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OVERLAPPING BATCH MEANS: SOMETHING MORE FOR NOTHING?

Christos Alexopoulos
David Goldsman

H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0205, USA

James R. Wilson

Edward P. Fitts Department of Industrial and Systems Engineering
North Carolina State University
Raleigh, NC 27695-7906, USA

ABSTRACT

Output analysis methods that provide reliable point and confidence-interval estimators for system performance characteristics are critical elements of any modern simulation project. Remarkable advances in simulation output analysis have been achieved over the last thirty years, in part owing to the application of data-reuse techniques designed to improve estimator accuracy and efficiency. Many of the key insights regarding data reuse are given in the seminal 1984 Winter Simulation Conference paper by Meketon and Schmeiser that is titled "Overlapping Batch Means: Something for Nothing?" and that introduced the method of overlapping batch means (OBM). We trace the development of OBM from the original work of Meketon and Schmeiser, and we discuss some recent extensions of the method.

1 INTRODUCTION

Stochastic simulations, such as those arising in queueing and manufacturing applications, typically generate output processes that have both deterministic and stochastic components exhibiting highly anomalous behavior. In such cases, successive simulation-generated observations are rarely independent, identically distributed, or normal, thus making the use of classical statistical analysis methods questionable. A number of output analysis technologies have been developed in the literature that attempt to mitigate these difficulties. In terms of output analysis for *steady-state* simulation, Conway, Johnson, and Maxwell (1959) and Conway (1963) are the first papers to address the problem via the method of nonoverlapping batch means (NBM). Schmeiser (1982) makes a major contribution to the theory of NBM, as discussed by Nelson (2011) elsewhere in this *Proceedings*. Other important output analysis methods along the way have included regenerative analysis (Crane and Iglehart 1975, Crane and Lemoine 1977), standardized time series (STS) (Schruben 1983), spectral-based methods (Fishman and Kiviat 1967), and overlapping batch means (OBM) (Meketon and Schmeiser 1984), the latter of which motivates this paper.

The archetypal steady-state simulation output analysis problem is that of estimating the mean μ of a discrete-time process $\{Y_1, Y_2, \ldots\}$, where, for example, Y_i could denote the waiting time of the ith customer in a stationary queueing system. The obvious estimator for μ is of course the sample mean, $\overline{Y}_n \equiv n^{-1} \sum_{i=1}^{n} Y_i$, which is based on a sample of n consecutive observations. In keeping with proper statistical analysis, one often provides a corresponding measure of the sample mean's precision by calculating an estimator for $\sigma_n^2 \equiv n\mathrm{Var}(\overline{Y}_n)$ or for what is known as the variance parameter, $\sigma^2 \equiv \lim_{n\to\infty} \sigma_n^2$. These quantities are of direct interest in and of themselves, and they are useful in constructing confidence intervals (CIs) for μ. In fact, the objective of a major line of research in simulation output analysis has for years been to develop improved estimators for σ^2 with, for example, lower variance, lower bias, and lower mean squared error (MSE).

Assuming that the simulation has been warmed up adequately so that it is indeed in steady-state operation, we will henceforth regard the observed time series $\{Y_i : i = 1, 2, \ldots, n\}$ as a realization of a stationary process. We are motivated to perform a single run whose length n is sufficiently large to ensure acceptable precision in the estimation of both μ and σ^2. In particular, a single long run is typically preferable to multiple independent replications with a proportionately smaller common run length, primarily because of the difficulty and inefficiency of dealing with a potential warm-up period within each replication (Alexopoulos and Goldsman 2004).

The NBM, OBM, and STS methods are usually carried out as single-run strategies. A common technique shared by these single-run methodologies is obviously that of *batching* the observations — instead of considering the entire simulation time series $\{Y_i : i = 1, 2, \ldots, n\}$ in one fell swoop (that is, performing the relevant analysis using all the observations in the data set at the same time), we break up this data set into smaller subseries (batches) that are composed of consecutive observations; these batches may be disjoint or overlapping, depending on the analysis method. We then perform the appropriate analysis on each batch separately. For instance, when applying the NBM method, as discussed in Section 2, we split the observations into adjacent nonoverlapping batches; then we assume that the resulting sample (batch) means computed from each batch are approximately independent and identically distributed (i.i.d.) normal random variables; and finally we apply "standard" variance-estimation techniques to the batch means. In the STS method, batched estimators also use adjacent nonoverlapping batches. The idea is to compute a separate STS estimator from each batch, assume the resulting STS estimators are i.i.d., and then average those estimators.

As explained in Section 3, the OBM method works with *overlapping* batches, under the full realization that the associated overlapping batch means are *not* independent, but they are identically distributed and asymptotically normal with increasing batch size. We then compute an estimator of σ^2 based on the sample variance of the overlapping batch means. This seemingly problematic technique exploits key results from the theory of spectral analysis to yield an estimator of the variance parameter σ^2 that is provably superior to the NBM variance estimator, at least asymptotically for certain types of serially correlated simulation output processes. This is the key insight of Meketon and Schmeiser (1984).

Section 4 discusses variance estimators based on STS applied to overlapping batches, which is a natural extension of the OBM method. In addition to formulating the first two moments of the overlapping STS variance estimators, we give approximations to the asymptotic distribution of these variance estimators as the batch size increases; and we exploit these approximations to construct asymptotically valid CIs for both μ and σ^2. For more details on the results presented in this paper, see Aktaran-Kalaycı et al. (2009) and Alexopoulos et al. (2007a, 2007b). The slides for the oral presentation of this article are available online via www.ise.ncsu.edu/jwilson/files/wsc11obm.pdf [accessed October 25, 2011].

2 NONOVERLAPPING ESTIMATORS

In this section, we will work with b contiguous, nonoverlapping batches of observations, each of length (batch size) m, from the simulation-generated time series $\{Y_j : j = 1, 2, \ldots, n\}$ of length n, where we assume that $n = bm$. In particular, the observations $\{Y_{(i-1)m+k} : k = 1, 2, \ldots, m\}$ constitute the ith nonoverlapping batch for $i = 1, 2, \ldots, b$. For the remainder of this article, we take $b \equiv n/m$ so that b always represents the ratio of the sample size to the batch size; and when we work with nonoverlapping batches, b also equals the number of batches.

2.1 Nonoverlapping Batch Means Estimator

The quantities $\overline{Y}_{i,m} \equiv m^{-1}\sum_{k=1}^{m} Y_{(i-1)m+k}$, $i = 1,2,\ldots,b$, are the nonoverlapping batch means, and are often assumed to be i.i.d. normal random variables, at least for large enough values of the batch size m. The i.i.d. normality assumption suggests that for fixed b we consider the NBM estimator for σ^2,

$$\mathscr{N}(b,m) \;\equiv\; \frac{m}{b-1}\sum_{i=1}^{b}(\overline{Y}_{i,m}-\overline{Y}_n)^2 \;\Rightarrow\; \frac{\sigma^2\chi^2_{b-1}}{b-1},$$

where χ^2_ν denotes a chi-squared random variable with ν degrees of freedom; and the symbol $\Rightarrow$ denotes convergence in distribution as the batch size $m \to \infty$ (Glynn and Whitt 1991, Steiger and Wilson 2001). The statistic $\mathscr{N}(b,m)$ is one of the most popular estimators for σ^2, and it serves as a benchmark for comparison with the other estimators discussed in this article. Under mild conditions, Chien, Goldsman, and Melamed (1997), Goldsman and Meketon (1986), and Song and Schmeiser (1995) show that the expected value of the NBM estimator converges to σ^2 moderately quickly,

$$\mathrm{E}[\mathscr{N}(b,m)] \;=\; \sigma^2 + \frac{\gamma(b+1)}{bm} + O(1/m^2), \tag{1}$$

where we use standard "big-Oh" notation to denote a term that goes to zero quickly, and the parameter $\gamma \equiv -2\sum_{k=1}^{\infty} k\mathrm{Cov}[Y_1,Y_{1+k}]$ is determined by the correlation structure of the underlying stationary stochastic process. Thus, we see that the bias of $\mathscr{N}(b,m)$ as an estimator of σ^2 is approximately $\gamma(b+1)/(bm)$. As for the NBM estimator's variance, Glynn and Whitt (1991) (among others) find that, for fixed b,

$$\lim_{m\to\infty}(b-1)\,\mathrm{Var}[\mathscr{N}(b,m)] \;=\; 2\sigma^4.$$

The question that now arises is: Can we find other estimators for σ^2 that have better bias or variance properties than those of $\mathscr{N}(b,m)$? The answer is in the affirmative. We now discuss alternative standardized time series estimators based on nonoverlapping batches; and in the following two sections, we finally present overlapping versions of these estimators.

2.2 Nonoverlapping Standardized Time Series Estimators

The *standardized time series* based on nonoverlapping batch i of size m is (Schruben 1983)

$$T_{i,m}(t) \;\equiv\; \frac{\lfloor mt\rfloor(\overline{Y}_{i,m}-\overline{Y}_{i,\lfloor mt\rfloor})}{\sigma\sqrt{m}} \quad \text{for } t\in[0,1],$$

where $\lfloor\cdot\rfloor$ denotes the floor function and $\overline{Y}_{i,j} \equiv j^{-1}\sum_{k=1}^{j} Y_{(i-1)m+k}$ denotes the jth cumulative sample mean from batch i for $i = 1,2,\ldots,b$ and $j = 1,2,\ldots,m$. Under a mild functional central limit theorem assumption, described, e.g., in Alexopoulos et al. (2007b), it can be shown that

$$\big[T_{1,m}(\cdot),\ldots,T_{b,m}(\cdot)\big] \;\Rightarrow\; \big[\mathscr{B}_0(\cdot),\ldots,\mathscr{B}_{b-1}(\cdot)\big],$$

where $\mathscr{B}_0(\cdot),\ldots,\mathscr{B}_{b-1}(\cdot)$ are independent standard Brownian bridge processes on $[0,1]$. In particular, we define

$$\mathscr{B}_s(t) \;\equiv\; t[\mathscr{W}(s+1)-\mathscr{W}(s)] - [\mathscr{W}(s+t)-\mathscr{W}(s)] \quad \text{for } t\in[0,1] \text{ and } s\in[0,b-1],$$

where $\mathscr{W}(\cdot)$ is itself a standard Brownian motion process.

We let $A_i(f;m)$ denote the area estimator computed exclusively from nonoverlapping batch i, and we let $A_i(f)$ denote the associated limiting functional as the batch size $m\to\infty$ so that we have

$$A_i(f;m) \;\equiv\; \left[\frac{1}{m}\sum_{k=1}^{m} f(k/m)\sigma T_{i,m}(k/m)\right]^2 \quad \text{and} \quad A_i(f) \;\equiv\; \left[\int_0^1 f(t)\sigma\mathscr{B}_{i-1}(t)\,dt\right]^2,$$

Table 1: Approximate asymptotic bias and variance for different estimators.

Nonoverlapping Estimators	(m/γ) Bias	(b/σ^4) Var	Overlapping Estimators	(m/γ) Bias	(b/σ^4) Var
$\mathscr{N}(b,m)$	1	2	$\mathscr{O}(b,m)$	1	1.333
$\mathscr{A}(f;b,m)$	Eq. (3)	2	$\mathscr{A}^{\mathrm{O}}(f;b,m)$	Eq. (7)	Eq. (8)
$\mathscr{A}(f_0;b,m)$	3	2	$\mathscr{A}^{\mathrm{O}}(f_0;b,m)$	3	0.686
$\mathscr{A}(f_2;b,m)$	$o(1)$	2	$\mathscr{A}^{\mathrm{O}}(f_2;b,m)$	$o(1)$	0.819
$\mathscr{A}(f_{\cos,j};b,m)$	$o(1)$	2	$\mathscr{A}^{\mathrm{O}}(f_{\cos,j};b,m)$	$o(1)$	$(8\pi^2 j^2+15)/(12\pi^2 j^2)$
			$\mathscr{A}^{\mathrm{O}}(f_{\cos,1};b,m)$	$o(1)$	0.793

respectively, for $i=1,2,\ldots,b$, where the weight function $f(\cdot)$ satisfies the conditions

$$\mathrm{E}[A_i(f)] = \sigma^2 \quad \text{and} \quad \tfrac{d^2}{dt^2} f(t) \text{ is continuous at every } t \in [0,1]. \tag{2}$$

Clearly, $A_i(f;m)$ is the weighted area under the STS of that batch; and it is easily shown in the cited references that $A_1(f),\ldots,A_b(f)$ are i.i.d. $\sigma^2\chi_1^2$ random variables.

Equation (2) immediately suggests the *batched area* estimator for σ^2, which is simply the average of these estimators taken over all the nonoverlapping batches,

$$\mathscr{A}(f;b,m) \equiv \frac{1}{b}\sum_{i=1}^{b} A_i(f;m),$$

which, owing to the independence of $A_1(f),\ldots,A_b(f)$, converges to a $\sigma^2\chi_b^2/b$ random variable as $m\to\infty$. Further, Goldsman, Meketon, and Schruben (1990) show that

$$\mathrm{E}[\mathscr{A}(f;b,m)] = \sigma^2 + \frac{[(F-\overline{F})^2+\overline{F}^2]\gamma}{2m} + O(1/m^2), \tag{3}$$

where

$$F(s) \equiv \int_0^s f(t)\,dt \text{ for } s\in[0,1],\ F\equiv F(1),\ \overline{F}(u)\equiv\int_0^u F(s)\,ds \text{ for } u\in[0,1], \text{ and } \overline{F}\equiv\overline{F}(1).$$

Further, under mild conditions, we have

$$\lim_{m\to\infty} b\,\mathrm{Var}[\mathscr{A}(f;b,m)] = 2\sigma^4.$$

Example 1 Schruben (1983) studied the area estimator with constant weight function $f_0(t)\equiv\sqrt{12}$ for all $t\in[0,1]$; in this case, Equation (3) implies that $\mathrm{E}[\mathscr{A}(f_0;b,m)] = \sigma^2+3\gamma/m+O(1/m^2)$. If we choose a weight function for which $F=\overline{F}=0$, then the resulting estimator is *first-order unbiased* for σ^2; i.e., $\mathscr{A}(f;b,m)$ has bias of the form $O(1/m^2)$. An example of such a weight function is $f_2(t)\equiv\sqrt{840}\big(3t^2-3t+1/2\big)$ (Goldsman, Meketon, and Schruben 1990). Other weight functions that yield first-order unbiased estimators for σ^2 are given by the family $\{f_{\cos,j}(t)\equiv\sqrt{8}\pi j\cos(2\pi jt) : j=1,2,\ldots\}$. Foley and Goldsman (1999) show that this sequence of weights produces area estimators $\{\mathscr{A}(f_{\cos,j};b,m) : j=1,2,\ldots\}$ that are not only first-order unbiased, but also asymptotically i.i.d. $\sigma^2\chi_b^2/b$. For comparison purposes, see the nonoverlapping results on the left-hand side of Table 1, which summarizes bias and variance properties for many of the estimators studied herein.

Remark 1 Goldsman, Kang, and Seila (1999) study estimators for σ^2 based on nonoverlapping Cramér–von Mises (CvM) functionals of standardized time series. It turns out that it is possible to construct first-order unbiased CvM estimators having lower variance than typical nonoverlapping area estimators. Yet for ease of exposition, we simply refer the reader to the cited references for more details.

3 OVERLAPPING BATCH MEANS ESTIMATOR — SOMETHING FOR NOTHING

Given the output of a single run of a steady-state simulation, one might seek to reuse that data set effectively so as to calculate an improved estimator of σ^2. This is the fundamental insight of Meketon and Schmeiser (1984), an article that has garnered 191 citations in Google Scholar as of July 9, 2011. Among all the articles published in the *Proceedings of the Winter Simulation Conference* since 1968, Meketon and Schmeiser (1984) was one of ten articles to be recognized in 2007 with WSC's Fortieth Anniversary Landmark Paper Award.

To elaborate the fundamental insight of Meketon and Schmeiser (1984), we now consider the use of estimators based on *overlapping* batches. Here we form $n-m+1$ overlapping batches, each of size m, from the time series $\{Y_j : j = 1,2,\ldots,n\}$. In particular, the observations $\{Y_{i+k} : k = 0,\ldots,m-1\}$ constitute the ith overlapping batch for $i = 1,\ldots,n-m+1$. Recall that we continue to use $b \equiv n/m$ as before. We define the ith overlapping batch mean as $\overline{Y}^{\mathrm{O}}_{i,m} \equiv m^{-1}\sum_{k=0}^{m-1} Y_{i+k}$, for $i = 1,2,\ldots,n-m+1$; clearly, these overlapping batch means are strongly correlated, in contrast to the asymptotically independent nonoverlapping batch means. In any case, the OBM estimator for σ^2, originally studied by Meketon and Schmeiser (1984) (with a slightly different scaling constant), is

$$\mathscr{O}(b,m) \equiv \frac{nm}{(n-m+1)(n-m)} \sum_{i=1}^{n-m+1} (\overline{Y}^{\mathrm{O}}_{i,m} - \overline{Y}_n)^2.$$

Under mild conditions, it can be shown that

$$\mathrm{E}[\mathscr{O}(b,m)] = \sigma^2 + \frac{\gamma(b^2+1)}{mb(b-1)} + O(1/m^2), \tag{4}$$

which is a close match with the corresponding NBM result given by Equation (1). For simple derivations of (4), see Alexopoulos et al. (2007b), Goldsman and Meketon (1986), and Song and Schmeiser (1995). As for the OBM estimator's variance, Meketon and Schmeiser (1984) found that for sufficiently large batch size m and sample-to-batch-size ratio b,

$$\frac{\mathrm{Var}[\mathscr{O}(b,m)]}{\mathrm{Var}[\mathscr{N}(b,m)]} \approx 2/3. \tag{5}$$

In other words, the OBM estimator has about the same bias as, but only 2/3 the variance of, the NBM estimator! Equation (5) is perhaps the key result of Meketon and Schmeiser (1984), and it is of course the driving force for several follow-up papers in the literature. For instance, Damerdji (1995) derives the sharp result

$$\lim_{m\to\infty} \mathrm{Var}[\mathscr{O}(b,m)] = \frac{(4b^3-11b^2+4b+6)\sigma^4}{3(b-1)^4} \sim \frac{4\sigma^4}{3b} \quad \text{as } b \to \infty.$$

Using Equations (4) and (5), one can show that for a sufficiently large sample size n, the batch size that minimizes the mean squared error $\mathrm{MSE}[\mathscr{O}(b,m)] \equiv \mathrm{Bias}^2[\mathscr{O}(b,m)] + \mathrm{Var}[\mathscr{O}(b,m)]$ is given by

$$m^* = \left(\frac{3\gamma^2 n}{2\sigma^4}\right)^{1/3}. \tag{6}$$

Song (1996) developed methods for estimating the ratio γ^2/σ^4 for a variety of processes, including moving average processes and autoregressive processes. Then one can obtain an estimator for m^* by plugging the estimator of γ^2/σ^4 into Equation (6). Sherman (1995) proposed a method that does not rely on the estimation of γ^2/σ^4.

A second major feature of Meketon and Schmeiser (1984) is that this is one of the very first articles to account carefully for the computational requirements of estimators for the steady-state variance parameter

σ^2. After all, a procedure that requires $O(n^2)$ calculations to deliver an estimate of σ^2 is useless in any typical large-sample application. The good news is that the Meketon and Schmeiser (1984) provide an efficient algorithm to calculate the OBM estimator, essentially in $O(n)$ effort. To recapitulate: OBM produces estimators with reasonable bias, yet significantly smaller variance than NBM, while taking about the same computational effort; and numerous subsequent analytical and Monte Carlo studies have borne out OBM's efficacy. Thus, OBM literally gives us *something for nothing* — a pithy description of OBM's main advantages!

Since Meketon and Schmeiser (1984) appeared, there has been significant progress in the study of overlapping estimators. Welch (1987) relates OBM to certain spectral estimators and looks into the effects of partial overlapping. Goldsman and Meketon (1986) and Song and Schmeiser (1993, 1995) derive bias and variance properties of OBM estimators, among others. Song and Schmeiser (1993) also give additional insight by plotting the coefficients of the estimators' quadratic-form representations; included in their presentation are the OBM estimators as well as overlapped versions of the STS area estimators, which we shall describe next. Pedrosa and Schmeiser (1993, 1994) establish covariance properties between OBM estimators and then propose a batch-size determination algorithm. In a terrific series of papers, Damerdji (1991, 1994, 1995) establishes consistency results (both in the strong and mean-square senses) for a variety of variance estimators, including OBM and an overlapping version of a certain STS estimator. In the spirit of Welch (1987), Damerdji also establishes a formal linkage between the spectral method and simulation analysis methods based on overlapping batches.

4 OTHER OVERLAPPING ESTIMATORS — SOMETHING MORE FOR NOTHING

The discussion in Section 3 raises an interesting question: what happens if we apply the technique of overlapping batches to other estimators, say STS estimators? The results are surprisingly good — as in the case of OBM, the STS estimators computed from overlapping batches possess the same bias as, but asymptotically substantially smaller variance than, their counterparts computed from nonoverlapping batches; and as a bonus, certain overlapping STS variance estimators outperform the original OBM estimator.

4.1 Moments of Overlapping STS Area Estimators

In parallel to our discussion in Section 2.2, we define the STS from overlapping batch i as

$$T^{\mathrm{O}}_{i,m}(t) \equiv \frac{\lfloor mt \rfloor (\overline{Y}^{\mathrm{O}}_{i,m} - \overline{Y}^{\mathrm{O}}_{i,\lfloor mt \rfloor})}{\sigma\sqrt{m}} \quad \text{for } t \in [0,1] \text{ and } i = 1,2,\ldots,n-m+1,$$

where

$$\overline{Y}^{\mathrm{O}}_{i,j} \equiv \frac{1}{j}\sum_{k=0}^{j-1} Y_{i+k}, \quad \text{for } i = 1,2,\ldots,n-m+1 \text{ and } j = 1,2,\ldots,m.$$

Again under the mild functional central limit theorem assumption as described in Alexopoulos et al. (2007b), we have

$$T^{\mathrm{O}}_{\lfloor sm \rfloor,m}(\cdot) \Rightarrow \mathscr{B}_s(\cdot) \quad \text{for fixed } s \in [0,b-1].$$

We define the area estimator computed exclusively from overlapping batch i by

$$A^{\mathrm{O}}_i(f;m) \equiv \left[\frac{1}{m}\sum_{k=1}^{m} f(k/m)\sigma T^{\mathrm{O}}_{i,m}(k/m)\right]^2 \quad \text{for } i = 1,2,\ldots,n-m+1.$$

The *overlapping area* estimator for σ^2 is then given by the average of the area estimators taken over all of the overlapping batches,

$$\mathscr{A}^{\mathrm{O}}(f;b,m) \equiv \frac{1}{n-m+1}\sum_{i=1}^{n-m+1} A^{\mathrm{O}}_i(f;m).$$

One can then show that as $m \to \infty$,

$$\mathscr{A}^{\mathrm{O}}(f;b,m) \Rightarrow \mathscr{A}^{\mathrm{O}}(f;b) \equiv \frac{1}{b-1}\int_0^{b-1}\left[\sigma\int_0^1 f(u)\mathscr{B}_s(u)\,du\right]^2 ds,$$

along with the obvious result

$$\mathrm{E}[\mathscr{A}^{\mathrm{O}}(f;b,m)] = \sigma^2 + \frac{[(F-\overline{F})^2+\overline{F}^2]\gamma}{2m} + O(1/m^2), \tag{7}$$

which follows in light of Equation (3) and the fact that, for fixed batch size m and weight function $f(\cdot)$, the area estimators from all of the batches have the same expected value.

So far so good — the expected values of the nonoverlapping and overlapping versions of the area estimator match up. The discussion finally gets interesting when we consider the variance of the overlapping area estimator. First of all, Alexopoulos et al. (2007b) show that, under mild conditions,

$$\mathrm{Var}[\mathscr{A}^{\mathrm{O}}(f;b,m)] \to \mathrm{Var}[\mathscr{A}^{\mathrm{O}}(f;b)], \quad \text{as } m \to \infty.$$

But how do we calculate this asymptotic variance? In terms of the functions $F(\cdot)$ and $\overline{F}(\cdot)$ defined in the unnumbered display immediately below Equation (3), we formulate the auxiliary function

$$p(y) \equiv \overline{F}(1)\big[\overline{F}(y)-\overline{F}(1-y)-\overline{F}(1)y\big] + \int_0^{1-y} F(u)F(y+u)\,du \quad \text{for } y \in [0,1].$$

The first-order unbiased weight functions $f_2(\cdot)$ and $f_{\cos,j}(\cdot)$ satisfy the condition $\overline{F}(1)=0$, making the calculation of $p(y)$ for those weights particularly easy. In any case, for any weight $f(\cdot)$ satisfying (2), and fixed $b \geq 2$, a corvée (i.e., an awful lot) of algebra yields

$$\mathrm{Var}[\mathscr{A}^{\mathrm{O}}(f;b)] = \frac{4\sigma^4}{(b-1)^2}\int_0^1 (b-1-y)p^2(y)\,dy. \tag{8}$$

Contrary to our findings in Section 2.2, where we did not use overlapping batches, some examples show that the variance of the overlapping area estimator *does* depend on the choice of weight function. In particular, as revealed in Table 1, the variances of the overlapping versions of the area estimator are significantly lower than the corresponding nonoverlapping versions — and are even a great deal lower than the variance of the OBM estimator — at least for those weight functions studied here. Although not addressed in the current paper, the same type of improvement is also achieved with the overlapped versions of the CvM estimators for σ^2. After noting that Alexopoulos et al. (2007a) present an $O(n)$ algorithm to calculate these overlapping estimators, we see that we have achieved something more for nothing.

4.2 Density Estimation for Overlapping Variance Estimators

One can apply the technique of Satterthwaite (1941), as detailed in Alexopoulos et al. (2007b), to obtain an approximation to the distribution of $\mathscr{V}^{\mathrm{O}}(b,m)$, a "generic" overlapping variance estimator, using the following approach:

$$\mathscr{V}^{\mathrm{O}}(b,m) \mathrel{\dot\sim} \mathrm{E}[\mathscr{V}^{\mathrm{O}}(b,m)]\chi^2_{\nu_{\mathrm{eff}}}/\nu_{\mathrm{eff}}, \quad \text{where} \quad \nu_{\mathrm{eff}} = \left[\!\!\left[\frac{2\mathrm{E}^2[\mathscr{V}^{\mathrm{O}}(b,m)]}{\mathrm{Var}[\mathscr{V}^{\mathrm{O}}(b,m)]}\right]\!\!\right], \tag{9}$$

the symbol $\dot\sim$ means "is approximately distributed as," and $[\![z]\!]$ denotes rounding of the real number z towards the nearest integer. The quantity ν_{eff} is called the "effective" degrees of freedom for the overlapping variance estimator $\mathscr{V}^{\mathrm{O}}(b,m)$.

Figure 1 (taken from Alexopoulos et al. 2007b) illustrates the accuracy of the approximation (9) when the particular STS variance estimators $\mathscr{A}^{\mathrm{O}}(f_0;20,1000)$ and $\mathscr{A}^{\mathrm{O}}(f_2;20,1000)$ are applied to a stationary first-order autoregressive (AR(1)) process with autoregressive parameter $\phi = 0.9$, steady-state mean $\mu = 0$, and variance parameter $\sigma^2 = 19$. We generated 1,000,000 i.i.d. sample paths of this test process, and each sample path contained $n = 20{,}000$ observations organized into overlapping batches of size $m = 1{,}000$ so that $b = 20$. Two estimates of the probability density function (p.d.f.) of $\mathscr{A}^{\mathrm{O}}(f_0;20,1000)$ are plotted in the left-hand panel of Figure 1: (i) the dashed red line shows the fitted p.d.f. based on Equation (9), which yielded a scaled chi-squared distribution with $\nu_{\mathrm{eff}} = 53$ degrees of freedom; and (ii) the solid blue line shows the associated frequency polygon based on our random sample of size 1,000,000 from the distribution of $\mathscr{A}^{\mathrm{O}}(f_0;20,1000)$. The right-hand panel of Figure 1 shows the corresponding approximations to the distribution of $\mathscr{A}^{\mathrm{O}}(f_2;20,1000)$; and for this estimator, Equation (9) yielded a scaled chi-squared distribution with $\nu_{\mathrm{eff}} = 47$ degrees of freedom. In all our experimental work, we have obtained excellent approximations to the distributions of overlapping variance estimators based on Equation (9).

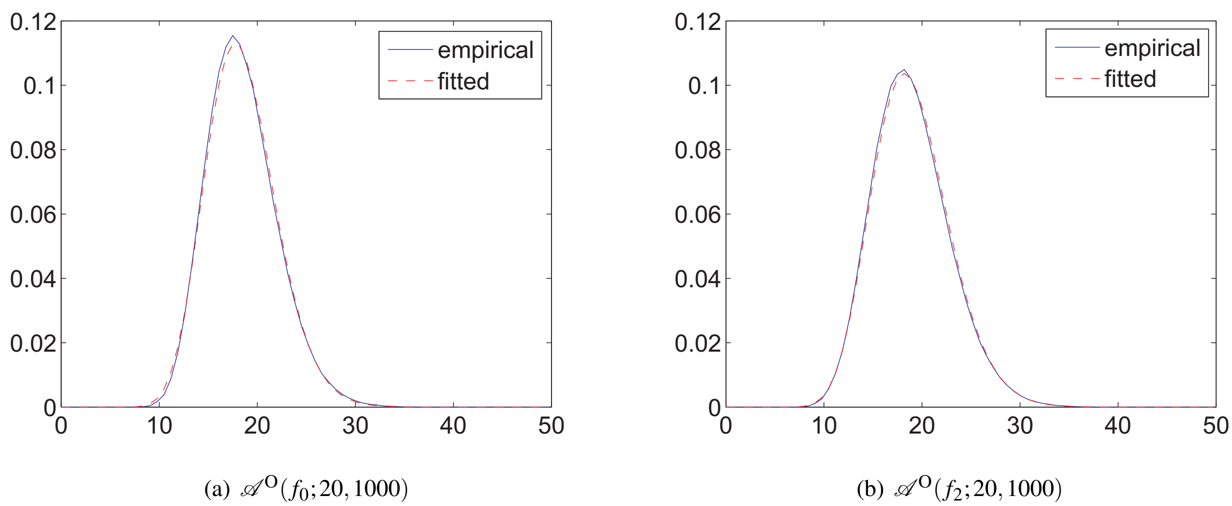

Figure 1: Fitted and empirical p.d.f.'s for two STS variance estimators computed from an AR(1) process with autoregressive parameter $\phi = 0.9$, steady-state mean $\mu = 0$, and variance parameter $\sigma^2 = 19$.

4.3 Confidence-Interval Estimation for μ and σ^2

Equation (9) can also be used to obtain approximate CIs for σ^2 and μ, provided that the batch size m is sufficiently large. Based on the generic overlapping variance estimator $\mathscr{V}^{\mathrm{O}}(b,m)$, we have the following approximate $100(1-\alpha)\%$ two-sided CI for the variance parameter σ^2:

$$\frac{\nu_{\mathrm{eff}}\,\mathscr{V}^{\mathrm{O}}(b,m)}{\chi^2_{\nu_{\mathrm{eff}},1-\alpha/2}} \le \sigma^2 \le \frac{\nu_{\mathrm{eff}}\,\mathscr{V}^{\mathrm{O}}(b,m)}{\chi^2_{\nu_{\mathrm{eff}},\alpha/2}}; \tag{10}$$

and the corresponding approximate $100(1-\alpha)\%$ two-sided CI for the steady-state mean μ is

$$\overline{X}_n - t_{\nu_{\mathrm{eff}},1-\alpha/2}\sqrt{\mathscr{V}^{\mathrm{O}}(b,m)/n} \le \mu \le \overline{X}_n + t_{\nu_{\mathrm{eff}},1-\alpha/2}\sqrt{\mathscr{V}^{\mathrm{O}}(b,m)/n}. \tag{11}$$

Here $\chi^2_{\nu,\beta}$ denotes the β-quantile of the chi-squared distribution with ν degrees of freedom, and $t_{\nu,\beta}$ denotes the β-quantile of Student's t-distribution with ν degrees of freedom for $\beta \in (0,1)$.

Using the same data from the AR(1) test process that was used to produce Figure 1, we evaluated the empirical coverage probabilities for nominal 90% CIs of the form (10) and (11) based on the overlapping variance estimators $\mathscr{A}^{\mathrm{O}}(f_0;20,1000)$ and $\mathscr{A}^{\mathrm{O}}(f_2;20,1000)$. Table 2 summarizes the results. Because the estimated coverage for each type of CI is based on 1,000,000 independent CIs of that type, the standard error of each estimated coverage is approximately 0.0003; and thus each entry in the table is reported to three significant figures.

Table 2: Performance of two nominal 90% STS CIs for μ and σ^2 for an AR(1) process with $\phi = 0.9$.

	CI Eqn.	Empirical Coverage of CIs Based on	
Parameter	Number	$\mathscr{A}^{\mathrm{O}}(f_0;20,1000)$	$\mathscr{A}^{\mathrm{O}}(f_2;20,1000)$
μ	(11)	0.895	0.899
σ^2	(10)	0.905	0.901

In all our experimental work with CIs of the form (11) and (10) for μ and σ^2, respectively, we have obtained excellent results. These developments have important implications for practical applications of steady-state simulation output analysis.

4.4 Optimal Linear Combinations of Overlapping Variance Estimators

Additional benefits can be obtained by combining overlapping variance estimators. Aktaran-Kalaycı et al. (2009) considered optimal linear combinations of overlapping variance estimators (OLCOVEs). The constituent estimators are OBMs or overlapping STS area estimators. Each estimator's batch size is a fixed multiple (at least unity) of a base batch size, appropriately rounded; hence the overall sample size is a fixed integral multiple of the base batch size. The control-variates method was used to obtain coefficients of the linear combination so as to yield a minimum variance OLCOVE. The paper also established asymptotic properties of the bias and variance of OLCOVEs, as the base batch size increases, and constructed CIs for μ and σ^2.

5 CONCLUSIONS

The ground-breaking Winter Simulation Conference paper Meketon and Schmeiser (1984) on the method overlapping batch means has led to numerous advances in both the theory and practice of simulation output analysis over the past three decades — in particular, many key insights that have subsequently set the stage for more-efficient estimators for use in simulations. This article has surveyed some of these developments. We anticipate much future work in the area of resampling and data reuse and its application to the design and analysis of stochastic simulation experiments.

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AUTHOR BIOGRAPHIES

CHRISTOS ALEXOPOULOS is an associate professor in the H. Milton Stewart School of Industrial and Systems Engineering at the Georgia Institute of Technology. His research interests are in the areas of simulation, statistics, and optimization of stochastic systems. He is a member of INFORMS and an active participant in the Winter Simulation Conference, having been Proceedings Co-Editor in 1995, Associate Program Chair in 2006, and a member of the Board of Directors since 2008. He is also an Area Editor of the *ACM Transactions on Modeling and Computer Simulation*. His e-mail address is christos@isye.gatech.edu, and his Web page is www.isye.gatech.edu/∼christos.

DAVID GOLDSMAN is a professor in the H. Milton Stewart School of Industrial and Systems Engineering at the Georgia Institute of Technology. His research interests include simulation output analysis, ranking and selection, and healthcare simulation. He is a long-standing participant in the Winter Simulation Conference, having been Program Chair in 1995 and a member of the WSC Board of Directors between 2001–2009. His e-mail address is sman@gatech.edu, and his Web page is www.isye.gatech.edu/∼sman.

JAMES R. WILSON is a professor in the Edward P. Fitts Department of Industrial and Systems Engineering at North Carolina State University. His current research interests are focused on probabilistic and statistical issues in the design and analysis of simulation experiments. He has held the following editorial positions: departmental editor of *Management Science* (1988–1996); area editor of *ACM Transactions on Modeling and Computer Simulation* (1997–2002); guest editor of a special issue of *IIE Transactions* honoring Alan Pritsker (1999–2001); and Editor-in-Chief of *ACM Transactions on Modeling and Computer Simulation* (2004–2010). He served The Institute of Management Sciences College on Simulation (now the INFORMS Simulation Society) as secretary-treasurer (1984–1986), vice president (1986–1988), and president (1988–1990). As a participant in the Winter Simulation Conference (WSC), he served as proceedings editor (1986), associate program chair (1991), and program chair (1992). As a member of the WSC Board of Directors corepresenting the INFORMS Simulation Society during the period 1997–2004, he served as secretary (2001), vice chair (2002), and chair (2003). He was a trustee of the WSC Foundation during the period 2006–2010. He is a member of ACM, ASA, and SCS; and he is a Fellow of IIE and INFORMS. His e-mail address is jwilson@ncsu.edu, and his Web page is www.ise.ncsu.edu/jwilson.