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NOVEL USE OF SINGULARITY FUNCTIONS TO MODEL PERIODIC PHENOMENA IN CASH FLOW ANALYSIS

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ABSTRACT

When seeking to properly consider the time value of money, typical periodic cash flows such as payments and interest charges are difficult to model. This paper explores a new signal function that employs singularity functions to express such intermittent phenomena. This flexible signal function allows manipulating the parameters of start and finish, amplitude, and period of the signal efficiently, so that payments and interest charges can be modeled accurately. This novel approach is beneficial in several ways. First, the new model can effectively incorporate shift and delay effects that may affect an activity. Second, it applies an exact interest calculation. Third, it can handle compounding in its accumulation. Finally, a comprehensive model is created that returns the cumulative balance including interest charges at all times. It is concluded that signal functions are a promising area for future research on modeling and optimizing the cash flows.

1 INTRODUCTION

Cash flows must be managed carefully in any construction project. However, it is difficult to model those cash flows in detail because their balance typically possesses an uneven jagged shape (Elazouni and Metwally 2005). The interactions between cash outflow and inflow elements (Lucko 2011) are "disparate and shifted phenomena [that] give[s] an uneven 'sawtooth' pattern" (Lucko and Thompson 2010, p. 2). Periodic phenomena are typical of cash flows, but are complicated by such interactions and their cumulative nature (Lucko 2011). Research is needed on how to effectively and efficiently model this periodicity.

1.1 Periodic Cash Inflows

Payments are the main cash inflows for contractors. Regular progress payments are periodic under the contract between owner and contractor. After the contractor sends a monthly bill to the owner, a billing-to-payment-delay occurs. The owner also keeps a retainage from each payment to incentivize the contractor to finish the project timely and with good quality (Cui et al. 2010). The delay and retainage influence the timing and amount of periodic payments. If hierarchical pay when/if paid clauses exist between contractor and subcontractors or suppliers, such total delay could extend beyond 90 days (Setzer 2009).

1.2 Periodic Cash Outflows

In the construction industry, cash outflows including labor, materials, equipment, and various site expenses are approximated as a continuous pattern (Kenley 2003). However, they occur intermittently; e.g. worker salary may be paid monthly, weekly, or even daily; equipment rent and depreciation are assessed periodically; and overhead may be fixed or vary. Financing fees are charged typically at the end of each month (Lucko and Thompson 2010). Overall, there exists a major need to model such period phenomena of cash flows in detail. In the following, compound interest is charged monthly without considering that the number of calendar days varies monthly. For brevity, retainage and unused credit fee are excluded.

2 LITERATURE REVIEW

2.1 Limitations of Previous Cash Flow Models

Previous researchers have proposed many cash flow models. Most models implement time lags and note them in their assumptions. A time lag is the "difference between the time a resource is used on the site and the time it is paid for" (Navon 1997, p. 1056). Fondahl's (1973) model required forecasting the average time lag from cost to payment. Ashley and Teicholz (1977) set up monthly payments, earnings, and billing lags. McCaffer (1979) used monthly and weekly periods. Au and Hendrickson (1986) included an inflation factor to measure the gross operating profit and financing costs of multi-year projects and incorporated a lag of one year. Elazouni (2009) used workdays as the periodic unit in his model. Many other researchers made similar assumptions (Liu and Wang 2008; Park et al. 2005; Barbosa and Pimental 2001; Singh 2001; Khosrowshahi 2000; Navon 1997). But impacts of shifts and delays on starts and finishes of activities and effects on the lagged cost and payment were typically ignored. Khosrowshahi (1988) used regression to fit a mathematical model of cash flow based on the data of interim certified payments from 480 projects. In that model, the periodic expenditure profile was expressed by an exponential function in an approximation. Halpin and Woodhead (1998) used a decomposed chronological approach to calculate month-end balances. An infinitesimal offset ε is implicitly used for payments and charging interest; first calculating the overdraft, then adding interest, and finally receiving payment. Others used the same method for their balances, either implicitly (Elazouni and Metwally 2005; Elazouni and Gab-Allah 2004) or explicitly (Lucko 2011). It returns the correct balance, but severely lacks flexibility and required manipulating many factors. Cui et al. (2010) applied system dynamics. Feedback loops in their cash flow model illustrate periodic phenomena to test cash flow management strategies, but fall short regarding accuracy. Yet payment terms like lag and frequency significantly impact the accuracy of cash flow models (Chen et al. 2005). A major need exists for a comprehensive, accurate, and flexible model of periodic cash flows.

2.2 Previous Periodic Functions

In mathematics, periodic functions, e.g. Fourier series, can repeat their values. In physics, the Dirac delta function is zero except at a single point. In signal processing, an impulse response function is used in areas like "radar sonar[,] cartography, [and] seismology" (Szabó et al. 1985, p. 266). In economics, it "measures the time profile of the effect of a shock on the behaviour of a series" (Koop et al. 1996, p. 120). These periodic functions are well-suited for their application but remain insufficient to fully express periodic cash flows, whose disparate and shifted elements interact with each other and are cumulative.

2.3 Need for Research on Signal Functions

Disparate and shifted effects influence both starts and finishes of activities; the amount of periodic cash flows is changed accordingly. Cumulative interactions between periodic cash flows exacerbate the difficulty of modeling and determine the sign of the balance. From the contractor's perspective at the start of a single project, cumulative cash outflows exceed inflows, likely causing a negative balance. As payments are received, it may reach a breakeven point and become positive. This assumes no residual funds from prior projects. Previous approaches applied a decomposed chronological approach, but lacked flexibility. Especially in sensitivity and optimization research, starts and finishes or durations of activities are systematically modified, as are the billing-to-payment delay, or periodic cash flows. Even small changes may create a substantially different cash flow profile. A decomposed approach is not only arduous, but also impossible in some cases. Patterns in periodic cash flows are thus better modeled by singularity functions.

3 SIGNAL FUNCTION

Equation (1) is the basic formula for singularity functions, where y is the independent variable and $z(y)$ is the dependent variable. The term in the pointed brackets performs a switching operation. For example, if

y is smaller than the cutoff a, then the singularity functions returns zero. If it is equal to or larger than a, then the term with round brackets will be returned. If the exponent $n = 0$, then $z(y)$ is a constant for $y \geq a$, and the scale parameter s can change its intercept on the z-axis. But if $n = 1$, then $z(y)$ is a linear function for $y \geq a$, and s will change its slope. There could exist higher exponents, e.g. $n = 2$ for a parabolic case, but this is uncommon in cash flow models. Multiple basic terms in singularity functions can be added and subtracted. When y is increasing, these basic terms switch on one after another at their respective a, which mimics that cash flows grow in a chronological manner. Equations (2) and (3) are singularity functions with floor and ceiling operators to assist in modeling a signal function. The floor $\lfloor\ \rfloor$ and ceiling $\lceil\ \rceil$ operators return rounded down and up integer values of the operand, respectively. Equation (4) calculates the difference between the subtrahend $z_1(y)$ and the minuend $z_2(y)$. If $n_1 = n_2 = 1$ and $a_1 + 1 = a_2$, then Equation (5) is the general signal equation. If $s_1 = s_2 = 1$ and $a = 1$, then $z_{signal}(y) = \langle\lfloor y\rfloor - 0\rangle^1 - \langle\lceil y\rceil - 1\rangle^1$. In other words, it returns an infinite series spaced at intervals of multiples of one. Figure (1) displays this new signal function and Table 1(a) lists verification calculations of its behavior. The signal function is an impulse function, which fluctuates at intermittent points. Its start and finish, amplitude, and period are affected by its parameters, i.e. the scaling factor s, cutoff a, and operands in the floor and ceiling operators.

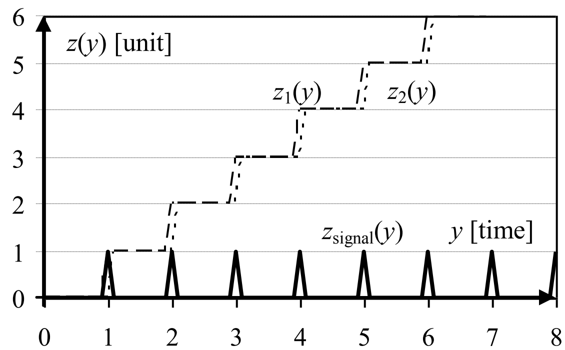

Figure 1: Unit signal function.

$$z(y) = s \cdot \langle y - a\rangle^n = \begin{cases} 0 & for\ y < a \\ s \cdot (y-a)^n & for\ y \geq a \end{cases}. \tag{1}$$

$$z_1(y) = s_1 \cdot \langle\lfloor y\rfloor - a_1\rangle^{n_1} = \begin{cases} 0 & for \lfloor y\rfloor < a_1 \\ s_1 \cdot (\lfloor y\rfloor - a_1)^{n_1} & for \lfloor y\rfloor \geq a_1 \end{cases}. \tag{2}$$

$$z_2(y) = s_2 \cdot \langle\lceil y\rceil - a_2\rangle^{n_2} = \begin{cases} 0 & for \lceil y\rceil < a_2 \\ s_2 \cdot (\lceil y\rceil - a)^{n_2} & for \lceil y\rceil \geq a_2 \end{cases}. \tag{3}$$

$$z_3(y) = z_1(y) - z_2(y) = s_1 \cdot \langle\lfloor y\rfloor - a_1\rangle^{n_1} - s_2 \cdot \langle\lceil y\rceil - a_2\rangle^{n_2}. \tag{4}$$

$$z_{signal}(y) = s_1 \cdot \langle\lfloor y\rfloor - (a-1)\rangle^1 - s_2 \cdot \langle\lceil y\rceil - a\rangle^1. \tag{5}$$

3.1 Starts and Finishes

As each activity has a start and finish, as do payments and charging interest; thus a signal function models an interval. Two stop terms are added by singularity functions per Equation (6), where a and b are the signal start and finish and $s_1 = s_2 = 1$. For example, if a signal starts at $y = 2$ and finishes at $y = 4$, its equation is $z_{signal}(y) = (\langle\lfloor y\rfloor - 1\rangle^1 - \langle\lfloor y\rfloor - 4\rangle^1) - (\langle\lceil y\rceil - 2\rangle^1 - \langle\lceil y\rceil - 5\rangle^1)$. Figure (2) displays this signal function and Table 1(b) lists its verification calculation. The activities of construction projects could start and finish at any time; signal function should also have this ability. Equation (7) expresses a signal function that starts and finishes at non-integer times. The signal starts at $a + \lambda$ and finishes at $b + \lambda$ time units. For example, if a signal function starts at $y = 2.5$ and finishes at $y = 4.5$, its equation will be $z_{\text{signal}}(y) = (\langle\lfloor y - 0.5\rfloor - 1\rangle^1 - \langle\lfloor y - 0.5\rfloor - 4\rangle^1) - (\langle\lceil y - 0.5\rceil - 2\rangle^1 - \langle\lceil y - 0.5\rceil - 5\rangle^1)$ per Figure (3); Table 1(c) lists its verification.

$$z_{signal}(y) = s_1 \cdot \left(\langle\lfloor y\rfloor - (a-1)\rangle^1 - \langle\lfloor y\rfloor - b\rangle^1\right) - s_2 \cdot \left(\langle\lceil y\rceil - a\rangle^1 - \langle\lceil y\rceil - (b+1)\rangle^1\right). \tag{6}$$

$$z_{signal}(y) = s_1 \cdot \left\langle\langle\lfloor y-\lambda\rfloor - (a-1)\rangle^1 - \langle\lfloor y-\lambda\rfloor - b\rangle^1\right\rangle - s_2 \cdot \left(\langle\lceil y-\lambda\rceil - a\rangle^1 - \langle\lceil y-\lambda\rceil - (b+1)\rangle^1\right). \tag{7}$$

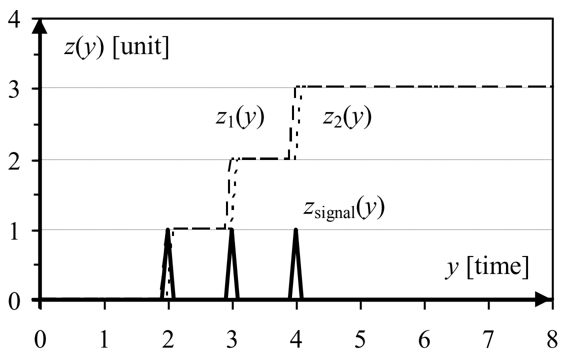

Figure 2: Signal with integer start and finish.

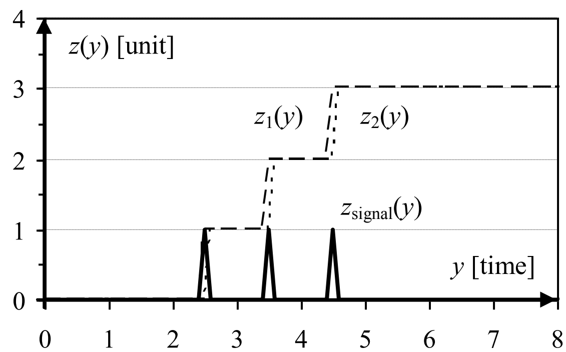

Figure 3: Signal with non-integer start and finish.

3.2 Amplitude

The signal function can be scaled as needed. The parameter s controls its amplitude. Same as the example in Figure (2), but with different $s = 2$, Figure 4 shows $z_{\text{signal}}(y) = 2 \cdot (\langle\lfloor y\rfloor - 1\rangle^1 - \langle\lfloor y\rfloor - 4\rangle^1) - 2 \cdot (\langle\lceil y\rceil - 2\rangle^1 - \langle\lceil y\rceil - 5\rangle^1)$. Another powerful feature of the signal function is that its amplitude can change if the signal starts and finishes at non-integer values. For example, for non-integer a and b it could be $z_{\text{signal}}(y) = (\langle\lfloor y\rfloor - 0.6\rangle^1 - \langle\lfloor y\rfloor - 3.8\rangle^1) - (\langle\lceil y\rceil - 1.6\rangle^1 - \langle\lceil y\rceil - 4.8\rangle^1)$. Figure (5) displays it and Table 1(d) lists its verification.

Table 1: Calculation verification.

	(a) Unit Signal			(b) Integer S and F			(c) Non-Integer S and F			(d) Non-Integer Cutoffs		
y	$z_1(y)$	$z_2(y)$	$z_{\text{signal}}(y)$	$z_1(y)$	$z_2(y)$	$z_{\text{signal}}(y)$	$z_1(y)$	$z_2(y)$	$z_{\text{signal}}(y)$	$z_1(y)$	$z_2(y)$	$z_{\text{signal}}(y)$
0.0	0	0	0	0	0	0	0	0	0	0.0	0.0	0.0
0.5	0	0	0	0	0	0	0	0	0	0.0	0.0	0.0
1.0	1	0	1	0	0	0	0	0	0	0.4	0.0	0.4
1.5	1	1	0	0	0	0	0	0	0	0.4	0.4	0.0
2.0	2	1	1	1	0	1	0	0	0	1.4	0.4	1.0
2.5	2	2	0	1	1	0	1	0	1	1.4	1.4	0.0
3.0	3	2	1	2	1	1	1	1	0	2.4	1.4	1.0
3.5	3	3	0	2	2	0	2	1	1	2.4	2.4	0.0
4.0	4	3	1	3	2	1	2	2	0	3.2	2.4	0.8
4.5	4	4	0	3	3	0	3	2	1	3.2	3.2	0.0
5.0	5	4	1	3	3	0	3	3	0	3.2	3.2	0.0
5.5	5	5	0	3	3	0	3	3	0	3.2	3.2	0.0

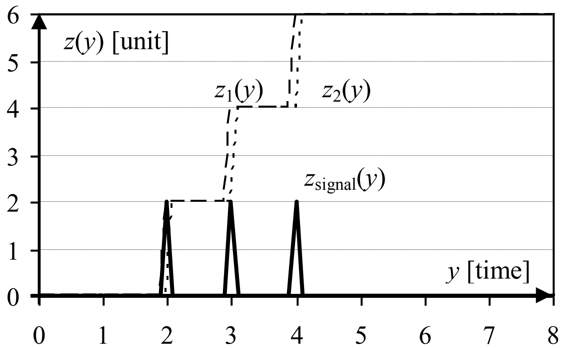

Figure 4: Signal function with amplitude 2.

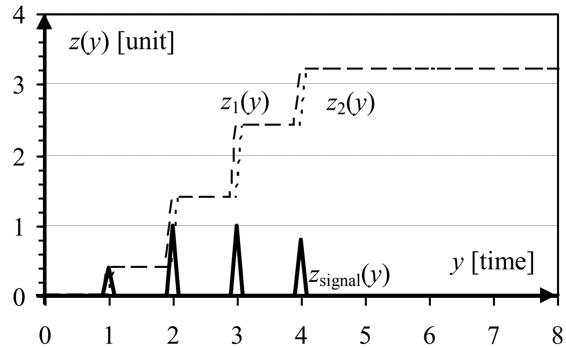

Figure 5: Signal function with non-integer cutoff.

3.3 Period

The period of the signal function could take on any repetitive value. Equation (8) shows the general signal function where n is its cycle time, which can be integer or non-integer duration. Note that the signal starts at time $n \cdot a + \lambda$ and finishes at time $n \cdot b + \lambda$. If the period is larger than one, e.g. equal to two, and the signal starts at $y = 2.5$ and finishes at $y = 8.5$, its equation is $z_{signal}(y) = (\langle\lfloor (y - 0.5) / 2\rfloor - 0\rangle^1 - \langle\lfloor (y - 0.5) / 2\rfloor - 4\rangle^1) - (\langle\lceil (y - 0.5) / 2\rceil - 1\rangle^1 - \langle\lceil (y - 0.5) / 2\rceil - 5\rangle^1)$ as shown in Figure (6). If a period is smaller than one, e.g. one half, and the signal starts at $y = 1$ and finishes at $y = 2.5$, its equation is $z_{signal} = (\langle\lfloor (y - 0.5) / 0.5\rfloor - 0\rangle^1 - \langle\lfloor (y - 0.5) / 0.5\rfloor - 4\rangle^1) - (\langle\lceil (y - 0.5) / 0.5\rceil - 1\rangle^1 - \langle\lceil (y - 0.5) / 0.5\rceil - 5\rangle^1)$ as Figure (7) shows.

$$z_{signal}(y) = \left(\left\langle \left\lfloor \frac{y-\lambda}{n} \right\rfloor - (a-1) \right\rangle^1 - \left\langle \left\lfloor \frac{y-\lambda}{n} \right\rfloor - b \right\rangle^1 \right) - \left(\left\langle \left\lceil \frac{y-\lambda}{n} \right\rceil - a \right\rangle^1 - \left\langle \left\lceil \frac{y-\lambda}{n} \right\rceil - (b+1) \right\rangle^1 \right). \quad (8)$$

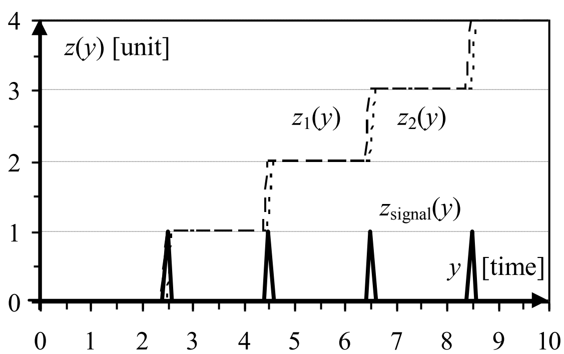

Figure 6: Signal function with period 2.

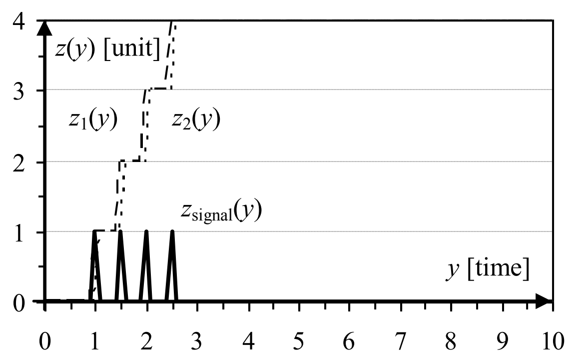

Figure 7: Signal function with period 0.5.

4 APPLICATIONS IN CASH FLOW MODELS

This paper assumes that cash flows are modeled from the contractor's perspective and that the activity cost grows linearly over its duration (Elazouni and Metwally 2005). Each activity has a known planned start, finish, and duration a_S, a_F, and D, cost C, markup rate M, and billing-to-payment-delay b that depends on contract terms and conditions. Shifts d_1 and delays d_2 change dates: A shift affects a_S and a_F, but not D. A delay modifies a_F and D. Actual start and finish thus are $a_S^* = a_S + d_1$ and $a_F^* = a_F + d_1 + d_2$.

4.1 Payment Model

Indices are z_{signal} for the signal function, $z_{\text{pay\_signal}}$ for the signal at each payment time, $z_{\text{each\_pay}}$ for the non-cumulative payment at each payment time, $z_{\text{sum\_pay}}$ for the sum of all payments without considering future value (FV), $z_{\text{FV\_pay}}$ for the cumulative FV of all payments at each payment time, z_{int} for the future value of interest, $z_{\text{int\_signal}}$ for the signal at each charge interest time, $z_{\text{cost\_int}}$ for the cumulative cost with interest at each charge interest time, and $z_{\text{bal\_int}}$ for the cumulative balance with interest at each charge interest time.

4.1.1 Current Value Model

A signal function can be used for equally spaced sampling purposes when multiplied with other functions, e.g. a payment function. If $n = 1$, s is the slope of the linear function. Based on the assumption that cost grows linearly throughout an activity, its bills and payments have a slope, albeit stepped. Note that operator $\lfloor\ \rfloor$ rounds them down to yield that stepped growth, because they are issued incrementally. Multiplying this slope with a signal function can sample the cost, bill, and payment at specific occurrence times. The current value for each payment can be modeled by Equations (9) and (10). Equation (9) is the signal function for payments, which is active from the integer at or directly after an activity start (because bills are issued at the ends of each month) plus the billing-to-payment-delay b to the integer at or directly after its finish plus b. For example, for $b = 1$, if $a_S^* = 0.5$, then the first bill at $y = 1$ is to be paid at $y = 2$; if $a_F^* = 1.5$, then the last bill at $y = 2$ is to be paid at $y = 3$. The term $C \cdot (1 + M) / (D + d_2)$ in Equation (10) is the bill slope to yield the amount of payment in each subsequent period. Note that Equation (10) returns the current value at each payment time. Equation (11) sums all payments and returns their cumulative value.

$$z_{pay\_signal}(y)=\left(\left\langle\lfloor y\rfloor-\left(a_s^*+b\right)\right\rangle^1-\left\langle\lfloor y\rfloor-\left(a_F^*+b\right)\right\rangle^1\right)-\left(\left\langle\lceil y\rceil-\left(a_s^*+b+1\right)\right\rangle^1-\left\langle\lceil y\rceil-\left(a_F^*+b+1\right)\right\rangle^1\right). \quad (9)$$

$$z_{each\_pay}(y)=\frac{C\cdot(1+M)}{D+d_2}\cdot z_{pay\_signal}(y). \quad (10)$$

$$z_{sum\_pay}(y)=\sum z_{each\_pay}(y). \quad (11)$$

4.1.2 Future Value Model

Construction projects may extend over multiple years. It is therefore necessary to consider the *time value of money*. The goal is to calculate the FV of cumulative payments at any time. The *FV of an annuity* of accounting is similar to this problem, but assumes equal payment amounts, which is not guaranteed here. A periodic payment may have a fractional start and finish amount due to a fractional start and finish of an activity, e.g. if cost of \$100 are incurred from time 0.5 to 2.5 but billed monthly, then their bills and subsequent payments (assume M = 0%) would become \$25, \$50, and \$25. Equations (12) and (13) model the FV mathematically. In Equation (12), $z_{\text{each\_pay}}$ is the principal at each payment time, $(1+i)^{\langle\lfloor y\rfloor-(\ldots)\rangle}$ calculates compound interest, $\langle y-(\ldots)\rangle^0$ is a control term that returns zero if y is earlier than the payment time, but one if y is later than it. The first payment time is $\lfloor a_S^*\rfloor + 1 + b$. If an activity e.g. starts at $a_S^* = 0.25$, then the first payment is at $\lfloor 0.25\rfloor + 1$ plus any billing-to-payment-delay b. The second payment time adds two periods, andsoforth. The term $\lceil a_F^*\rceil + b$ is the last payment time. Equation (12) calculates the FV at any time y: The FV at the first payment time is the first payment; the FV at the second payment time is the second plus the first compounded over one period; the FV at the third payment time is the third plus the previous two compounded separately, andsoforth. As cost grows linearly over the activity duration, the periodic payments from the second to the second-last cover complete periods and are equal in amount. The first and last ones may be fractional, however. Equation (13) exploits the *collapsing* effect of mutually canceling out the middle terms to simplify Equation (12), leaving only the start and finish terms.

$$
\begin{aligned}
z_{FV\_pay}(y) = {} & z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 1\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^0 \\
& + z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 2\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 2\right)\right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + b + 2\right)\right\rangle^0 \\
& + z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 3\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 3\right)\right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + b + 3\right)\right\rangle^0 \\
& \vdots \\
& + z_{each\_pay}\left(\lceil a_F^* \rceil + b - 1\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b - 1\right)\right\rangle^1} \cdot \left\langle y - \left(\lceil a_F^* \rceil + b - 1\right)\right\rangle^0 \\
& + z_{each\_pay}\left(\lceil a_F^* \rceil + b\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b\right)\right\rangle^1} \cdot \left\langle y - \left(\lceil a_F^* \rceil + b\right)\right\rangle^0 .
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
z_{FV\_pay}(y) = {} & z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 1\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^0 \\
& + z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 2\right) \cdot \frac{1}{i} \cdot \left[(1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} - (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b - 1\right)\right\rangle^1} \right] \\
& + z_{each\_pay}\left(\lceil a_F^* \rceil + b\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b\right)\right\rangle^1} \cdot \left\langle y - \left(\lceil a_F^* \rceil + b\right)\right\rangle^0 .
\end{aligned} \tag{13}
$$

The second term [$(1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} - (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b - 1\right)\right\rangle^1}$] in Equation (13) creates two advantages; first, if y is smaller than the second payment time, the exponents of $(1 + i)$ terms in subtrahend and minuend are both zero, which returns $(1 + i)^0 - (1 + i)^0 = 0$; second, if y is larger than the second payment time, then the factor $1 / i \cdot$ [$(1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} - (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b - 1\right)\right\rangle^1}$] calculates the compound interest of payments in complete 'middle' periods. At e.g. the second payment time, it returns the second payment, $(1 + i)^0 = 1 / i \cdot [(1 + i)^1 - (1 + i)^0] = 1$; at the third payment time, it returns the third plus the second with compound interest, $(1 + i)^1 + (1 + i)^0 = 1 / i \cdot [(1 + i)^2 - (1 + i)^0]$; at the second-last payment time, it returns the sum of the middle payments, $(1 + i)^n + \ldots + (1 + i)^1 + (1 + i)^0 = 1 / i \cdot [(1 + i)^{n+1} - (1 + i)^0]$; and at the last payment time, it returns the sum of the middle payments with compound interest, $(1 + i)^n + 1 + \ldots + (1 + i)^2 + (1 + i)^1 = 1 / i \cdot [(1 + i)^{n+2} - (1 + i)^1]$. Note that the interest will be different if other time lags apply.

If known values e.g. are $D = 2$ months, $C = \$200{,}000$, $a_S = 0$ months, $d_1 = 0.25$ months, $d_2 = 1.\overline{3}$ months, $b = 1$ month, interest $i = 5\%$/month and $M = 20\%$ of the cost, then the bill slope is $C \cdot (1 + M) / (D + d_2) = \$72{,}000$, i.e. the actual total cost divided by actual total duration of the activity. Cost is typically considered negative and its profile should be drawn below the horizontal axis in a cash flow diagram; however, this paper uses the absolute value of cost to keep the figures compact. Figure (8) shows each payment, which is a pulse, as a thin solid line, the profiles of cost and the sum of the payments as dashed lines, the FV of payments as a thick solid line, and the balance profile as a dashed line below the time axis, which is the combined effects of cost with interest (introduced in the following section) and the FV of payments. Table (2) lists periodic values of Equations (9), (10), and (13) for this payment model.

Table 2: Payment model values of Equations (9), (10), and (13).

y [months]	$z_{pay\ signal}(y)$ [unit]	$z_{pay}(y)$ [\$k]	$z_{FV\ pay}$ [\$k]
1	0.000	00	0
2	0.750	54	54
3	1.000	72	$54 \cdot (1 + i) + 72 = 128.7$
4	1.000	72	$54 \cdot (1 + i)^2 + 72 \cdot (1 + i)^1 + 72 = 207.135$
5	0.583	42	$54 \cdot (1 + i)^3 + 72 \cdot (1 + i)^2 + 72 \cdot (1 + i) + 42 = 259.4917$

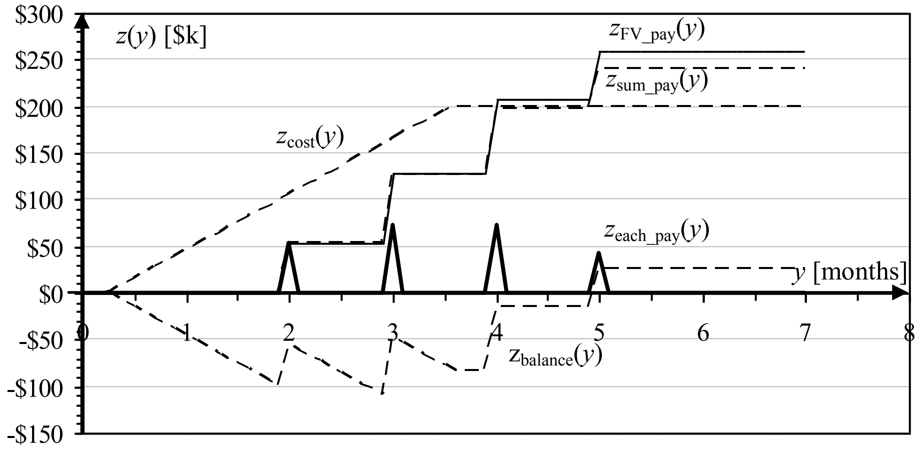

Figure 8: Cash flow model profiles for example.

4.2 Charging Interest Model

To model periodic cash outflows, a need exists to determine the interest that must be charged at the end of each period. An exact interest calculation for linearly growing cost is used per Equations (14) through (17) (Lucko 2013), where *A* is the payment (akin to an annuity) that is repeated at the ends of integer time periods, *D* is the activity duration, and $\delta = \lceil a_F \rceil - a_F$ is the partial period from a_F to the next integer time.

$$FV_{int} = A \cdot \left[(1+i)^{D+\delta} - (1+i)^{\delta}\right] / \ln(1+i). \tag{14}$$

$$FV_{int} = A \cdot \left[(1+i)^{D} - 1\right] / \ln(1+i) \text{ if balance grows in second half of one period.} \tag{15}$$

$$FV_{int} = A \cdot \left[(1+i) - (1+i)^{\delta}\right] / \ln(1+i) \text{ if balance grows only in first half, then remains constant.} \tag{16}$$

$$FV_{int} = A \cdot i / \ln(1+i) \text{ if linearly growing balance across one period.} \tag{17}$$

Equation (18) is the signal function to charge interest on cost, which is active from the integer at or directly after an activity start (because interest is charged at the ends of each period), to the integer at or directly after its finish. It is powerful and can charge interest on any balance after costs occur, if payments have not yet fully reduced it to a zero value. Equation (19) yields the FV of cost with compound interest.

$$z_{int\_signal}(y) = \left[\left\langle \lfloor y \rfloor - a_S^* \right\rangle^1 - \left\langle \lfloor y \rfloor - a_F^* \right\rangle^1\right] - \left[\left\langle \lceil y \rceil - \left(a_S^* + 1\right) \right\rangle^1 - \left\langle \lceil y \rceil - \left(a_F^* + 1\right) \right\rangle^1\right]. \tag{18}$$

$$\begin{aligned} z_{cost\_int}(y) = {} & \frac{C}{D+d_2} \cdot \frac{1}{\ln(1+i)} \cdot \Bigg(\left[(1+i)^{z_{int\_signal}\left(\lfloor a_S^* \rfloor + 1\right)} - 1 \right] \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + 1\right) \right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + 1\right) \right\rangle^0 \\ & + \left[(1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + 1\right) \right\rangle^1} - (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil - 1\right) \right\rangle^1} \right] + \left[(1+i) - (1+i)^{1 - z_{int\_signal}\left(\lceil a_F^* \rceil\right)} \right] \cdot (1+i)^{\left\langle \lfloor y \rfloor - \lceil a_F^* \rceil \right\rangle^1} \cdot \left\langle y - \lceil a_F^* \rceil \right\rangle^0 \Bigg). \end{aligned} \tag{19}$$

Equation (19) has three terms to calculate the exact compound interest for the start, middle, and finish part of the cost, respectively, where $C / (D + d_2)$ is the cost slope. The factor $1 / \ln(1 + i)$ can be extracted from them. Equations (15) and (16) express the cost with exact interest and compounds it for fractional start and finish. The term $C / (D + d_2) \cdot 1 / \ln(1 + i) \cdot [(1+i)^{z_{int\_signal}\left(\lfloor a_S^* \rfloor + 1\right)} - 1]$ is the principal at the charge first interest time, $(1+i)^{\langle \lfloor y \rfloor - (\ldots) \rangle}$ calculates compound interest, and $\langle y - (\ldots) \rangle^0$ is a control term that is zero unless y is larger than the charge first interest time Note that if the start and finish are integers, then $\delta = 0$ and Equations (15) and (16) become equal to Equation (17). The middle term in Equation (19) calculates the compound interest of the middle periods. Note that i in the numerator of Equation (17) and $1 / i$ in the denominator of Equation (19) cancel out, so that only $\left(1+i\right)^{\langle \lfloor y \rfloor - (\lfloor a_S^* \rfloor + b + 1) \rangle^1} - \left(1+i\right)^{\langle \lfloor y \rfloor - (\lceil a_F^* \rceil + b - 1) \rangle^1}$ remains.

Continuing this example, Figure (9) shows the charge interest signal of Equation (18). Note its non-integer activity start and finish, i.e. fractional signals for them. Figure (10) shows the profile of the cost with interest. Its stepped shape is caused by interest being charged only at the end of each month; being constant within periods. Table (3) lists values of Equations (18) and (19) for this charge interest model.

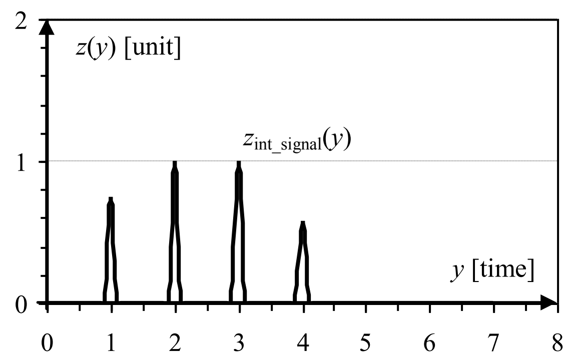

Figure 9: Profile for charge interest signal.

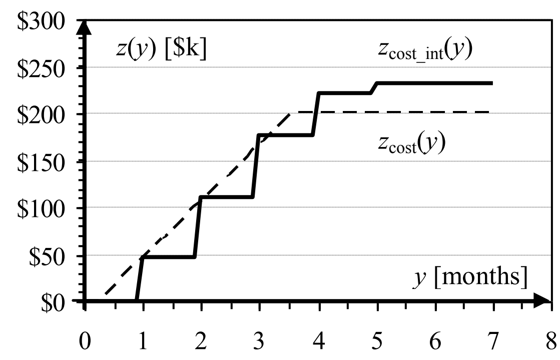

Figure 10: Cost with interest for example.

Table 3: Charge interest model values of Equations (18) and (19).

y [months]	$z_{\text{int\_signal}}(y)$ [unit]	$z_{\text{cost\_int}}(y)$ [\$k]
1	0.750	$60 \cdot [(1+i)^{0.75} - 1] / \ln(1+i) = 45.8335$
2	1.000	$45.8335 \cdot (1+i)^1 + 60 \cdot i / \ln(1+i) = 109.6129$
3	1.000	$45.8335 \cdot (1+i)^2 + 60 \cdot i / \ln(1+i) \cdot [(1+i)^1 + 1] = 176.5814$
4	1.000	$45.8335 \cdot (1+i)^3 + 60 \cdot i / \ln(1+i) \cdot [(1+i)^2 + (1+i)^1 + 1] = 221.6424$
5	0.583	$45.8335 \cdot (1+i)^4 + 60 \cdot i / \ln(1+i) \cdot [(1+i)^3 + (1+i)^2 + (1+i)^1 + 1]$ $+ 60 \cdot [(1+i) - (1+i)^{1-0.583}] / \ln(1+i) = 232.7245$

4.3 Balance Model

The cumulative interactions between cash outflows and inflows elements create a sawtooth shaped balance. Said balance is the difference between cash outflows (costs and interest) and subsequent inflows (payments). Subtracting the cost with interest from the FV of payment is the balance with interest. Equation (20) is the difference of Equations (13) and (19), which returns the value of each balance with interest at each charge interest time. To verify its correctness, it is compared against the aforementioned approach that uses an infinitesimal offset ε and calculates each value individually. Table (4) lists results of the former decomposed chronological approach, which requires arduous intermediate steps and is therefore significantly longer; while Table (5) lists the results of the latter, which uses the new signal function.

Integer times of period ends, here 1, 2, 3, 4, and 5, are when cost is assessed without interest. At an integer time plus ε, e.g. $1 + \varepsilon$, $2 + \varepsilon$, etc. the interest is charged on the balance at said integer time. Otherwise, the problem would be encountered that two different function values would exist at the same time. The exact interest calculation (Lucko 2013) is used once again. Finally, at the integer time plus $2 \cdot \varepsilon$, e.g. $1 + 2 \cdot \varepsilon$, $2 + 2 \cdot \varepsilon$, the payment is added to reduce the balance. This methods yields the correct result, but is arduous, especially when many cash flow elements interact. Another problem is its inflexibility that restricts the ability to perform optimization research. Changing just one parameter may yield a significantly different balance in the cash flow profile, and would require the entire calculation to be repeated. As there may exist thousands of permutation of how parameters can be combined, this becomes a near impossible task. The new signal function using singularity functions, on the other hand, enables a direct calculation.

$$
\begin{aligned}
&z_{bal\_int}(y) = z_{FV\_pay}(y) - z_{\cos t\_int}(y) \\
&= \left(\begin{aligned} &z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 1\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^0 \\
&+ z_{each\_pay}\left(\lfloor a_S^* \rfloor + b + 2\right) \cdot \frac{1}{i} \cdot \left[(1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + b + 1\right)\right\rangle^1} - (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b - 1\right)\right\rangle^1} \right] \\
&+ z_{each\_pay}\left(\lceil a_F^* \rceil + b\right) \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil + b\right)\right\rangle^1} \cdot \left\langle y - \left(\lceil a_F^* \rceil + b\right)\right\rangle \end{aligned} \right) \\
&- \left(\begin{aligned} &\frac{C}{D + d_2} \cdot \frac{1}{\ln(1+i)} \cdot \left(\left[(1+i)^{z_{int\_signal}\left(\lfloor a_S^* \rfloor + 1\right)} - 1 \right] \cdot (1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + 1\right)\right\rangle^1} \cdot \left\langle y - \left(\lfloor a_S^* \rfloor + 1\right)\right\rangle^0 \right. \\
&+ \left[(1+i)^{\left\langle \lfloor y \rfloor - \left(\lfloor a_S^* \rfloor + 1\right)\right\rangle^1} - (1+i)^{\left\langle \lfloor y \rfloor - \left(\lceil a_F^* \rceil - 1\right)\right\rangle^1} \right] \\
&\left. + \left[(1+i) - (1+i)^{1 - z_{int\_signal}\left(\lceil a_F^* \rceil + 1\right)} \right] \cdot (1+i)^{\left\langle \lfloor y \rfloor - \lceil a_F^* \rceil \right\rangle^1} \cdot \left\langle y - \lceil a_F^* \rceil \right\rangle^0 \right) \end{aligned} \right).
\end{aligned} \tag{20}
$$

Table 4: Balance with interest using infinitesimal offset ε.

y [months]	Cost [$k]	Interest [$k]	Payment [$k]	Balance [$k]
1	-45	0	0	-45.0000
$1 + \varepsilon$	0	$60 \cdot [(1 + i)^{0.75} - 1] / \ln(1 + i) - 45 = 0.8335$	0	**-45.8335**
$1 + 2\varepsilon$	0	0	0	-45.8335
2	-60	0	0	-105.8335
$2 + \varepsilon$	0	$45.8335 \cdot i + 60 \cdot i / \ln(1 + i) - 60 = 3.779475$	0	-109.6129
$2 + 2\varepsilon$	0	0	54	**-55.6129**
3	-60	0	0	-115.6129
$3 + \varepsilon$	0	$55.6129 \cdot i + 60 \cdot i / \ln(1 + i) - 60 = 4.2684479$	0	-119.8814
$3 + 2\varepsilon$	0	0	72	**-47.8814**
4	-35	0	0	-82.8814
$4 + \varepsilon$	0	$47.8814 \cdot i + 60 \cdot [(1+i) - (1+i)^{1-0.583}] / \ln(1+i) - 35 = 3.6260$	0	-86.5074
$4 + 2\varepsilon$	0	0	72	**-14.5074**
5	0	0	0	-14.5074
$5 + \varepsilon$	0	$14.5074 \cdot i = 0.72537$	0	-15.23277
$5 + 2\varepsilon$	0	0	42	**26.7672**

Table 5: Balance with interest using new signal function.

y [months]	$z_{\text{FV-pay}}(y)$ [$k]	$z_{\text{cost int}}(y)$ [$k]	$z_{\text{bal int}}(y)$ [$k]
1	000.0000	045.8335	-45.8335
2	054.0000	109.6129	-55.6129
3	128.7000	176.5814	-47.8814
4	207.1350	221.6424	-14.5074
5	259.4917	232.7245	26.7672

5 CONCLUSIONS

Modeling, analyzing, and optimizing the many periodic cash outflows and inflows are typical of the financial management challenges on projects in the construction industry. The disparate, shifted, and cumulative interactions between their elements create an uneven 'sawtooth' shape of their balances that has proved to be extremely difficult to model. Previous researchers had resorted to an infinitesimal offset ε to model such phenomena in an accurate but very piecewise and disjointed manner. But its inflexibility seriously hampers optimization research on cash flows. The contributions to the body of knowledge of this paper revolve around the introduction of signal functions that are based on singularity functions. Specifically, they provides new integrated modeling capabilities. Periodic phenomena that frequently occur in both cash outflows and inflows can be newly modeled comprehensively and accurately using this method. Their various parameters of start, finish, and duration, as well as shift and delay at the activity level, and the billing-to-payment delay, amplitude, and periods at the project level can be easily customized within the various signal functions that have been derived. Combining them with an exact interest calculation, a structured manner of compounding it, cumulative balances, and future values thereof creates a comprehensive method to efficiently analyze and optimize cash flows. It is recommended that future research on cash flows explore signal functions even further. They have a noteworthy potential to facilitate optimization, specifically when there is a need to investigate the effects of interactions between their elements.

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