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A NONPARAMETRIC METHOD FOR PRICING AND HEDGING AMERICAN OPTIONS

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ABSTRACT

In this paper, we study the problem of estimating the price of an American option and its price sensitivities via Monte Carlo simulation. Compared to estimating the option price which satisfies a backward recursion, estimating the price sensitivities is more challenging. With the readily-computable pathwise derivatives in a simulation run, we derive a backward recursion for the price sensitivities. We then propose nonparametric estimators, the *k*-nearest neighbor estimators, to estimate conditional expectations involved in the backward recursion, leading to estimates of the option price and its sensitivities in the same simulation run. Numerical experiments indicate that the proposed method works well and is promising for practical problems.

1 INTRODUCTION

An American option is a derivative contract in which the holder has the right to exercise it at any time prior to the expiry. Pricing American options is one of the most challenging problems in financial engineering. Typically, no closed-form formulas are available for prices of the American options except in the simplest cases. Thus one has to resort to numerical methods. Partial differential equation (PDE) approaches and Monte Carlo simulation are among the most popular numerical methods. Compared to PDE approaches that are often fast and accurate in solving low-dimensional problems, Monte Carlo simulation allows for more general model settings and may work for high-dimensional problems. In this paper, we focus on simulation methods.

Over the past two decades, significant progress has been made towards computing the price of an American option. Notable examples include the regression-based methods of Carrière (1996), Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001), the stochastic mesh method of Broadie and Glasserman (2004). These methods estimate the continuation value of the optimal stopping problem with some simulation procedures, based on which the option price can be approximated. Another stream of research focuses on the dual formulation of the problem, which was established by Haugh and Kogan (2004) and Rogers (2002). The dual minimizes an expectation over a class of martingales, thus leading to an upper bound of the option price; see, e.g., Anderson and Broadie (2004) for simulation algorithms. For a comprehensive overview of the existing methods, interested readers are referred to Glasserman (2004).

While computing the price of an American option is one objective in Monte Carlo simulation, the accurate estimation of the price sensitivities with respect to some market parameters is arguably equally important but much more challenging. These price sensitivities are also referred to as Greek letters or hedging parameters, which play a crucial role in risk management. For instance, the sensitivity of the option price with respect to the initial value of the underlying asset is called *delta*, which indicates the quantity of the underlying asset one should hold in order to hedge the option position (Hull 2009). The sensitivity of the option price with respect to the volatility of the underlying asset is called *vega*, which reflects how changes in the volatility will affect the option price.

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Compared to the vast literature on computing American option price, the computation of price sensitivities remains a challenging task, and has received increasing attention among researchers recently. Kaniel, Tompaidis, and Zemlianov (2008) construct confidence intervals for *delta* and *gamma* by combining the likelihood ratio method and the duality formulation of the optimal stopping problem. But it is not clear how their method can be applied to estimate other Greek letters such as vega. Wang and Caflisch (2010) propose a perturbed version of the least-squares method of Longstaff and Schwartz (2001) to estimate the price sensitivities, which is especially suitable for computing *delta*. Recently, Chen and Liu (2012) generalize the classical pathwise method to American-style options. Given suboptimal exercise policies, their method performs sensitivity estimation in a straightforward manner and resolves the difficulty caused by discontinuity of the optimal decision with respect to the market parameter.

In this paper, we consider the estimation of the option price and its sensitivities in a single simulation run. In particular, by employing the pathwise derivatives that are readily computable from the simulation run, we derive a backward recursion for the price sensitivities. While the pathwise method (also referred to as infinitesimal perturbation analysis (IPA); see Ho and Cao (1983)) has been studied extensively for European options (see, e.g., Broadie and Glasserman (1996)), the backward recursion derived in this paper can be viewed as an extension of the pathwise method to American-style options. We then propose a nonparametric estimation method, the k-nearest neighbor method, to estimate conditional expectations involved in the backward recursion, leading to a nonparametric simulation algorithm for estimating both the option price and its sensitivities.

The rest of the paper is organized as follows. We develop the mathematical framework and formulate the problem in Section 2. A backward recursion formulation of price sensitivities is derived in Section 3, and a nonparametric method based on the backward recursion is proposed in 4. Numerical results are illustrated in Section 5, followed by conclusions in Section 6.

2 MATHEMATICAL FRAMEWORK

In this section we specify the mathematical framework for the price of an American option and its sensitivity with respect to a market parameter.

Consider an American option that is written on one or more underlying assets. Let S_t be a Markov process representing the price dynamics of the underlying assets. Suppose that S_t depends on a market parameter θ, which may represent, e.g., the initial value or the volatility of an underlying asset. Without loss of generality, we assume that θ is one-dimensional with $\theta \in \Theta$, where Θ is an open set. When θ is multi-dimensional, one may consider each dimension separately while fixing other dimensions constants. To allow for more clarity, the price dynamics is written as $S_t(\theta)$. We consider a general setting in which the payoff of the option depends on $S_t(\theta)$ and a path-dependent function $J_t(\theta)$, which is a function of $\{S_u(\theta) : 0 \le u \le t\}$, if the option is exercised at time t. In other words, the option is allowed to be path dependent. Examples of the path-dependent function J_t in practical problems include the average value and running maximum (minimum) of an underlying asset. It can be seen that the Markov process $\{(S_t(\theta), J_t(\theta))\}$ governs the evolution of the price of the American option.

Throughout the paper we work with American options that can be exercised at a finite number of time points. Sometimes these options are also referred to as Bermudan options in the literature. Suppose that the American option can be exercised at $\{t_i, 1 \le i \le m\}$ with $0 < t_1 < t_2 < \ldots < t_m = T$, where T is the maturity date of the option. Following the convention in the literature, we assume that the option cannot be exercised at time 0. For notational ease in the following presentation, we let (S_i, J_i) denote $(S_{t_i}(\theta), J_{t_i}(\theta))$ for $i = 1, \ldots, m$, where the explicit dependence of (S_t, J_t) on the market parameter θ is suppressed when there is no confusion. If exercised at time t_i, the discounted payoff to the holder of the option is denoted by $h_i(S_i, J_i)$ for $i = 1, \ldots, m$. Then the price of the American option at time 0 is

$$V_0(\theta) = \sup_{\tau \in \mathscr{T}} \mathrm{E}\left[h_\tau(S_\tau, J_\tau)\right],$$

where $\mathscr{T}$ is the class of stopping times taking values in $\{1,\dots,m\}$, and the expectation is taken under the pricing martingale measure. Note that the initial state of (S_t,J_t), (S_0,J_0), is fixed and known.

Computing the price of the American is usually the first task for participants in the financial markets, and a number of simulation methods have been proposed to address this issue. In addition to the price, one is also interested in computing the price sensitivities. In our setting, the price sensitivity of interest is represented as

$$\alpha = \frac{d}{d\theta}V_0(\theta).$$

The focus of this paper is on the accurate computation of $V_0(\theta)$ and $dV_0(\theta)/d\theta$.

3 A BACKWARD RECURSION FOR PRICE SENSITIVITIES

For $i=1,\dots,m$, let $V_i(x,y;\theta)$ denote the value function of the American option at time t_i given $(S_i,J_i)=(x,y)$. Then it is well known that a backward recursion formulation of the value functions is specified by

$$\begin{aligned} V_m(S_m,J_m;\theta) &= h_m(S_m,J_m), \\ V_i(S_i,J_i;\theta) &= \max\left(h_i(S_i,J_i), \mathrm{E}\left[V_{i+1}(S_{i+1},J_{i+1};\theta)|S_i,J_i\right]\right), \quad i=m-1,\dots,1. \end{aligned}$$

The price of the American option is then

$$V_0(\theta) = \mathrm{E}[V_1(S_1,J_1;\theta)].$$

Define the continuation value function

$$C_i(x,y;\theta) = \mathrm{E}\left[V_{i+1}(S_{i+1},J_{i+1};\theta)|(S_i,J_i)=(x,y)\right], \quad i=1,\dots,m-1.$$

Then $V_i(x,y;\theta)=\max(h_i(x,y),C_i(x,y;\theta))$.

While the option price can be computed by the above backward recursion formulation, it remains a question whether the price sensitivity α can be characterized by a similar backward recursion. Motivated by the pathwise method of Broadie and Glasserman (1996), we show that the answer is affirmative. We derive a backward recursion formulation for α in the rest of this section.

We first examine the simulation procedure in greater detail. Note that in a simulation run, for $i=1,\dots,m$, S_i can be viewed as a function of (S_{i-1},U_i,θ), where U_i denotes a uniform $(0,1)$ random vector that is independent of S_{i-1}. We denote this function by F, i.e., $S_i=F(S_{i-1},U_i;\theta)$[1]. Furthermore, we assume that J_i is a function of (S_i,J_{i-1}), and let ν denote this function, i.e., $J_i=\nu_i(S_i,J_{i-1})$.

For notational ease, we let $V_{i,1}(x,y;\theta)$, $V_{i,2}(x,y;\theta)$ and $V_{i,3}(x,y;\theta)$ denote the partial derivatives of $V_i(x,y;\theta)$ with respect to its three arguments, i.e., $\partial_x V_i(x,y;\theta)$, $\partial_y V_i(x,y;\theta)$ and $\partial_\theta V_i(x,y;\theta)$, respectively. We similarly define $C_{i,k}(x,y;\theta)$ as partial derivatives of $C_i(x,y;\theta)$ for $k=1,2,3$. We further let $h_{i,k}(x,y)$ and $\nu_{i,k}(x,y)$ denote the partial derivatives of $h_i(x,y)$ and $\nu_i(x,y)$ respectively for $k=1,2$, and $F_1(x,U;\theta)$ and $F_3(x,U;\theta)$ denote $\partial_x F(x,U;\theta)$ and $\partial_\theta F(x,U;\theta)$ respectively.

Let $\mathscr{X}$ and $\mathscr{Y}$ denote the state space of S_t and J_t respectively. To facilitate the analysis, we make the following assumptions.

Assumption 1 For any $i\in\{1,\dots,m\}$, there exists a random variable K that may depend on θ such that $\mathrm{E}(K)<\infty$, and with probability 1 (w.p.1),

$$|S_i(\theta+\Delta\theta)-S_i(\theta)| \le K|\Delta\theta|, \quad |J_i(\theta+\Delta\theta)-J_i(\theta)| \le K|\Delta\theta|,$$

when $|\Delta\theta|$ is small enough.

[1]For notational ease, we suppress the possible dependence of F on the time step i. Results in this paper can be easily adapted to this case.

Assumption 2 For any $i \in \{1,\dots,m\}$, there exist constants c_1 and c_2 such that

$$|h_i(x_1,y)-h_i(x_2,y)| \le c_1|x_1-x_2|, \quad |h_i(x,y_1)-h_i(x,y_2)| \le c_2|y_1-y_2|, \quad \forall x,x_1,x_2 \in \mathscr{X}, y,y_1,y_2 \in \mathscr{Y}.$$

Assumption 3 For any $i \in \{1,\dots,m\}$, there exist constants c_3 and c_4 such that

$$|\nu_i(x_1,y)-\nu_i(x_2,y)| \le c_3|x_1-x_2|, \quad |\nu_i(x,y_1)-\nu_i(x,y_2)| \le c_4|y_1-y_2|, \quad \forall x,x_1,x_2 \in \mathscr{X}, y,y_1,y_2 \in \mathscr{Y}.$$

Assumption 4 There exist functions $Q(\cdot)$ and $G(\cdot)$ such that $\forall x,x_1,x_2 \in \mathscr{X}, \theta,\theta_1,\theta_2 \in \Theta$,

$$|F(x_1,U;\theta)-F(x_2,U;\theta)| \le Q(U,\theta)|x_1-x_2|, \quad |F(x,U;\theta_1)-F(x,U;\theta_2)| \le G(x,U)|\theta_1-\theta_2|, \quad w.p.1.$$

Furthermore, $\mathrm{E}[g(S_i)] < \infty$ for $i=1,\dots,m$, and $\sup_{\theta\in\Theta}\mathrm{E}[Q(U,\theta)] \le \infty$, where $g(\cdot)$ is defined by $g(x) = \mathrm{E}[G(x,U)]$.

Assumption 1 is a Lipschitz continuity requirement on (S_i,J_i). It is a standard assumption in the sensitivity estimation literature; see, e.g, Broadie and Glasserman (1996), Liu and Hong (2011), and Chen and Liu (2012). Assumptions 2-4 essentially require the functions h_i, F and ν_i to be Lipschitz continuous. Given the payoff function of an American option, Assumptions 2 and 3 are easy to verified. Typically, for American options with continuous payoff functions, these two assumptions are satisfied. Assumption 4 is on the price dynamics of the underlying assets. It is typically satisfied for commonly used models such as the Black-Scholes model.

Given the above assumptions, a representation of the price sensitivity α can be obtained, which is summarized in the following theorem. Proof of the theorem is provided in the appendix.

Theorem 1 Suppose that for $i=1,\dots,m$, (S_i,J_i) is differentiable w.p.1, and h_i and ν_i are differentiable almost everywhere. If Assumptions 1-3 are satisfied, then

$$\alpha = \frac{d}{d\theta}V_0(\theta) = \mathrm{E}\left[V_{1,1}(S_1,J_1;\theta)\partial_\theta S_1 + V_{1,2}(S_1,J_1;\theta)\partial_\theta J_1 + V_{1,3}(S_1,J_1;\theta)\right],$$

where $\partial_\theta S_1$ and $\partial_\theta J_1$ denote respectively the pathwise derivatives of S_1 and J_1 with respect to θ.

Theorem 1 shows that the price sensitivity α can be written as an expectation, in which the integrand involves partial derivatives of the value functions and pathwise derivatives of the price dynamics of the underlying asset. Based on Theorem 1, α can be computed via the following backward recursion.

1. Initial condition: For all $(x,y) \in (\mathscr{X},\mathscr{Y})$, $V_m(x,y;\theta) = h_m(x,y)$, $V_{m,k}(x,y;\theta) = h_{m,k}(x,y)$ for $k=1,2$, $V_{m,3}(x,y;\theta) = 0$, and $h_{i,3}(x,y) \equiv 0$ for $i=1,\dots,m$.
2. For $i = m-1,\dots,1,$, define

$$\begin{aligned}
D_{i+1,1} &= V_{i+1,1}(S_{i+1},J_{i+1};\theta)F_1(S_i,U_{i+1};\theta) + V_{i+1,2}(S_{i+1},J_{i+1};\theta)\nu_{i+1,1}(S_{i+1},J_i)F_1(S_i,U_{i+1};\theta),\\
D_{i+1,2} &= V_{i+1,2}(S_{i+1},J_{i+1};\theta)\nu_{i+1,2}(S_{i+1},J_i),\\
D_{i+1,3} &= V_{i+1,1}(S_{i+1},J_{i+1};\theta)F_3(S_i,U_{i+1};\theta) + V_{i+1,2}(S_{i+1},J_{i+1};\theta)\nu_{i+1,1}(S_{i+1},J_i)F_3(S_i,U_{i+1};\theta)\\
&\quad + V_{i+1,3}(S_{i+1},J_{i+1};\theta),
\end{aligned}$$

   and set

$$\begin{aligned}
V_i(x,y;\theta) &= \max\left(h_i(x,y), C_i(x,y;\theta)\right),\\
C_i(x,y;\theta) &= \mathrm{E}\left[V_{i+1}(S_{i+1},J_{i+1};\theta)|(S_i,J_i)=(x,y)\right],\\
C_{i,k}(x,y;\theta) &= \mathrm{E}[D_{i+1,k}|(S_i,J_i)=(x,y)], \quad k=1,2,3,\\
V_{i,k}(x,y;\theta) &= h_{i,k}(x,y)\cdot 1_{\{h_i(x,y)\ge C_i(x,y;\theta)\}} + C_{i,k}(x,y;\theta)\cdot 1_{\{h_i(x,y)<C_i(x,y;\theta)\}}, \quad k=1,2,3,
\end{aligned}$$

   where $1_{\{A\}}$ denotes the indicator function that is equal to 1 if the event A occurs, and 0 otherwise.

3. The price of the American option is $V_0 = \mathrm{E}[V_1(S_1, J_1; \theta)]$, and the price sensitivity is

$$\alpha = \mathrm{E}\left[V_{1,1}(S_1, J_1; \theta)\partial_\theta S_1 + V_{1,2}(S_1, J_1; \theta)\partial_\theta J_1 + V_{1,3}(S_1, J_1; \theta)\right].$$

A key step in the above backward recursion is the computation of the conditional expectations $C_i(x,y;\theta)$ and $C_{i,k}(x,y;\theta)$ for $k = 1,2,3$. Once these conditional expectations can be computed, the price of the American option and its price sensitivities can be obtained in the same simulation run. Essentially, this backward recursion exploits the pathwise derivatives in a simulation run, and can be viewed as an extension of the pathwise method of Broadie and Glasserman (1996) to American-style options.

4 A NONPARAMETRIC ESTIMATION METHOD

The backward recursion established in Section 3 leads immediately to simulation algorithms for estimating the price of the American option and its price sensitivities. Key components of such algorithms are procedures to estimate the conditional expectations $C_i(x,y;\theta)$ and $C_{i,k}(x,y;\theta)$ for $k = 1,2,3$. A number of existing procedures are readily applicable to accomplish this task, notable ones including the regression-based method (see, e.g., Carrière (1996), Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001)) and the stochastic mesh method (Broadie and Glasserman 2004).

In this paper we study a nonparametric method to estimate these conditional expectations, the k-nearest neighbor (kNN) estimators; see, e.g., Härdle (1990) and Li and Racine (2007) for overviews. Essentially, the kNN method is a local smoothing method that approximates a regression function by the average of samples in a neighborhood.

Suppose we generate n samples paths of the price dynamics, and we let the superscript j denote the pathwise information recorded for the jth sample path. For a general function $r(\cdot,\cdot)$, the kNN estimator approximates the conditional expectation $\mathrm{E}[r(S_{i+1}, J_{i+1})|(S_i, J_i) = (x,y)]$ by using the average of k_n samples that are nearest to (x,y), where the positive integer k_n is the user-specified smoothing parameter. In particular, let $\mathscr{A}(x,y)$ denotes index set of the k_n samples among $\{(S_i^j, J_i^j), 1 \le j \le n\}$ that are nearest to (x,y) in terms of Euclidean distance. Then a kNN estimator of $\mathrm{E}[r(S_{i+1}, J_{i+1})|(S_i, J_i) = (x,y)]$ is specified by

$$\frac{1}{k_n} \sum_{j \in \mathscr{A}(x,y)} r(S_{i+1}^j, J_{i+1}^j).$$

We propose to use the kNN estimators to estimate the conditional expectations in the backward recursion established in Section 3. Specifically, sequentially for $i = m-1, \ldots, 1$, the conditional expectations $C_i(x,y;\theta)$ and $C_{i,k}(x,y;\theta)$ for $k = 1,2,3$ are estimated by

$$\widetilde{C}_i^n(x,y;\theta) = \frac{1}{k_n} \sum_{j \in \mathscr{A}(x,y)} \widetilde{V}_{i+1}^n(S_{i+1}^j, J_{i+1}^j; \theta),$$

and

$$\widetilde{C}_{i,k}^n(x,y;\theta) = \frac{1}{k_n} \sum_{j \in \mathscr{A}(x,y)} \widetilde{D}_{i+1,k}^j,$$

respectively, where

$$\begin{aligned}
\widetilde{V}_i^n(x,y;\theta) &= \max\left(h_i(x,y), \widetilde{C}_i^n(x,y;\theta)\right), \\
\widetilde{V}_{i,k}^n(x,y;\theta) &= h_{i,k}(x,y) \cdot 1_{\{h_i(x,y) \ge \widetilde{C}_i^n(x,y;\theta)\}} + \widetilde{C}_{i,k}^n(x,y;\theta) \cdot 1_{\{h_i(x,y) < \widetilde{C}_i^n(x,y;\theta)\}},
\end{aligned}$$

and $\widetilde{D}_{i+1,k}$ is an approximate counterpart of $D_{i+1,k}$, obtained by replacing $V_{i+1,k}$ with $\widetilde{V}_{i+1,k}^n$ in $D_{i+1,k}$.

Performance of the kNN method relies on appropriate selection of the smoothing parameter k_n. Typically, k_n is required to satisfy $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. In practical implementation, one may choose the "best" k_n among a number of user-specified candidates based on a least-squares cross-validation procedure. Intuitively, the cross-validation leads to a k_n that minimizes the least-squares error of the regression estimate. Due to page limit, details of the least-squares cross-validation procedure is omitted.

5 NUMERICAL EXPERIMENTS

In this section we conduct numerical experiments to examine the performance of the proposed nonparametric method. In particular, we measure the performance of an estimator by its relative root mean square error (RRMSE), which is defined as the percentage of the root mean square error to the absolute value of the quantity being estimated.[2] When implementing the kNN estimator, a least-squares cross-validation procedure is applied to choose the best k_n among a set of candidates. All RRMSEs reported are estimated based on 1000 independent replications.

We consider four examples to examine the performance of the proposed nonparametric method. Numerical results for these examples are summarized in Tables 1-4 respectively. The first example is an American put option with a large number of exercise dates. Results in Table 1 show that the proposed estimators work very well for both the option price and its sensitivities. The second example is an American lookback put option. Its payoff at an exercise date t_i is $(K-\hat{S}_i)^+$, where $\hat{S}_i$ denotes the running maximum of the underlying asset up to t_i and K denotes the strike price. Results in Table 2 shows that the proposed method leads to highly accurate estimates for the option price, *delta* and *vega*, where all relative errors are within 0.5%. The third example is an American max-call option written on two assets, where the payoff depends on the maximum of two assets. In particular, its payoff at an exercise date t_i is $(\max(S_i^1, S_i^2) - K)^+$, where S_i^1 and S_i^2 denote the values of the two underlying assets at t_i. For this max-call option, we consider the performance of the proposed nonparametric method for different coefficients of correlation. Results in Table 3 show that the estimators perform quite well. We further consider an American max-call option on five assets to test the performance of the proposed method when dimension becomes larger. Results in 4 show that the proposed method performs reasonably well, with relative errors ranging from 2% to 8%.

While it is well recognized that performance the kNN estimators may deteriorate as dimension increases, it is not fully clear what is the threshold on dimension for kNN estimators to be practically viable. Further investigation on this issue is desirable, and is left as a topic for future research. Given that dimensions of most practical problems are usually not very high, numerical experiments in this section indicate that the proposed nonparametric method could be a promising tool in practice.

Table 1: Performance of estimators for an American put option written on a single asset with the Black-Scholes model. Results are shown for different strike K. Parameters: $S_0 = 40$, risk-free interest rate $r = 4.88\%$, volatility $\sigma = 20\%$, maturity $T = 7/12$, and number of exercise dates $m = 400$. Sample size used in the estimation is $n = 10^5$.

	RRMSE (%)		
K	`price`	`delta`	`vega`
35	2.1	2.0	2.1
40	1.1	0.9	1.1
45	0.8	0.9	1.4

[2] During the implementation, we approximate the true value of the quantity being estimated using an accurate point estimate from the existing literature when available. When only interval estimates but not point estimates are available, we calculate the RRMSE using either the lower or the upper estimates of the intervals, depending on which one lead to larger RRMSEs.

Table 2: Performance of estimators for an American lookback put option written on a single asset with the Black-Scholes model. Results are shown for different strike K. Parameters: $S_0 = 40$, risk-free interest rate $r = 8\%$, volatility $\sigma = 20\%$, maturity $T = 1/4$, and number of exercise dates $m = 10$. Sample size used in the estimation is $n = 10^5$.

	RRMSE (%)		
K	`price`	`delta`	`vega`
42	0.2	0.1	0.5
47	0.04	0.01	0.5
52	0.02	0.01	0.5

Table 3: Performance of estimators for an American max-call option written on two assets with the Black-Scholes model. Results are shown for different coefficients of correlation between these two assets, denoted by ρ. Parameters: initial stock prices are 100 for both assets, strike $K = 100$, risk-free interest rate $r = 5\%$, dividends $\delta_1 = \delta_2 = 10\%$, volatility $\sigma_1 = \sigma_2 = 20\%$, maturity $T = 3$, and number of exercise dates $m = 9$. Sample size used in the estimation is $n = 10^5$.

	RRMSE (%)		
ρ	`price`	`delta`	`vega`
−0.75	0.7	5.2	0.8
−0.5	1.2	3.0	1.1
−0.25	1.1	2.7	1.0
0	1.2	3.6	1.7
0.25	1.5	4.9	1.3
0.5	1.2	5.1	0.8
0.75	1.6	4.8	1.7

6 CONCLUSIONS

In this paper we study the problem of pricing and hedging American options via Monte Carlo simulation. We derive a backward recursion formulation for the price sensitivities of an American option, based on which we propose a nonparametric method to estimate both the option price and its sensitivities in the same simulation run. Numerical experiments show that the proposed method is promising for solving practical problems.

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Table 4: Performance of estimators for an American max-call option written on five assets with the Black-Scholes model. These five assets are assumed to be independent and follow the same distribution. Results are shown for different initial values of the underlying assets S_0. Parameters: strike $K = 100$, risk-free interest rate $r = 5\%$, dividend $\delta = 10\%$, volatility $\sigma = 20\%$, maturity $T = 3$, and number of exercise dates $m = 9$. Sample size used in the estimation is $n = 10^5$.

	RRMSE (%)		
S_0	`price`	`delta`	`vega`
90	5.5	2.7	6.2
100	3.6	4.1	7.4
110	2.4	4.2	7.5

A APPENDIX: Proof of Theorem 1

Proof. Based on the dominated convergence theorem (Durrett 2005) and Assumption 1, it suffices to show that $V_i(x,y;\theta)$ is Lipschitz continuous in (x,y) with a Lipschitz constant c_0, i.e., there exists a constant c_0 such that for all $x,x_1,x_2 \in \mathscr{X}$, $y,y_1,y_2 \in \mathscr{Y}$, and $\theta \in \Theta$

$$\begin{aligned} |V_i(x_1,y;\theta) - V_i(x_2,y;\theta)| &\leq c_0|x_1 - x_2|, && (1)\\ |V_i(x,y_1;\theta) - V_i(x,y_2;\theta)| &\leq c_0|y_1 - y_2|, && (2) \end{aligned}$$

for $i = 1$, and there exists a function $r(\cdot)$ satisfying $\mathrm{E}[r(S_1,J_1)] < \infty$ such that for all $(x,y) \in (\mathscr{X},\mathscr{Y})$,

$$|V_i(x,y;\theta+\Delta\theta) - V_i(x,y;\theta)| \leq r(x,y)|\Delta\theta|, \qquad (3)$$

for $i = 1$, when $|\Delta\theta|$ is small enough.

We first show Eqn.(1)-(2) by backward recursion. Note that $V_m(x,y;\theta) = h_m(x,y)$ for any $(x,y) \in (\mathscr{X},\mathscr{Y})$. Then by Assumption 2, it can be seen that $V_m(x,y;\theta)$ satisfies Eqn.(1)-(2) by setting $c_0 = \max(c_1,c_2)$. Suppose that $V_{i+1}(x,y;\theta)$ satisfies Eqn.(1)-(2) with Lipschitz constant c_0'. Then for any $x_1,x_2 \in \mathscr{X}$ and $y \in \mathscr{Y}$,

$$\begin{aligned} &|C_i(x_1,y;\theta) - C_i(x_2,y;\theta)|\\ &\quad= |\mathrm{E}[V_{i+1}(F(x_1,U_{i+1};\theta),\nu_{i+1}(F(x_1,U_{i+1};\theta),y);\theta) - V_{i+1}(F(x_2,U_{i+1};\theta),\nu_{i+1}(F(x_2,U_{i+1};\theta),y);\theta)]|\\ &\quad\leq \mathrm{E}\left(c_0'|F(x_1,U_{i+1};\theta) - F(x_2,U_{i+1};\theta)| + c_0'|\nu_{i+1}(F(x_1,U_{i+1};\theta),y) - \nu_{i+1}(F(x_2,U_{i+1};\theta),y)|\right)\\ &\quad\leq \mathrm{E}\left[(c_0' + c_0'c_3)Q(U_{i+1},\theta)|x_1 - x_2|\right] \leq (c_0' + c_0'c_3)\sup_{\theta\in\Theta}\mathrm{E}[Q(U,\theta)]|x_1 - x_2|, \end{aligned}$$

where the second inequality follows from Assumptions 4 and 3.

Similarly we can show that for any $x \in \mathscr{X}$ and $y_1,y_2 \in \mathscr{Y}$,

$$|C_i(x,y_1;\theta) - C_i(x,y_2;\theta)| \leq c_0'c_4|y_1 - y_2|.$$

Because $V_i(x,y;\theta) = \max(h_i(x,y),C_i(x,y;\theta))$, it can be seen that for any $x,x_1,x_2 \in \mathscr{X}$ and $y,y_1,y_2 \in \mathscr{Y}$,

$$\begin{aligned} &|V_i(x_1,y;\theta) - V_i(x_2,y;\theta)|\\ &\quad\leq |h_i(x_1,y) - h_i(x_2,y)| + |C_i(x_1,y;\theta) - C_i(x_2,y;\theta)| \leq \left\{c_1 + (c_0' + c_0'c_3)\sup_{\theta\in\Theta}\mathrm{E}[Q(U,\theta)]\right\}|x_1 - x_2|, \end{aligned}$$

and

$$|V_i(x,y_1;\theta) - V_i(x,y_2;\theta)| \leq |h_i(x,y_1) - h_i(x,y_2)| + |C_i(x,y_1;\theta) - C_i(x,y_2;\theta)| \leq (c_2 + c_0'c_4)|y_1 - y_2|.$$

If we set $c_0'' = \max\left(c_1 + (c_0' + c_0' c_3)\sup_{\theta\in\Theta} \mathrm{E}[Q(U,\theta)], c_2 + c_0' c_4\right)$, then it can be seen that $V_i(x,y;\theta)$ satisfies Eqn.(1)-(2) with the Lipschitz constant c_0''. Recursively we can show that Eqn.(1)-(2) are satisfied for $i = 1,\ldots,m$.

Then we only need to show Eqn.(3) for $i = 1,\ldots,m$. Note that $V_m(x,y;\theta)$ satisfies Eqn.(3) by letting $r(x,y) = 0$. Then by Eqn.(1)-(2) and Assumptions 4,

$$\begin{aligned}
&|C_{m-1}(x,y;\theta+\Delta\theta) - C_{m-1}(x,y;\theta)| \\
&\quad = \ \left|\mathrm{E}\left[V_m(F(x,U_m;\theta+\Delta\theta), \nu_m(F(x,U_m;\theta+\Delta\theta),y);\theta+\Delta\theta) - V_m(F(x,U_m;\theta), \nu_m(F(x,U_m;\theta),y);\theta)\right]\right| \\
&\quad \leq \ \mathrm{E}\left(c_0|F(x,U_m;\theta+\Delta\theta) - F(x,U_m;\theta)| + c_0|\nu_m(F(x,U_m;\theta+\Delta\theta),y) - \nu_m(F(x,U_m;\theta),y)|\right) \\
&\quad \leq \ (c_0 + c_0 c_3)\mathrm{E}\left[G(x,U_m)\right]|\Delta\theta|.
\end{aligned}$$

Recall that for $i = 1,\ldots,m$,

$$\begin{aligned}
&|V_i(x,y;\theta+\Delta\theta) - V_i(x,y;\theta)| \\
&\quad = \ |\max(h_i(x,y), C_i(x,y;\theta+\Delta\theta)) - \max(h_i(x,y), C_i(x,y;\theta))| \leq |C_i(x,y;\theta+\Delta\theta) - C_i(x,y;\theta)|.
\end{aligned}$$

Thus for any $(x,y) \in (\mathscr{X},\mathscr{Y})$,

$$|V_{m-1}(x,y;\theta+\Delta\theta) - V_{m-1}(x,y;\theta)| \leq (c_0 + c_0 c_3)\mathrm{E}\left[G(x,U_m)\right]|\Delta\theta|.$$

In a similar manner, we can show that

$$|V_{m-2}(x,y;\theta+\Delta\theta) - V_{m-2}(x,y;\theta)| \leq (c_0 + c_0 c_3)\left(\mathrm{E}[G(x,U_{m-1})] + \mathrm{E}\left[G(F(x,U_{m-1};\theta),U_m)\right]\right)|\Delta\theta|.$$

Recursively, we have

$$|V_1(x,y;\theta+\Delta\theta) - V_1(x,y;\theta)| \leq (c_0 + c_0 c_3)\left(\mathrm{E}[G(x,U_2)] + \sum_{k=2}^{m-1}\mathrm{E}\left[G(S_k,U_{k+1})\right]\right)|\Delta\theta|,$$

basing on the fact that $S_i = F(S_{i-1},U_i;\theta)$ for $i = 1,\ldots,m$ where U_i is independent of S_{i-1}.

Let $r(x,y) = (c_0 + c_0 c_3)\left(\mathrm{E}[G(x,U_2)] + \sum_{k=2}^{m-1}\mathrm{E}\left[G(S_k,U_{k+1})\right]\right)$. Then by Assumption 4 it can be seen that $\mathrm{E}[r(S_1,J_1)] < \infty$. Therefore, Eqn.(3) is satisfied for $i = 1$, which completes the proof. □

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