect to the steady-state distribution of networks, thus we can use it as an initial state of a simulation of behavior of customers.

2 PRODUCT-FORM SOLUTION

We denote the set of real numbers (non-negative, positive real numbers) by $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$), and the set of integers (non-negative, positive integers) by $\mathbb{Z}$ ($\mathbb{Z}_+$, $\mathbb{Z}_{++}$), respectively. In queueing network theory, it is well known that a closed Jackson network has a product form solution. Let $n \in \mathbb{Z}_{++}$ be the number of nodes and $K \in \mathbb{Z}_+$ be the number of customers in a closed Jackson network. Let us consider the set of non-negative integer points

$$\Xi \stackrel{\text{def.}}{=} \left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_+^n \mid \textstyle\sum_{i=1}^n x_i = K \right\}$$

in an $n-1$ dimensional simplex. Let W be the transition probability matrix for a closed Jackson network system. Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}_{++}^n$ be an eigenvector for W with corresponding to the eigenvalue 1, i.e., $\boldsymbol{\theta} W = \boldsymbol{\theta}$. Here we note that W is ergodic. Given a vector $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_{++}^n$ of the inverse of mean of exponentially distributed service time on nodes, and a vector $(s_1, s_2, \ldots, s_n) \in \mathbb{Z}_+^n$ of number of servers on nodes, a product form solution of the closed Jackson network $J : \Xi \to \mathbb{R}_{++}$ is

$$J(\boldsymbol{x}) \stackrel{\text{def.}}{=} \frac{1}{G} \prod_{i=1}^n \frac{1}{\prod_{j=1}^{x_i} \min\{j, s_i\}} \left(\frac{\theta_i}{\mu_i} \right)^{x_i},$$

where

$$G \stackrel{\text{def.}}{=} \sum_{\boldsymbol{x} \in \Xi} \prod_{i=1}^n \frac{1}{\prod_{j=1}^{x_i} \min\{j, s_i\}} \left(\frac{\theta_i}{\mu_i} \right)^{x_i}$$

is the normalizing constant. In the rest of this paper, we define a function $\alpha_i : \mathbb{Z}_+ \to \mathbb{R}_{++}$ as follows,

$$\begin{aligned} \alpha_i(m) &\stackrel{\text{def.}}{=} \frac{1}{\prod_{j=1}^m \min\{j, s_i\}} \left(\frac{\theta_i}{\mu_i} \right)^m \\ &\equiv \begin{cases} \dfrac{1}{m!} \left(\dfrac{\theta_i}{\mu_i} \right)^m & (m \leq s_i), \\ \dfrac{1}{s_i^{m-s_i} s_i!} \left(\dfrac{\theta_i}{\mu_i} \right)^m & (m > s_i), \end{cases} \end{aligned} \tag{1}$$

for $m \in \mathbb{Z}_+$, thus the product form slution is described as

$$J(\boldsymbol{x}) = \frac{1}{G} \prod_{i=1}^n \alpha_i(x_i).$$

3 APPROXIMATE SAMPLER

Now we propose a new Markov chain $\mathcal{M}_\text{A}$ with state space Ξ. A transition of $\mathcal{M}_\text{A}$ from a current state $X \in \Xi$ to a next state X' is defined as follows. First, we chose a distinct pair of indices $\{j_1, j_2\}$ uniformly at random. Next, let $k = X_{j_1} + X_{j_2}$, and chose $l \in \{0, 1, \ldots, k\}$ with probability

$$\begin{aligned} &\frac{\alpha_{j_1}(l) \alpha_{j_2}(k-l)}{\sum_{s=0}^k \alpha_{j_1}(s) \alpha_{j_2}(k-s)} \\ &\qquad \equiv \frac{\alpha_{j_1}(l) \alpha_{j_2}(k-l) \prod_{j \notin \{j_1, j_2\}} \alpha_j(X_j)}{\sum_{s=0}^k \alpha_{j_1}(s) \alpha_{j_2}(k-s) \prod_{j \notin \{j_1, j_2\}} \alpha_j(X_j)} \end{aligned}$$

then set

$$X'_i = \begin{cases} l & (\text{for } i = j_1), \\ k - l & (\text{for } i = j_2), \\ X_i & (\text{otherwise}). \end{cases}$$

Since $\alpha_i(x)$ is a positive function, the Markov chain $\mathcal{M}_\text{A}$ is irreducible and aperiodic, so ergodic, hence has a unique stationary distribution. Also, $\mathcal{M}_\text{A}$ satisfies *detailed balance equation*

$$J(\boldsymbol{x}) P(\boldsymbol{x} \to \boldsymbol{y}) = J(\boldsymbol{y}) P(\boldsymbol{y} \to \boldsymbol{x})$$

for any states $\boldsymbol{x}, \boldsymbol{y} \in \Xi$, where $P(\boldsymbol{x} \to \boldsymbol{y})$ denotes the transition probability from $\boldsymbol{x}$ to $\boldsymbol{y}$. Thus the stationary distribution is the product form solution $J(\boldsymbol{x})$ of closed Jackson networks (Jackson 1957).

We can obtain a sample w.r.t. the product form solution $J(\boldsymbol{x})$ by simulating $\mathcal{M}_\text{A}$ sufficiently many times. Next we discuss the mixing time (defined in detail below) of $\mathcal{M}_\text{A}$.

Given a pair of probability distributions ν_1 and ν_2 on the finite state space Ω, the *total variation distance* between ν_1 and ν_2 is defined by

$$\mathrm{d_{TV}}(\nu_1, \nu_2) \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{x \in \Omega} |\nu_1(x) - \nu_2(x)|.$$

Given an arbitrary positive real ε, the *mixing time* of an ergodic Markov chain is defined by

$$\tau(\varepsilon) \stackrel{\text{def.}}{=} \max_{x \in \Omega} \{ \min\{ t \mid \forall s \geq t,\ \mathrm{d_{TV}}(\pi, P_x^s) \leq \varepsilon \} \}$$

where π is the stationary distribution and P_x^s is the probability distribution of the chain at time period $s \geq 0$ with initial state x (at time period 0). In the rest of this section, we show the following theorem

Theorem 1 *For $0 < \forall\varepsilon < 1$, the mixing time $\tau(\varepsilon)$ of Markov chain $\mathcal{M}_{\mathrm{A}}(K)$ satisfies*

$$\tau(\varepsilon) \le \frac{n(n-1)}{2}\ln(K\varepsilon^{-1}).$$

Here, we consider the cumulative distribution function $g_{ij}^k : \{0, 1, \ldots, k\} \to \mathbb{R}_+$ defined by

$$g_{ij}^k(l) \stackrel{\text{def.}}{=} \frac{\sum_{s=0}^{l}\alpha_i(s)\alpha_j(k-s)}{\sum_{s=0}^{k}\alpha_i(s)\alpha_j(k-s)}.$$

We also define $g_{ij}^k(-1) \stackrel{\text{def.}}{=} 0$, for convenience. We can simulate the Markov chain $\mathcal{M}_{\mathrm{A}}$ with the function g_{ij}^k. First, choose a pair $\{i, j\}$ of indices with the probability $2/(n(n-1))$. Next, put $k = X_i + X_j$, generate an uniformly random real number $\Lambda \in [0, 1)$, choose l satisfying $g_{ij}^k(l-1) \le \Lambda \le g_{ij}^k(l)$, and set $X_i' = l$ and $X_j' = k - l$.

The following Lemma is important for our main theorem.

Lemma 2 *The each function α_i $(i \in \{1, 2, \ldots, n\})$ is a log-concave function, which means*

$$\ln\alpha_i(m) - \ln\alpha_i(m-1) \ge \ln\alpha_i(m+1) - \ln\alpha_i(m) \qquad (2)$$

for any $m \in \mathbb{Z}_{++}$. Then, for any pair of distinct indices (i, j) $(i, j \in \{1, 2, \ldots, n\})$ and for any $k \in \mathbb{Z}_+$, the alternating inequalities

$$g_{ij}^{k+1}(l) \le g_{ij}^k(l) \le g_{ij}^{k+1}(l+1) \qquad (3)$$

holds for any $l \in \mathbb{Z}_+$.

Figure 1 is a figure of alternating inequalities. In the figure, $A \stackrel{\text{def.}}{=} \sum_{s=0}^{k}\alpha_i(s)\alpha_j(k-s)$ and $A' \stackrel{\text{def.}}{=} \sum_{s=0}^{k+1}\alpha_i(s)\alpha_j(k+1-s)$ are normalizing constants.

Proof: First, we show that the function $\alpha_i(m)$ is log-concave. From the equations (1) of the function $\alpha_i(m)$,

$$\ln\alpha_i(m) = \begin{cases} m\ln\left(\frac{\theta_i}{\mu_i}\right) - \sum_{j=1}^{m}\ln j & (m \le s_i), \\ m\ln\left(\frac{\theta_i}{\mu_i}\right) - (m-s_i)\ln s_i - \sum_{j=1}^{s_i}\ln j & (m > s_i), \end{cases}$$

hold. Thus $\ln\alpha_i(m) - \ln\alpha_i(m-1)$

$$= \begin{cases} \ln\left(\frac{\theta_i}{\mu_i}\right) - \ln m & (m \le s_i), \\ \ln\left(\frac{\theta_i}{\mu_i}\right) - \ln s_i & (m > s_i). \end{cases}$$

From the above, the function $\alpha_i(m)$ satisfies (2), thus $\alpha_i(m)$ is log-concave.

Next, we show the latter statement. When $k = 0$, it is obvious. When we fix $k \in \mathbb{Z}_{++}$, the alternating inequalities (3) hold for any $l \in \{0, 1, \ldots, k\}$, if and only if

$$\begin{aligned} &\left(\sum_{s=0}^{l}\alpha_i(s)\alpha_j(k+1-s)\right) \\ &\quad\cdot\left(\sum_{s'=l+1}^{k}\alpha_i(s')\alpha_j(k-s')\right) \\ \le\ &\left(\sum_{s=0}^{l}\alpha_i(s)\alpha_j(k-s)\right) \\ &\quad\cdot\left(\sum_{s'=l+1}^{k+1}\alpha_i(s')\alpha_j(k+1-s')\right), \end{aligned} \qquad (4)$$

and

$$\begin{aligned} &\left(\sum_{s=0}^{l}\alpha_i(k+1-s)\alpha_j(s)\right) \\ &\quad\cdot\left(\sum_{s'=l+1}^{k}\alpha_i(k-s')\alpha_j(s')\right) \\ \le\ &\left(\sum_{s=0}^{l}\alpha_i(k-s)\alpha_j(s)\right) \\ &\quad\cdot\left(\sum_{s'=l+1}^{k+1}\alpha_i(k+1-s')\alpha_j(s')\right), \end{aligned} \qquad (5)$$

hold for any $l \in \{0, 1, \ldots, k-1\}$. With considering the expansion of (4), it is enough to show that $\forall s, \forall s' \in \{0, 1, \ldots, k\}$ satisfying $0 \le s < s' \le k$,

$$\begin{aligned} &\alpha_i(s)\alpha_j(k+1-s)\alpha_i(s')\alpha_j(k-s') \\ &\quad\le\ \alpha_i(s)f_j(k-s)\alpha_i(s')\alpha_j(k+1-s'). \end{aligned} \qquad (6)$$

Since, α_j is a log-concave function for any index $j \in \{1, 2, \ldots, n\}$, the inequalities $(k-s') < (k-s'+1) \le (k-s) < (k-s+1)$ imply that

$$\begin{aligned} &\ln\alpha_j(k-s') + \ln\alpha_j(k-s+1) \\ &\quad\le\ \ln\alpha_j(k-s'+1) + \ln\alpha_j(k-s) \end{aligned}$$

holds. From the above, the inequality (6) hold $\forall s, \forall s' \in \{0, 1, \ldots, k\}$ satisfying $0 \le s < s' \le k$. For the inequality (5), we obtain the claim in the same way as (4) by interchanging i and j. □

We will show Theorem 1 by using the path coupling technique. The following path coupling theorem proposed by Bubley and Dyer (1997) is useful for bounding the mixing time.

Theorem 3 (Path coupling Bubley and Dyer (1997)) *Let $\mathcal{M}$ be a finite ergodic Markov chain with a state space Ω. Let $H = (\Omega, \mathcal{E})$ be a connected undirected graph with vertex set Ω and edge set $\mathcal{E} \subset \Omega^2$. Let the length of all edges be 1, and let the distance between x and y, denoted by $d(x, y)$ and/or $d(y, x)$, be the length of a shortest path between x and y. Suppose that there exists a joint process $(X, Y) \mapsto (X', Y')$ with respect to $\mathcal{M}$ satisfying that whose marginals are a faithful copy of $\mathcal{M}$. If there exists a positive*

0	$\alpha_i(0)\alpha_j(k)/A$	$\alpha_i(1)\alpha_j(k-1)/A$	$\cdots$	$\alpha_i(k)\alpha_j(0)/A$	1

0	$\alpha_i(0)\alpha_j(k+1)/A'$	$\alpha_i(1)\alpha_j(k)/A'$	$\alpha_i(2)\alpha_j(k-1)/A'$	$\cdots$	$\alpha_i(k+1)\alpha_j(0)/A'$	1

Figure 1: A figure of alternating inequalities for a pair of indices (i, j) and a non-negative integer k.

real β, exactly less than one, satisfying

$$\mathrm{E}[d(X', Y')] \leq \beta d(X, Y)$$

for any edge $\{X, Y\} \in \mathcal{E}$ of H, then the mixing time $\tau(\varepsilon)$ of the Markov chain $\mathcal{M}$ satisfies

$$\tau(\varepsilon) \leq (1-\beta)^{-1} \ln(\varepsilon^{-1} D),$$

where $D \stackrel{\text{def.}}{=} \max\{d(x, y) \mid \forall x, \forall y \in \Omega\}$ is the diameter of the graph H.

Proof of Theorem 1 Let $H = (\Xi, \mathcal{E})$ be an undirected simple graph with vertex set Ξ and edge set $\mathcal{E}$ defined as follows. A pair of vertices $\{x, y\}$ is an edge of H if and only if $(1/2)\sum_{i=1}^{n} |x_i - y_i| = 1$. Clearly the graph H is connected. We define the length of an edge $e \in \mathcal{E}$ as 1, and the distance $d(x, y)$ for each pair $(x, y) \in \Xi^2$ as the length of a shortest path between x and y on H. Clearly, the diameter of H defined by $\max_{x,y\in\Xi}\{d(x, y)\}$, is bounded by K.

We define a joint process $(X, Y) \mapsto (X', Y')$ for any pair $\{X, Y\} \in \mathcal{E}$. Pick a distinct pair of indices $\{i_1, i_2\}$ uniformly at random. Then put $k_X = X_{i_1} + X_{i_2}$ and $k_Y = Y_{i_1} + Y_{i_2}$, generate an uniform random number $\Lambda \in [0, 1)$, chose $l_X \in \{0, 1, \ldots, k_X\}$ and $l_Y \in \{0, 1, \ldots, k_Y\}$ which satisfy $g^{k_X}_{i_1 i_2}(l_X - 1) \leq \Lambda < g^{k_X}_{i_1 i_2}(l_X)$ and $g^{k_Y}_{i_1 i_2}(l_Y - 1) \leq \Lambda < g^{k_Y}_{i_1 i_2}(l_Y)$, and set $X'_{i_1} = l_X$, $X'_{i_2} = k_X - l_X$, $Y'_{i_1} = l_Y$ and $Y'_{i_2} = k_Y - l_Y$.

Now we show that

$$\beta = 1 - \frac{2}{n(n-1)}$$

satisfies

$$\mathrm{E}[d(Y', Y')] \leq \beta d(X, Y)$$

for any pair $\{X, Y\} \in \mathcal{E}$. Here we suppose that $X, Y \in \mathcal{E}$ satisfies $|X_j - X_j| = 1$ for $j \in \{j_1, j_2\}$, and $|X_j - X_j| = 0$ for $j \notin \{j_1, j_2\}$.

<u>**Case 1**</u>: In case that the neither of indices j_1 nor j_2 are chosen, i.e., $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$. Put $k = X_{i_1} + X_{i_2}$, then it is easy to see that $\Pr(X'_{i_1} = l) = \Pr(Y'_{i_1} = l)$ for any $l \in \{0, \ldots, k\}$ since $Y_{i_1} + Y_{i_2} = k$. By setting $X'_{i_1} = Y'_{i_1}$ and $X'_{i_2} = Y'_{i_2}$, we have $d(X', Y') = d(X, Y)$.

<u>**Case 2**</u>: In case that the both of indices j_1 and j_2 are chosen, i.e., $\{i_1, i_2\} = \{j_1, j_2\}$. In the same way as Case 1, we can set $X'_{i_1} = Y'_{i_1}$ and $X'_{i_2} = Y'_{i_2}$. Hence $d(X', Y') = 0$.

<u>**Case 3**</u>: In case that exactly one of j_1 and j_2 is chosen, i.e., $|\{i_1, i_2\} \cap \{j_1, j_2\}| = 1$. Without loss of generality, we can assume that $i_1 = j_1$ and that $X_{i_1} + 1 = Y_{i_1}$. Let $k = X_{i_1} + X_{i_2}$. Then $Y_{i_1} + Y_{i_2} = k + 1$ obviously. We consider the joint process as a random number $\Lambda \in [0, 1)$ is given. Let $l \in \{0, 1, \ldots, k\}$ satisfies $g^{k}_{i_1 i_2}(l-1) \leq \Lambda < g^{k}_{i_1 i_2}(l)$, then alternating inequalities imply that $g^{k+1}_{i_1 i_2}(l-1) \leq \Lambda < g^{k+1}_{i_1 i_2}(l+1)$. Therefore, if $X'_{i_1} = l$ then Y'_{i_1} should be in $\{l, l+1\}$ by the joint process. Thus we always obtain that $[X'_{i_1} = Y'_{i_1}$ and $X'_{i_2} + 1 = Y'_{i_2}]$ or $[X'_{i_1} + 1 = Y'_{i_1}$ and $X'_{i_2} = Y'_{i_2}]$. Hence $d(X', Y') = d(X, Y)$.

With considering that Case 2 occurs with probability $2/(n(n-1))$, we obtain that

$$\mathrm{E}[d(X', Y')] \leq \left(1 - \frac{2}{n(n-1)}\right) d(X, Y).$$

Since the diameter of H is bounded by K, Theorem 3 (Path Coupling Theorem) implies that the mixing time $\tau(\varepsilon)$ satisfies

$$\tau(\varepsilon) \leq \frac{n(n-1)}{2} \ln(K\varepsilon^{-1}).$$

□

4 PERFECT SAMPLER

4.1 Monotone Markov Chain

In this section we propose new Markov chain $\mathcal{M}_{\mathrm{P}}$. The transition rule of $\mathcal{M}_{\mathrm{P}}$ is defined by the following *update function* $\phi : \Xi \times [1, n) \to \Xi$. For a current state $X \in \Xi$, the next state $X' = \phi(X, \lambda) \in \Xi$ with respect to a random number $\lambda \in [1, n)$ is defined by

$$X'_i = \begin{cases} l & (\text{for } i = \lfloor\lambda\rfloor), \\ k - l & (\text{for } i = \lfloor\lambda\rfloor + 1), \\ X_i & (\text{otherwise}), \end{cases}$$

where $k = X_{\lfloor\lambda\rfloor} + X_{\lfloor\lambda\rfloor+1}$ and $l \in \{0, 1, \ldots, k\}$ satisfies

$$g^{k}_{\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1)}(l-1) < \lambda - \lfloor\lambda\rfloor \leq g^{k}_{\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1)}(l).$$

Our chain $\mathcal{M}_\mathrm{P}$ is a modification of $\mathcal{M}_\mathrm{A}$, obtained by restricting to choose only a consecutive pair of indices. Clearly, $\mathcal{M}_\mathrm{P}$ is ergodic. The chain has a unique stationary distribution $J(\boldsymbol{x})$ defined in Section 2.

In the following, we show the monotonicity of $\mathcal{M}_\mathrm{P}$. Here we introduce a partial order "$\succeq$" on Ξ. For any state $\boldsymbol{x} \in \Xi$, we introduce *cumulative sum vector* $c_{\boldsymbol{x}} = (c_{\boldsymbol{x}}(0), c_{\boldsymbol{x}}(1), \ldots, c_{\boldsymbol{x}}(n)) \in \mathbb{Z}_+^{n+1}$ defined by

$$c_{\boldsymbol{x}}(i) \stackrel{\text{def.}}{=} \begin{cases} 0 & (\text{for } i = 0), \\ \sum_{j=1}^{i} x_j & (\text{for } i \in \{1, 2, \ldots, n\}). \end{cases}$$

For any pair of states $\boldsymbol{x}, \boldsymbol{y} \in \Xi$, we say $\boldsymbol{x} \succeq \boldsymbol{y}$ if and only if $c_{\boldsymbol{x}} \geq c_{\boldsymbol{y}}$. Next, we define two special states $x_\mathrm{U}, x_\mathrm{L} \in \Xi(K)$ by $x_\mathrm{U} \stackrel{\text{def.}}{=} (K, 0, \cdots, 0)$ and $x_\mathrm{L} \stackrel{\text{def.}}{=} (0, \ldots, 0, K)$. Then we can see easily that $\forall \boldsymbol{x} \in \Xi(K)$, $x_\mathrm{U} \succeq \boldsymbol{x} \succeq x_\mathrm{L}$.

Theorem 4 *Markov chain $\mathcal{M}_\mathrm{P}$ is monotone on the partially ordered set $(\Xi(K), \succeq)$, i.e., $\forall \lambda \in [1, n)$, $\forall X, \forall Y \in \Xi(K)$, $X \succeq Y \Rightarrow \phi(X, \lambda) \succeq \phi(Y, \lambda)$.*

Proof: We say that a state $X \in \Xi$ *covers* $Y \in \Xi$ (at j), denoted by $X \cdot\!\succ Y$ (or $X \cdot\!\succ_j Y$), when

$$X_i - Y_i = \begin{cases} +1 & (\text{for } i = j), \\ -1 & (\text{for } i = j+1), \\ 0 & (\text{otherwise}). \end{cases}$$

We show that if a pair of states $X, Y \in \Xi$ satisfies $X \cdot\!\succ_j Y$, then $\forall \lambda \in [1, n)$, $\phi(X, \lambda) \succeq \phi(Y, \lambda)$. We denote $\phi(X, \lambda)$ by X' and $\phi(Y, \lambda)$ by Y' for simplicity. For any index $i \neq \lfloor \lambda \rfloor$, it is easy to see that $c_X(i) = c_{X'}(i)$ and $c_Y(i) = c_{Y'}(i)$, and so $c_{X'}(i) - c_{Y'}(i) = c_X(i) - c_Y(i) \geq 0$ since $X \succeq Y$. In the following, we show that $c_{X'}(\lfloor \lambda \rfloor) \geq c_{Y'}(\lfloor \lambda \rfloor)$.

<u>Case 1</u>: If $\lfloor \lambda \rfloor \neq j-1$ and $\lfloor \lambda \rfloor \neq j+1$. Let $k = X_{\lfloor \lambda \rfloor} + X_{\lfloor \lambda \rfloor + 1}$, then it is easy to see that $Y_{\lfloor \lambda \rfloor} + Y_{\lfloor \lambda \rfloor + 1} = k$. Accordingly $X'_{\lfloor \lambda \rfloor} = Y'_{\lfloor \lambda \rfloor} = l$ where l satisfies

$$g^k_{\lfloor \lambda \rfloor (\lfloor \lambda \rfloor + 1)}(l-1) \leq \lambda - \lfloor \lambda \rfloor < g^k_{\lfloor \lambda \rfloor (\lfloor \lambda \rfloor + 1)}(l),$$

hence $c_{X'}(\lfloor \lambda \rfloor) = c_{Y'}(\lfloor \lambda \rfloor)$.

<u>Case 2</u>: Consider the case that $\lfloor \lambda \rfloor = j-1$. Let $k+1 = X_{j-1} + X_j$, then $Y_{j-1} + Y_j = k$, since $X \cdot\!\succ_j Y$. From the definition of cumulative sum vector,

$$\begin{aligned} & c_{X'}(\lfloor \lambda \rfloor) - c_{Y'}(\lfloor \lambda \rfloor) \\ &= c_{X'}(j-1) - c_{Y'}(j-1) \\ &= c_{X'}(j-2) + X'_{j-1} - c_{Y'}(j-2) - Y'_{j-1} \\ &= c_X(j-2) + X'_{j-1} - c_Y(j-2) - Y'_{j-1} \\ &= X'_{j-1} - Y'_{j-1}. \end{aligned}$$

Thus, it is enough to show that $X'_{j-1} \geq Y'_{j-1}$. Now suppose that $l \in \{0, 1, \ldots, k\}$ satisfies $g^k_{(j-1)j}(l-1) \leq \lambda - \lfloor \lambda \rfloor < g^k_{(j-1)j}(l)$ for λ. Then $g^{k+1}_{(j-1)j}(l-1) \leq \lambda - \lfloor \lambda \rfloor < g^{k+1}_{(j-1)j}(l+1)$, since the alternating inequalities imply that $g^{k+1}_{(j-1)j}(l-1) \leq g^k_{(j-1)j}(l-1) < g^{k+1}_{(j-1)j}(l) \leq g^{k+1}_{(j-1)j}(l+1)$. Thus we have that if $Y'_{j-1} = l$ then X'_{j-1} is equal to l or $l+1$. In other words,

$$\begin{pmatrix} X'_{j-1} \\ Y'_{j-1} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} k \\ k+1 \end{pmatrix} \right\}$$

and $X'_{j-1} \geq Y'_{j-1}$ holds in all cases. Accordingly, we have that $c_{X'}(\lfloor \lambda \rfloor) \geq c_{Y'}(\lfloor \lambda \rfloor)$.

<u>Case 3</u>: Consider the case that $\lfloor \lambda \rfloor = j+1$. We can show $c_{X'}(\lfloor \lambda \rfloor) \geq c_{Y'}(\lfloor \lambda \rfloor)$ in a similar way to Case 2.

For any pair of states X, Y satisfying $X \succeq Y$, it is easy to see that there exists a sequence of states $Z_1, Z_2, \ldots, Z_r$ with appropriate length satisfying $X = Z_1 \cdot\!\succ Z_2 \cdot\!\succ \cdots \cdot\!\succ Z_r = Y$. Then applying the above claim repeatedly, we obtain that $\phi(X, \lambda) = \phi(Z_1, \lambda) \succeq \phi(Z_2, \lambda) \succeq \cdots \succeq \phi(Z_r, \lambda) = \phi(Y, \lambda)$. □

Since $\mathcal{M}_\mathrm{P}$ is a monotone chain, we can design a perfect sampler based on monotone CFTP, which we will introduce in the next subsection. Here we note that we could also employ Wilson's read once algorithm (Wilson 2000) and Fill's interruptible algorithm (Fill 1998, Fill et al. 2000), each of which also gives a perfect sampler.

4.2 Coupling from the Past

Suppose that we have an ergodic Markov chain $\mathcal{M}$ with a finite state space Ω and a transition matrix P. The transition rule of the Markov chain $X \mapsto X'$ can be described by a deterministic function $\phi : \Omega \times [0, 1) \to \Omega$, called *update function*, as follows. Given a random number Λ uniformly distributed over $[0, 1)$, update function ϕ satisfies that $\Pr(\phi(x, \Lambda) = y) = P(x, y)$ for any $x, y \in \Omega$. We can realize the Markov chain by setting $X' = \phi(X, \Lambda)$. Clearly, update functions corresponding to the given transition matrix P are not unique. The result of transitions of the chain from the time t_1 to t_2 ($t_1 < t_2$) with a sequence of random numbers $\boldsymbol{\lambda} = (\lambda[t_1], \lambda[t_1+1], \ldots, \lambda[t_2-1]) \in [0, 1)^{t_2 - t_1}$ is denoted by $\Phi^{t_2}_{t_1}(x, \boldsymbol{\lambda}) : \Omega \times [0, 1)^{t_2 - t_1} \to \Omega$ where $\Phi^{t_2}_{t_1}(x, \boldsymbol{\lambda}) \stackrel{\text{def.}}{=} \phi(\phi(\cdots(\phi(x, \lambda[t_1]), \ldots, \lambda[t_2 - 2]), \lambda[t_2 - 1])$. We say that a sequence $\lambda \in [0, 1)^{|T|}$ satisfies the *coalescence condition*, when $\exists y \in \Omega$, $\forall x \in \Omega$, $y = \Phi^0_T(x, \boldsymbol{\lambda})$.

Suppose that there exists a partial order "$\succeq$" on the set of states Ω. A transition rule expressed by a deterministic update function ϕ is called *monotone* (with respect to "$\succeq$") if $\forall \lambda \in [0, 1)$, $\forall x, \forall y \in \Omega$, $x \succeq y \Rightarrow \phi(x, \lambda) \succeq \phi(y, \lambda)$. We also say that a chain is *monotone* if the chain has a *monotone* update function. Here we suppose that there exists a unique pair of states $(x_\mathrm{U}, x_\mathrm{L})$ in partially ordered set $(\Omega, \succeq)$, satisfying $x_\mathrm{U} \succeq x \succeq x_\mathrm{L}$, $\forall x \in \Omega$.

With these preparations, a standard monotone Coupling From The Past algorithm is expressed as follows.

Algorithm 1 (monotone CFTP (Propp and Wilson 1996))

Step 1: Set the starting time period $T := -1$ to go back, and set $\boldsymbol{\lambda}$ be the empty sequence.

Step 2: Generate random real numbers $\lambda[T], \lambda[T+1], \dots, \lambda[\lceil T/2 \rceil - 1] \in [0, 1)$, and insert them to the head of $\boldsymbol{\lambda}$ in order, i.e., put $\boldsymbol{\lambda} := (\lambda[T], \lambda[T+1], \dots, \lambda[-1])$.

Step 3: Start two chains from x_{U} and x_{L}, respectively, at time period T, and run each chain to time period 0 according to the update function ϕ with the sequence of numbers in $\boldsymbol{\lambda}$. (Here we note that every chain uses the common sequence $\boldsymbol{\lambda}$.)

Step 4: [Coalescence check] The state obtained at time period 0 is denoted by $\Phi_T^0(x, \boldsymbol{\lambda})$.

(a) If $\exists y \in \Omega$, $y = \Phi_T^0(x_{\mathrm{U}}, \boldsymbol{\lambda}) = \Phi_T^0(x_{\mathrm{L}}, \boldsymbol{\lambda})$, then return y and stop.

(b) Else, update the starting time period $T := 2T$, and go to Step 2.

Theorem 5 (CFTP Theorem Propp and Wilson (1996)) *Let $\mathcal{M}$ be an ergodic finite Markov chain with state space Ω, defined by an update function $\phi : \Omega \times [0, 1) \to \Omega$. If the CFTP algorithm (Algorithm 1) terminates with probability 1, then the obtained value is a realization of a random variable exactly distributed according to the stationary distribution.*

Theorem 5 gives a (probabilistically) finite time algorithm for infinite time simulation.

5 CONCLUDING REMARKS

We propose two samplers based on Markov chain which produce a sample of the product form solution of closed Jackson networks. We show that the chain for approximate sampling mixes in $\mathrm{O}(n^2 \ln K)$ time. By using this result, we can design a polynomial-time randomized approximation scheme for computing normalizing constant and throughput, by combining with Monte Carlo method, it is so called MCMC (Markov chain Monte Carlo). The approximation scheme can be designed in standard method (see (Jerrum and Sinclair 1996)), but many discussion points are remained.

Though we did not discussed the mixing time of Markov chain $\mathcal{M}_{\mathrm{P}}$ for perfect sampling, we can show that $\mathcal{M}_{\mathrm{P}}$ rapidly mixes in $\mathrm{O}(n^3 \ln K)$ in some cases, which includes closed Jackson networks with single servers (i.e., $s_i = 1, \forall i$) and with infinite servers. (i.e., $s_i = +\infty, \forall i$). It is, however, open if the Markov chain $\mathcal{M}_{\mathrm{P}}$ mixes in polynomial time of n and $\ln K$ when the numbers of servers $(s_1, s_2, \dots, s_n)$ are arbitrary positive integers.

Major remaining problem is to extend to more generalized models, for example, with multiple classes of customers model or BCMP networks (Baskett et al. 1975) and so on.

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