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OUT-OF-THE-MONEY MONTE CARLO SIMULATION OPTION PRICING: THE JOINT USE OF IMPORTANCE SAMPLING AND DESCRIPTIVE SAMPLING

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ABSTRACT

As in any Monte Carlo application, simulation option valuation produces imprecise estimates. In such an application, Descriptive Sampling (DS) has proven to be a powerful Variance Reduction Technique. However, this performance deteriorates as the probability of exercising an option decreases. In the case of out-of-the-money options, the solution is to use Importance Sampling (IS). Following this track, the joint use of IS and DS is deserving of attention. Here, we evaluate and compare the benefits of using standard IS method with the joint use of IS and DS. We also investigate the influence of the problem dimensionality in the variance reduction achieved. Although the combination IS+DS showed gains over the standard IS implementation, the benefits in the case of out-of-the-money options were mainly due to the IS effect. On the other hand, the problem dimensionality did not affect the gains. Possible reasons for such results are discussed.

1 INTRODUCTION

A well-known weakness of Monte Carlo simulation is the lack of precision in the estimates. Naturally, this is also true in Monte Carlo Simulation option valuation. Variance Reduction Techniques (VRT) are generally recommended to minimize this problem, as suggested by Bratley, Fox, and Schrage (1987), and Charnes (2000). One of these techniques, Descriptive Sampling, proposed in Saliby (1990), has proven to be very efficient when compared with other direct sampling techniques. By direct sampling, we mean the usual approach where samples are directly drawn from model distributions, unlike the less common case where samples are draw from transformed distributions, as in Importance Sampling (IS). DS is a rather new and not very well-known Variance Reduction Technique based on a fully deterministic selection of the sample values and their random permutation. In general, DS produces more precise estimates than the standard Monte Carlo and other improved direct sampling schemes such as Latin Hypercube Sampling (LHS), as reported in Saliby (1997). Therefore, DS is a good choice in option pricing simulation. However, in the case of out-of-the-money options, where the exercise probability is quite low, all direct sampling methods, including DS, deteriorate. In such cases, the solution is to use Importance Sampling (IS). Following this track, the joint use of IS and DS is likely to be fruitful. This work evaluates and compares the benefits from using the standard IS method, based on a Simple Random Sampling (SRS) implementation, with the joint use of IS and DS. We also investigate the influence of the problem dimensionality in the variance reduction achieved.

European calls can be analytically priced through the well-known Black and Scholes (1973) model. Nevertheless, Monte Carlo simulation can also be used to price European options, mainly by serving as a reference when the simulation procedure is extended to other kinds of options without any known analytical solution. Another advantage in the simulation valuation of European options, specifically for purposes of this study, is the possibility of varying the problem dimensionality, e.g. the number of simulated time steps, without changing the responses and estimates being studied.

Although there is no great appeal in simulating European options, since a closed solution is available, it is expected that most simulation features in this standard case are likely to be extendable to other cases such as Path-Dependent and other kinds of exotic options.

A particular case of interest concerns out-of-the-money options, like European calls with strike prices far higher than the current asset price. As already mentioned, the estimate's precision deteriorates when using any direct sampling method; this applies to basic sampling methods such as Simple Random Sampling (SRS), as well as to more controlled sample schemes; for example, LHS, DS and Quasi Monte Carlo (QMC). Such is the case because, when the probability of exercise is too low and direct sampling methods are used, the problem becomes a rare event simulation case with most simulated payoff values being zero, and, consequently, very few positive payoff values will result. Since the payoff distribution is a mixed type distribution, i.e. discrete for zero values and continuous and tailed for positive values, the option's fair price will be poorly estimated when the two kinds of results are unbalanced present in the simulated payoffs. To improve the quality of simulation estimates when rare events are relevant, the use of Importance Sampling is, in principle, a good choice.

Importance Sampling (IS) is a variance reduction technique that changes the parameters of the original problem in a case where original rare events are no longer rare and, with proper adjustments, it provides unbiased and more precise estimates. In the present case, the parameters are changed in order to substantially increase the probability of exercising the option, so that the transformed option is no longer out-of-the-money. In principle, the gains with IS over SRS and other VRTs are higher as rare events become less likely. In fact, the use of IS in such cases is suggested by Charnes (2000) and Staum (2003), among others.

Another Variance Reduction Technique used herein, Descriptive Sampling, can be seen as an improvement over Latin Hypercube Sampling as described in Saliby (1997). The only practical difference between both methods is the deterministic selection of the sample values inside each stratum in the DS case, instead of a still random draw in each stratum in the LHS case. One key issue related to DS efficiency is problem dimensionality, i.e. the number of random variables in the simulation model. In the trivial one dimension case (dim = 1), DS produces determinist results, usually a good numerical approximation to the theoretical solution. This follows because, in such a case, the random permutation of the input values is irrelevant for the final simulation estimates. An example of this case is European call or put option pricing, where the final asset price is generated in just one time step. However, when dim >1, the random permutation of the input vector of values will vary the simulation estimates between different runs, even with a fixed set of input values. Therefore, apart from the trivial dim = 1 case, where the DS improvement is 100%, a question to be answered is how the problem dimensionality may affect the DS performance when dim >1.

In order to investigate the influence of the exercise probability in the IS efficiency, with and without DS, three different deep out-of-the-money European calls were simulated. The problem dimensionality also varied for the three cases by using different numbers of time steps to generate the final asset price. The quality of the estimates was evaluated by the standard error reduction over the standard Monte Carlo sampling method together with the Root Mean Squared Error (RMSE) reduction based on the Black and Scholes solution.

The remainder of this paper is organized as follows: Section 2 describes the methodology, briefly presenting the Variance Reduction Techniques in use; Section 3 shows the main results from the simulation experiments; finally, Section 4 concludes with a short discussion of the main findings.

2 METHODOLOGY

2.1 European Calls and The Black and Scholes Solution

A European call presents a simple payoff function, given as

$$Payoff = \max(0; S_T - K), \tag{1}$$

where:

- S_T = the underlying asset price at the maturity of the option,
- K = the exercise price of the option.

A call option is out-of-the-money when the current underlying asset price is below the strike price. The higher the exercise price, the lower the probability that the option will be exercised. When this probability is too low, the option is said to be deep out-of-the-money.

The price of a European call is defined by the present value of its expected payoff. The Black and Scholes (B&S) model presents a closed-form solution for this price:

$$c = S_0 N(d_1) - K e^{-R_f T/252} N(d_2), \tag{2}$$

where:

- $d_1 = \dfrac{\ln\left(S_0/K\right) + \left(R_f + \sigma^2/2\right) \cdot T/252}{\sigma\sqrt{T/252}}$,
- $d_2 = \dfrac{\ln\left(S_0/K\right) + \left(R_f - \sigma^2/2\right) \cdot T/252}{\sigma\sqrt{T/252}} = d_1 - \sigma\sqrt{T/252}$,
- c = European call price according to the Black and Scholes solution,
- S_0 = initial underlying asset price,
- R_f = annual risk-free interest rate,
- σ = annual asset volatility,

- T = option's maturity in working days (1 year equals 252 working days),
- K = exercise price of the option,
- $N(d_1)$ = value of the standard normal cumulative distribution function at point d_1,
- $N(d_2)$ = value of the standard normal cumulative distribution function at point d_2.

2.2 The Monte Carlo Simulation Model

A Monte Carlo simulation model is implemented to generate paths for the underlying asset price, and then to obtain estimates for the payoff of a European call. The average of the estimated payoffs is then calculated and brought to the present date value using the risk-free interest rate as the discount rate. In this study, the simulation prices along each path were generated in steps, defined by the number of dimensions used. As in the Black and Scholes model, we assumed that the underlying asset path of prices follows a Brownian geometric motion, defined by the differential stochastic equation:

$$\frac{dS}{S} = \mu dt + \sigma dW \quad , \tag{3}$$

where:

- dS = underlying asset price change during time interval dt,
- μ = asset return,
- σ = asset volatility,
- dW = Wiener process.

Rewriting Equation (3) in discrete time and adopting the risk neutrality assumption (asset return equals risk-free interest rate) and using Ito's Lemma, one obtains the following equation for the underlying asset price at time t (Hull 1999):

$$S_t = S_{t-1}\, e^{\left[(R_f - \sigma^2/2)*dt + \sigma*\sqrt{dt}*Z_t\right]} \quad , \tag{4}$$

where:

- S_t = underlying asset price in instant t,
- S_{t-1} = underlying asset price in instant $t-1$,
- dt = option's maturity (T) / number of dimensions (dim),
- Z_t = standard normal random variable in instant t.

In the empirical studies, each path was simulated up to the option's maturity date T at the 252nd day, based on Equation (4) and according to the number of dimensions (dim) chosen. The number of dimensions varied from 5 to 100. For example, when 15 dimensions were chosen, each path was simulated in 15 time steps. In each simulation run, n = 1000 paths were generated for the underlying asset price. The simulation experiment for each parameter combination comprised m = 40 simulation runs. In matrix representation, the experiment is described as follows:

For j = 1 to m runs:

$$j^{th}\ Random\ Matrix\ (Z^j) = \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,\dim} \\ \vdots & \ddots & \vdots \\ Z_{n,1} & \cdots & Z_{n,\dim} \end{bmatrix},$$

$$j^{th}\ Asset\ Price\ Matrix\ (S^j) = \begin{bmatrix} S_{1,1} & \cdots & S_{1,\dim} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \cdots & S_{n,\dim} \end{bmatrix},$$

$$j^{th}\ Payoffs\ Vector = \begin{bmatrix} Max\left[0; \left(S_{1,\dim} - K\right)\right] \\ \vdots \\ Max\left[0; \left(S_{n,\dim} - K\right)\right] \end{bmatrix},$$

$$j^{th}\ Payoffs\ PV\ Vector = \begin{bmatrix} Payoff_1 * \exp\left(-R_f * T/252\right) \\ \vdots \\ Payoff_n * \exp\left(-R_f * T/252\right) \end{bmatrix}.$$

The j^{th} call price estimate is the mean of the 1000 components of j^{th} Payoffs' PV (Present Value) Vector. The call price's final estimate is the mean of the 40 call price estimates. The standard error is given by the standard deviation of the 40 call price estimates.

Other simulation parameters, as used in the experiments, are presented in Table 1:

Table 1: Simulation Parameters Used in the Experiments.

S_0	Initial underlying asset price (at t=0)	\$100
R_f	Annual risk-free interest rate	5%
K	Exercise price	\$160, \$180, \$200
σ	Annual asset volatility	20%
dim	Number of dimensions	From 5 to 100 (increment of 5)
T	Option's maturity date	252 nd
N	Number of observations per run (number of generated paths per run)	1000
m	Number of runs	40

Each different K value above defined an out-of-the-money European call to be priced, with a theoretical exercise probability of 1.390% (K=160), 0.264% (K=180) and 0.046% (K=200).

2.3 Simple Random Sampling (SRS)

The SRS simulation was a straight implementation, based on the Inverse Transform Technique, generating random values for Z_t in Equation (4).

Variance Reduction Techniques as used in this paper are based on different sampling schemes.

2.4 Variance Reduction Techniques

2.4.1 Importance Sampling (IS)

When simulation observations are directly generated, as in the SRS case, many observations may fall into regions of no or little interest as, for example, a zero payoff. In the presence of relevant rare events, this may disrupt the estimate's precision.

When dealing with out-of-the-money options, few price paths with positive payoffs will be simulated, although such an option's price will be evaluated by combining both kinds of results: zero and nonzero payoffs. This unbalanced set of results leads to imprecise estimates. The IS's purpose is to restore this balance using a proper modification of the problem.

As such, IS usually changes the simulation problem parameters, but not the model, so that the option is not out-of-the-money anymore. This idea, as applied to option pricing, is described in Boyle, Broadie and Glasserman (1997). After the change, the usual IS approach is to continue using the standard SRS Monte Carlo simulation for the modified problem. In this work, a drift increase was applied by increasing the asset return rate, thus shifting the asset price distribution to the right. Therefore, instead of using random Z_t values from the standard normal distribution, Z'_t values were randomly drawn from a shifted normal distribution with mean μ and unitary standard deviation. At the end of the process, the simulated payoff was then adjusted to give proper answers to the original problem. This was achieved by multiplying each simulated result by the likelihood ratio, given by:

$$Ratio = e^{-0,5*\left[\sum_{i=1}^{\dim} Z'^{2}_{i} - \sum_{i=1}^{\dim}\left(Z'_{i} - \frac{\mu}{\sigma} * \sqrt{dt}\right)^{2}\right]}, \qquad (5)$$

where:

- $Z'_t \sim N(\mu,1)$,
- dim = problem dimensionality or time steps in price path.

2.4.2 Importance Sampling with Descriptive Sampling (IS + DS)

Instead of randomly drawing Zi values, this technique incorporates DS in the IS analysis, so that the Zd'_i values are deterministically chosen from the shifted normal distribution. Due to the selection procedure, input sample moments were fixed and very close to the respective theoretical values, thereby presenting no more variability between different runs.

The deterministic selection procedure consisted of stratifying the cumulative shifted normal distribution $N(\mu,1)$ into n parts of equal probability and using the median of each stratum. The selected n elements will compose the set of descriptive values, which will be randomly shuffled to produce a univariate descriptive sample. This method assures that all strata of the normal distribution $N(\mu,1)$ will be represented in the sample. In the multi-dimensional case, the set of descriptive values will be the same for each dimension or time step in the price path, but in a different random permutation.

Thus, the set of descriptive values (here identified as Zd_i, instead of Z_i), before shuffling, is given by:

$$Zd_i = F^{-1}\left(\frac{i-1+0.5}{n}\right) = F^{-1}\left[\frac{(i-0.5)}{n}\right], \qquad (6)$$

where:

- n = descriptive sample size,
- $i = 1,2,3..., n$,
- Zd_i = i[th] descriptive sample set value,
- F^{-1} = inverse transform of the input variable cumulative distribution; Inverse cumulative Normal in this study.

It is worth noticing that Descriptive Sampling and Quasi-Monte Carlo Methods are both based on a deterministic sample selection. However, unlike Quasi-Monte Carlo where sample sequences are also fixed, DS is based on a random permutation of the set values, thus resulting into different estimates for each simulation run.

3 RESULTS

Table 2 presents the simulated prices of the three out-of-the-money European calls considered in this paper, using IS in Monte Carlo simulation. Table 3 incorporates DS into the IS analysis. Various shift values (μ) were considered and four dimension levels (dim) were presented (5, 10, 20 and 100). The standard errors of the simulated prices are also presented. In Table 2, column $\mu = 0$ corresponds to Monte Carlo simulation using SRS, without any shift; in Table 3, it corresponds to the standard DS use, also without any shift. The tables also present the analytical prices of the three European calls according to the Black and Scholes solution.

Table 2: Estimated European Call Prices Using Standard Importance Sampling (IS+SRS), Standard Errors of the Estimates and the Black and Scholes Solution (B&S).

			μ															
Dim 5	**K**	**B&S**	**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1640	0.1674	0.1560	0.1592	0.1582	0.1592	0.1580	0.1592	0.1566	0.1549	0.1580	0.1626	0.1930	0.1143	0.1127	0.0355
	180	0.0286	0.0294	0.0313	0.0270	0.0291	0.0281	0.0285	0.0287	0.0286	0.0286	0.0287	0.0285	0.0283	0.0323	0.0298	0.0422	0.0205
	200	0.0048	0.0056	0.0054	0.0044	0.0049	0.0047	0.0048	0.0048	0.0048	0.0048	0.0049	0.0048	0.0048	0.0047	0.0047	0.0056	0.0056
S.E.	160	-	0.0573	0.0348	0.0151	0.0109	0.0059	0.0055	0.0052	0.0057	0.0080	0.0142	0.0275	0.0534	0.1393	0.2368	0.2590	0.1205
	180	-	0.0244	0.0117	0.0055	0.0028	0.0017	0.0013	0.0009	0.0009	0.0011	0.0016	0.0027	0.0050	0.0101	0.0234	0.0618	0.0654
	200	-	0.0105	0.0043	0.0019	0.0010	0.0004	0.0003	0.0002	0.0002	0.0002	0.0003	0.0003	0.0005	0.0009	0.0016	0.0041	0.0113
Dim 10			**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1693	0.1585	0.1567	0.1564	0.1601	0.1592	0.1591	0.1575	0.1597	0.1571	0.1540	0.1480	0.1805	0.1672	0.0556	0.0291
	180	0.0286	0.0311	0.0289	0.0284	0.0288	0.0288	0.0288	0.0289	0.0287	0.0289	0.0289	0.0288	0.0291	0.0306	0.0300	0.0270	0.0158
	200	0.0048	0.0038	0.0055	0.0049	0.0049	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0049	0.0048	0.0048	0.0052	0.0052	0.0025
S.E.	160	-	0.0681	0.0300	0.0116	0.0080	0.0065	0.0040	0.0051	0.0075	0.0082	0.0165	0.0274	0.0467	0.1465	0.2864	0.1104	0.1415
	180	-	0.0249	0.0113	0.0050	0.0026	0.0018	0.0013	0.0010	0.0012	0.0014	0.0021	0.0025	0.0054	0.0111	0.0270	0.0425	0.0724
	200	-	0.0070	0.0047	0.0019	0.0011	0.0004	0.0003	0.0002	0.0002	0.0002	0.0002	0.0003	0.0005	0.0009	0.0013	0.0033	0.0065
Dim 20			**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1626	0.1561	0.1585	0.1601	0.1575	0.1590	0.1587	0.1598	0.1607	0.1597	0.1530	0.1566	0.1345	0.1723	0.0723	0.1703
	180	0.0286	0.0306	0.0305	0.0278	0.0293	0.0279	0.0286	0.0286	0.0287	0.0287	0.0290	0.0283	0.0282	0.0285	0.0224	0.0317	0.0779
	200	0.0048	0.0053	0.0054	0.0044	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0047	0.0045	0.0089
S.E.	160	-	0.0544	0.0292	0.0178	0.0092	0.0075	0.0058	0.0051	0.0082	0.0111	0.0152	0.0270	0.0483	0.0908	0.3222	0.1712	0.5926
	180	-	0.0238	0.0111	0.0049	0.0032	0.0018	0.0013	0.0010	0.0011	0.0012	0.0018	0.0033	0.0044	0.0090	0.0167	0.0632	0.2368
	200	-	0.0090	0.0043	0.0015	0.0010	0.0005	0.0003	0.0002	0.0002	0.0002	0.0003	0.0003	0.0004	0.0009	0.0018	0.0044	0.0158
Dim 100			**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1483	0.1628	0.1620	0.1572	0.1594	0.1591	0.1594	0.1593	0.1600	0.1588	0.1636	0.1487	0.1316	0.1902	0.0946	0.0051
	180	0.0286	0.0244	0.0296	0.0295	0.0281	0.0282	0.0288	0.0288	0.0285	0.0285	0.0284	0.0288	0.0285	0.0272	0.0303	0.0331	0.0036
	200	0.0048	0.0030	0.0050	0.0048	0.0048	0.0047	0.0048	0.0048	0.0048	0.0048	0.0049	0.0048	0.0048	0.0047	0.0053	0.0043	0.0021
S.E.	160	-	0.0469	0.0219	0.0169	0.0085	0.0081	0.0054	0.0052	0.0054	0.0113	0.0124	0.0224	0.0410	0.1047	0.3582	0.2459	0.0108
	180	-	0.0225	0.0092	0.0051	0.0027	0.0017	0.0012	0.0013	0.0012	0.0014	0.0019	0.0028	0.0059	0.0102	0.0212	0.0666	0.0072
	200	-	0.0078	0.0035	0.0017	0.0008	0.0005	0.0003	0.0002	0.0002	0.0002	0.0002	0.0003	0.0005	0.0010	0.0020	0.0039	0.0036

Table 3: Estimated European Call Prices Using Importance Sampling with Descriptive Sampling (IS+DS), Standard Errors of the Estimates and the Black and Scholes Solution (B&S).

			μ															
Dim 5	**K**	**B&S**	**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1600	0.1646	0.1589	0.1596	0.1593	0.1593	0.1595	0.1584	0.1594	0.1615	0.1590	0.1555	0.1584	0.1456	0.7899	0.0071
	180	0.0286	0.0276	0.0311	0.0285	0.0285	0.0290	0.0283	0.0287	0.0285	0.0288	0.0286	0.0288	0.0291	0.0293	0.0286	0.0324	0.0049
	200	0.0048	0.0039	0.0060	0.0049	0.0047	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048	0.0047	0.0042	0.0048	0.0027
S.E.	160	-	0.0556	0.0240	0.0123	0.0074	0.0047	0.0047	0.0056	0.0051	0.0083	0.0131	0.0258	0.0556	0.1089	0.2476	3.1664	0.0218
	180	-	0.0226	0.0100	0.0057	0.0021	0.0017	0.0010	0.0009	0.0008	0.0013	0.0017	0.0025	0.0053	0.0101	0.0214	0.0523	0.0146
	200	-	0.0068	0.0038	0.0018	0.0007	0.0004	0.0003	0.0002	0.0002	0.0002	0.0002	0.0002	0.0005	0.0010	0.0015	0.0045	0.0073
Dim 10			**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1624	0.1591	0.1591	0.1600	0.1584	0.1577	0.1597	0.1585	0.1614	0.1555	0.1646	0.1602	0.1488	0.0895	0.0599	0.0778
	180	0.0286	0.0282	0.0277	0.0293	0.0283	0.0286	0.0284	0.0288	0.0288	0.0288	0.0285	0.0287	0.0261	0.0291	0.0267	0.0277	0.0409
	200	0.0048	0.0040	0.0039	0.0052	0.0046	0.0048	0.0048	0.0048	0.0048	0.0048	0.0049	0.0048	0.0048	0.0048	0.0047	0.0048	0.0059
S.E.	160	-	0.0662	0.0268	0.0140	0.0069	0.0054	0.0036	0.0048	0.0055	0.0069	0.0119	0.0278	0.0574	0.0979	0.1190	0.1327	0.3515
	180	-	0.0279	0.0104	0.0042	0.0022	0.0011	0.0010	0.0011	0.0008	0.0013	0.0018	0.0029	0.0048	0.0098	0.0168	0.0501	0.1740
	200	-	0.0127	0.0031	0.0017	0.0009	0.0004	0.0002	0.0002	0.0002	0.0002	0.0002	0.0003	0.0004	0.0008	0.0019	0.0038	0.0131
Dim 20			**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1462	0.1549	0.1619	0.1586	0.1587	0.1591	0.1587	0.1595	0.1605	0.1601	0.1598	0.1532	0.1749	0.1856	0.7557	0.0405
	180	0.0286	0.0248	0.0271	0.0306	0.0281	0.0288	0.0289	0.0288	0.0289	0.0288	0.0287	0.0281	0.0276	0.0272	0.0268	0.0312	0.0250
	200	0.0048	0.0029	0.0048	0.0050	0.0045	0.0048	0.0048	0.0048	0.0049	0.0047	0.0048	0.0048	0.0049	0.0046	0.0048	0.0055	0.0096
S.E.	160	-	0.0555	0.0252	0.0112	0.0064	0.0048	0.0050	0.0063	0.0060	0.0079	0.0147	0.0268	0.0578	0.1770	0.3881	2.3157	0.0881
	180	-	0.0194	0.0109	0.0047	0.0021	0.0014	0.0011	0.0010	0.0011	0.0010	0.0014	0.0026	0.0046	0.0092	0.0196	0.0569	0.0516
	200	-	0.0057	0.0048	0.0020	0.0008	0.0004	0.0002	0.0002	0.0002	0.0002	0.0002	0.0003	0.0005	0.0009	0.0018	0.0043	0.0158
Dim 100			**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**	**1.50**
Mean	160	0.1590	0.1582	0.1578	0.1593	0.1585	0.1583	0.1581	0.1588	0.1604	0.1590	0.1587	0.1618	0.1640	0.1788	0.1230	0.0808	0.0320
	180	0.0286	0.0279	0.0271	0.0284	0.0291	0.0287	0.0284	0.0287	0.0289	0.0288	0.0289	0.0281	0.0280	0.0287	0.0258	0.0317	0.0197
	200	0.0048	0.0047	0.0037	0.0046	0.0049	0.0048	0.0048	0.0048	0.0048	0.0047	0.0048	0.0049	0.0047	0.0048	0.0048	0.0042	0.0073
S.E.	160	-	0.0604	0.0231	0.0144	0.0075	0.0046	0.0041	0.0054	0.0057	0.0067	0.0115	0.0234	0.0529	0.1360	0.2700	0.2592	0.1064
	180	-	0.0243	0.0089	0.0055	0.0023	0.0014	0.0012	0.0011	0.0009	0.0012	0.0014	0.0024	0.0047	0.0086	0.0169	0.0795	0.0624
	200	-	0.0076	0.0026	0.0015	0.0009	0.0004	0.0003	0.0002	0.0002	0.0002	0.0002	0.0002	0.0005	0.0009	0.0016	0.0046	0.0185

One can observe that, as expected and required, the simulated call prices were in close agreement with their corresponding analytical prices, no matter the shift μ value. Both Importance Sampling variations (IS+SRS and IS+DS), with an adequate choice for the shift μ value, were also very efficient Variance Reduction Techniques. The more the call was out-of-the-money (or equivalently, the higher its exercise price, K), the higher was the standard error reduction.

For both IS variations, the calibration issue regarding the best shift value is present; an empirical approach is suggested. Figures 1 to 4 show the RMSE relative variation to the standard SRS, based on different μ values, here ranging up to μ = 1.20. Each figure refers to a particular dimensionality (5, 10, 20 and 100) and displays the RMSE relative variation for the three calls being studied (K=160, 180 and 200).

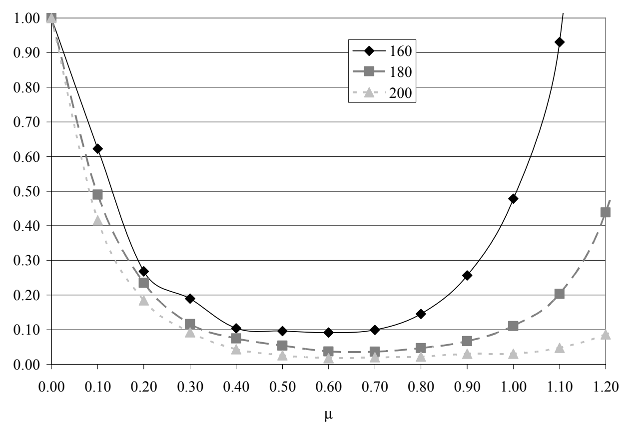

Figure 1: Importance Sampling RMSE Relative Variation with the Shift μ, for the 3 European Calls (Dimension = 5).

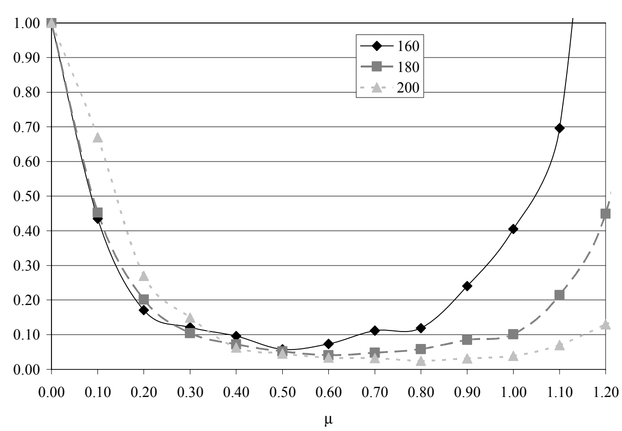

Figure 2: Importance Sampling RMSE Relative Variation with the Shift μ, for the 3 European Calls (Dimension = 10).

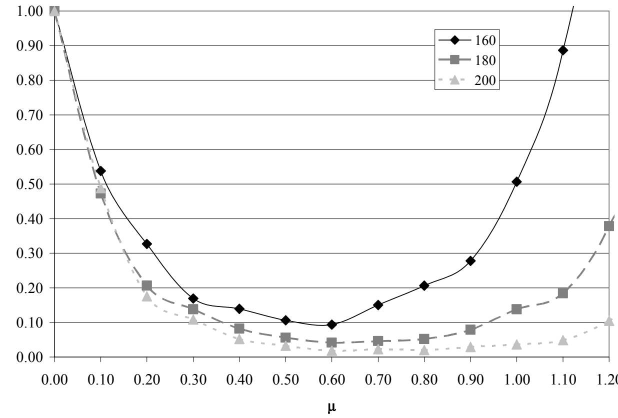

Figure 3: Importance Sampling RMSE Relative Variation with the Shift μ, for the 3 European Calls (Dimension = 20).

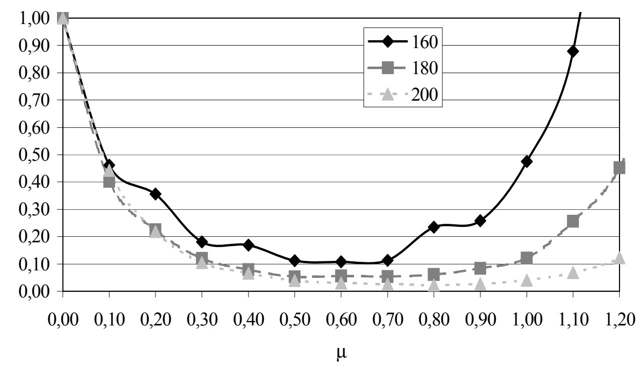

Figure 4: Importance Sampling RMSE Relative Variation with the Shift μ, for the 3 European Calls (Dimension = 100).

As shown, one may observe that, no matter the particular K value (160, 180 or 200), there are substantial gains from the use of Importance Sampling instead of Simple Random Sampling. It may also be observed that such gains are higher as the option becomes deeper out-of-the-money as K increases. Finally, as K increases, the optimum shift μ-value also increases, which can be explained by the need to keep the exercise probability of the transformed shifted option at a much higher level, usually somewhere around 70%. Concerning problem dimensionality, it seems that the number of points in the path price did not affect the above findings.

Although the IS benefit is noteworthy, DS improvements over the standard IS implementation were only marginal. Further results are needed to better evaluate the gains from the IS+DS combination and to better understand the case, but with foreknowledge that such gains are likely to be irrelevant in practical terms.

4 CONCLUSIONS

Although the use of variance reduction techniques in Monte Carlo option pricing is a common practice, the benefits from the joint use of such techniques is not well explored, in particular of IS and DS. In such a context, this paper presents some innovative results:

- as expected, it was advantageous to use IS as a variance reduction technique to price out-of-the-money European calls;
- the higher the exercise price considered, i.e. the lower the probability that the call would be exercised, the higher the gain provided by IS;
- the dimensionality of the simulation problem did not affect the gains achieved with IS;
- on the other hand, the combined use of IS + DS only produced marginal gains over the standard IS implementation. One possible reason for such a result, yet to be confirmed, is that IS also imposes a control over the input sample values, which is the purpose of DS.

These conclusions are likely to be extendable to other options, especially the ones that are difficult to price. Forthcoming steps from this research could be towards this generalization, in particular, the study of exotic options, such as Asian and barrier options.

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