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# IMPROVED MONTE CARLO AND QUASI–MONTE CARLO METHODS FOR THE PRICE AND THE GREEKS OF ASIAN OPTIONS

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## ABSTRACT

An improved variance reduction method for accurate estimation of the price, delta, and gamma of Asian options in a single simulation is presented. It combines randomized quasi–Monte Carlo with very efficient new control variates, that are especially successful in reducing the variance of the pathwise derivative method used to simulate delta and gamma. To improve the performance of randomized quasi–Monte Carlo, we smooth the integrands by employing conditional Monte Carlo and reduce the effective dimension of the smoothed integrands by using principal component analysis. Numerical results show that the new method yields significant variance reduction for the price, for delta and for gamma.

## 1 INTRODUCTION

Asian options are path dependent derivative securities whose payoff is calculated using the average of the prices of the underlying asset. Pricing Asian options requires efficient numerical methods, since there is no closed form solution for their prices. Monte Carlo (MC) and quasi–Monte Carlo (QMC) methods are among the most widely used methods for Asian options. In fact, pricing Asian options is a classical simulation problem much studied by the researchers in the simulation literature. Many studies suggesting new simulation methods use Asian options as an example to show the effectiveness of their methods. On the other hand, computing the option sensitivities, which are known as greeks, attracts less attention, although their accurate computation is as important as the computation of the prices.

We consider Asian call options with payoff functions $(A-K)^+$, where $A=\sum_{i=1}^{d} S(t_i)/d$ is the arithmetic average of the prices of the underlying asset observed at $d$ time points $\{S(t_i), 1 \le i \le d\}$ and $K$ is the strike price. The option price is given by the discounted risk neutral expectation of the payoff $P = e^{-rT}\,\mathrm{E}\left[(A-K)^+\right]$, where $r$ is the deterministic risk free interest rate and $T$ is the maturity of the option. The time points $0=t_0<t_1<\ldots<t_d=T$ are assumed to be equidistant, i.e. $t_i - t_{i-1} = \Delta t = T/d$ for $i=1,\ldots,d$. We assume that the underlying asset price process $S(t)$ follows geometric Brownian motion (GBM) with a constant volatility parameter $\sigma$, that is $S(t)=S(0)\exp((r-\sigma^2/2)t+\sigma W(t))$, where $\{W(t), t\ge 0\}$ denotes standard BM.

Delta ($\Delta$) and gamma ($\Gamma$) are the two most important greeks of options. Delta is the sensitivity of the option price ($P$) with respect to the initial stock price $S(0)$, i.e. $\Delta = \partial P/\partial S(0)$. Gamma is the sensitivity of delta with respect to $S(0)$. So, it is given by the second derivative of the option price with respect to $S(0)$, i.e. $\Gamma = \partial^2 P/\partial S(0)^2$. In this paper, we suggest a new and efficient method for Monte Carlo estimation of the price, delta and gamma of Asian options in a single simulation. Our algorithm combines quasi-Monte Carlo with control variates and conditional Monte Carlo.

By using pathwise derivative (PD) and likelihood ratio (LR) methods, it is possible to obtain the estimates of the price, delta and gamma of Asian options in a single simulation, see e.g. Glasserman (2004), Boyle and Potapchik (2008). The PD method is based on the interchange of differentiation and expectation, whereas the LR method differentiates the density. The PD estimate of delta is given by $e^{-rT}A/S(0)\mathbf{1}_{\{A>K\}}$. The PD method is not applicable to gamma. But, as suggested by Glasserman (2004), we can combine PD and LR methods. The resulting LR+PD gamma estimator is $e^{-rT}Z_1/(S(0)^2\sigma\sqrt{\Delta t})\mathbf{1}_{\{A>K\}}K$, where $Z_1$ is the standard normal variate used to simulate the increment of $W(t)$ over the first time interval $(0,t_1]$.

Dingeç and Hörmann (2013) suggest a very efficient variance reduction method for pricing basket and Asian options under GBM assumption. Their method utilizes the combination of control variate (CV) and conditional Monte Carlo (CMC) techniques. Dingeç, Sak, and Hörmann (2014) extend this method to more general models than GBM. They also show that it is possible to obtain PD estimators of delta and gamma under the new variance reduction method. The main aim in this study is to further improve the performance of these methods and to obtain the price, delta and gamma of Asian options under GBM in a single simulation.

For many simulation problems QMC sampling can achieve a rate of convergence close to $O(1/n)$, clearly higher than the $O(1/\sqrt{n})$ rate of MC. However, QMC sampling for delta and gamma is often less successful than for the prices, as the estimators of delta and gamma include the indicator functions. Still we hoped that a proper application of QMC could further enhance the efficiency of the algorithm. Unfortunately our first experiments with QMC did not show any improvement of the rate of convergence. This result seemed in line with common observations in the literature that QMC sampling under variance reduction methods is less successful than under naive simulation. Numerical examples of Lemieux (2009) and L'Ecuyer (2009) show for example, that for Asian options, the variance reduction obtained by QMC under the classical CV method of Kemna and Vorst (1990) is clearly smaller than that obtained by QMC under naive simulation. Nevertheless, since the estimators in our CV method are smoothed by CMC, we had hoped for more success. At that point we remembered that it is a well known fact in the literature that not only the smoothness of the integrand but also the so-called effective dimension of a problem has a strong impact on the performance of QMC methods. Using principal component analysis (PCA) we are able to reduce the effective dimension of the problem considerably. We also add several new CVs that further increase the performance of our randomized quasi-Monte Carlo (RQMC) algorithm especially for gamma.

In Section 2, we explain the methods of Dingeç and Hörmann (2013) and Dingeç, Sak, and Hörmann (2014). Section 3 presents our improved simulation method for the price and the greeks. Numerical results are reported in Section 4, whereas Section 5 contains our conclusions.

## 2 AN EFFICIENT VARIANCE REDUCTION METHOD

### 2.1 Using Lower Bound of Curran as CV

Curran (1994) decomposes the payoff function into two parts by using the geometric average $G=\prod_{i=1}^{d}S(t_i)$ as a conditioning variable, i.e. $(A-K)^+=(A-K)^+\mathbf{1}_{\{G<K\}}+(A-K)^+\mathbf{1}_{\{G\geq K\}}$ and suggests to use the expectation of the second term $e^{-rT}\mathrm{E}\left[(A-K)^+\mathbf{1}_{\{G\geq K\}}\right]$ as a lower bound for the price. Since $G<A$, $(A-K)^+\mathbf{1}_{\{G\geq K\}}$ is equal to $(A-K)\mathbf{1}_{\{G\geq K\}}$ and thus the lower bound $LB=e^{-rT}\mathrm{E}\left[(A-K)\mathbf{1}_{\{G\geq K\}}\right]$ is available in closed form. Due to the high correlation between $G$ and $A$, the expectation of the other term $\mathrm{E}\left[(A-K)^+\mathbf{1}_{\{G<K\}}\right]$ is close to zero, and $LB$ is an accurate approximation of the price.

Dingeç and Hörmann (2013) propose to use the lower bound of Curran (1994) as a CV. Their price estimator is $Y_{CV}=P_A-c(V-\mathrm{E}[V])$, where $P_A=e^{-rT}(A-K)^+$ is the naive simulation estimate, $V=e^{-rT}(A-K)\mathbf{1}_{\{G\geq K\}}$ is the new CV, and $c$ is the CV coefficient. The expectation of the CV was proposed by Curran (1994) as a lower bound for the price $LB=e^{-rT}\mathrm{E}\left[(A-K)\mathbf{1}_{\{G\geq K\}}\right]$. Due to the high correlation between $A$ and $G$, $(A-K)^+\mathbf{1}_{\{G<K\}}$ is equal to zero, and so $P_A=V$ in most of the replications. Thus the optimal coefficient $c^*=\mathrm{Cov}\left(P_A,V\right)/\mathrm{Var}(V)$ is very close to one, and fixing $c=1$ does not cause any significant loss of variance reduction. The resulting price estimator is $e^{-rT}(A-K)^+\mathbf{1}_{\{G<K\}}+LB$. The

closed formula of the *LB* is

$$LB = e^{-rT}\left(\frac{S(0)}{d}\sum_{i=1}^{d} e^{ri\Delta t}\,\Phi(-k+a_i) - K\Phi(-k)\right), \tag{1}$$

where $a_i = \sigma\sqrt{\Delta t}\, i\,(d+1-(i+1)/2)/\sqrt{d\,(d+1)\,(2d+1)/6}$ and $k = (\log K - \mu_{\tilde{s}})/\sigma_{\tilde{s}}$, where $\mu_{\tilde{s}} = \log S(0) + (r-\sigma^2/2)\Delta t(d+1)/2$, and $\sigma_{\tilde{s}} = \sigma/d\,\sqrt{\Delta t\, d\,(d+1)\,(2d+1)/6}$, see Curran (1994) and Dingeç and Hörmann (2013) for more details.

## 2.2 CMC

To reduce the variance of $Y = (A-K)^+\mathbf{1}_{\{G<K\}}$, Dingeç and Hörmann (2013) suggest a CMC method, which removes the variance contribution of $G$ by integrating it out. They first write the arithmetic average as a function of the standardized log geometric average, $X \sim N(0,1)$, and a $d$ dimensional standard normal vector, $Z \sim N(0, I_d)$ (here $I_d$ denotes the identity matrix of size $d$). That is $A(x,z) = 1/d\sum_{i=1}^{d} e^{a_i x} s_i(z)$, where $s_i(z)$'s are given by the recursion

$$s_i(z) = s_{i-1}(z)\exp\left((r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\,\xi_i\right), \qquad \text{for } i = 1,\ldots,d,$$

where $s_0(z) = S(0)$, $\xi = z - \upsilon(\upsilon^\top z)$, and

$$\upsilon_i = \frac{d-i+1}{\sqrt{\sum_{j=1}^{d}(d-j+1)^2}} = \frac{d-i+1}{\sqrt{d\,(d+1)\,(2d+1)/6}}, \quad i = 1,\ldots,d. \tag{2}$$

The smoothed estimator is given by the conditional expectation $\mathrm{E}[Y|Z=z] = \int_{-\infty}^{k}(A(x,z)-K)^+\phi(x)dx = \int_{b(z)}^{k}(A(x,z)-K)\phi(x)dx$, where $\phi(x)$ is the standard normal density and $b(z)$ is the unique root of $A(x,z)-K = 0$ for a given fixed $z$, i.e. $A(b(z),z)-K=0$. The smoothed estimator has the following semi-closed form expression

$$\mathrm{E}[Y|Z=z] = 1/d\sum_{i=1}^{d} s_i(z)\, e^{a_i^2/2}\left[\Phi(k-a_i) - \Phi(b(z)-a_i)\right] - K\left[\Phi(k) - \Phi(b(z))\right], \tag{3}$$

where $\Phi(x)$ denotes the cdf of the standard normal distribution. The root $b(z)$ is evaluated by means of Newton-Raphson method.

## 2.3 Quadratic CVs

Dingeç and Hörmann (2013) reduce the variance of the smoothed estimator $\mathrm{E}[Y|Z]$ by introducing new CVs. They observe that there is a strong relationship between the two conditional expectations $\mathrm{E}[Y|Z]$ and $\mathrm{E}[V_L|Z]$, where $Y = (A-K)^+\mathbf{1}_{\{X<k\}} = (A-K)\mathbf{1}_{\{b(Z)<X<k\}}$ and $V_L = (A-K)\mathbf{1}_{\{b^*<X<k\}}$ with $b^* = h^{-1}(K)$, the unique root of $h(x) - K = \mathrm{E}[A|X=x] - K = 0$. Nevertheless, the observed relation is highly nonlinear, and using $\mathrm{E}[V_L|Z]$ as a single CV does not yield a large variance reduction. Therefore, a bivariate CV method is suggested with a quadratic CV. The resulting price estimator is $e^{-rT}(\mathrm{E}[Y|Z] - c_1(\Psi - \mathrm{E}[\Psi]) - c_2(\Psi^2 - \mathrm{E}[\Psi^2])) + LB$, where $\Psi = \mathrm{E}[V_L|Z]$ and $\{c_1, c_2\}$ are the CV coefficients. Like for the conditional expectation of $Y$, there is a simple semi-closed form solution for the conditional expectation of $V_L$. It is given by $\Psi = \sum_{i=1}^{d}\gamma_i s_i(Z) - K\eta$, where $\gamma_i = (1/d)\, e^{a_i^2/2}\left[\Phi(k-a_i) - \Phi(b^*-a_i)\right]$ and $\eta = \Phi(k) - \Phi(b^*)$. The root $b^* = h^{-1}(K)$ can be easily evaluated, as $h(x) = \mathrm{E}[A|X=x]$ is simply equal to $S(0)/d\sum_{i=1}^{d}\exp\left(a_i x + ri\Delta t - a_i^2/2\right)$. The closed form expressions for the expectations of the CVs are

$$\mathrm{E}[\Psi] = \sum_{i=1}^{d}\gamma_i S(0) H_i - K\eta, \qquad \text{and} \qquad \mathrm{E}[\Psi^2] = \sum_{i=1}^{d}\sum_{j=1}^{d}\gamma_i\gamma_j S(0)^2 M_{ij} - 2K\eta\sum_{i=1}^{d}\gamma_i S(0) H_i + (K\eta)^2, \tag{4}$$

where

$$M_{ij} = \exp\left((r-\sigma^2/2)(i+j)\Delta t + \sigma^2 \Delta t\,(C_{ii}+C_{jj}+2C_{ij})/2\right),$$
$$H_i = \exp\left((r-\sigma^2/2)i\Delta t + \sigma^2 \Delta t\, C_{ii}/2\right),$$

and $C_{ij} = \min\{i,j\} - a_i a_j/(\sigma^2\Delta t)$.

### 2.4 Sensitivity Estimation

Dingeç, Sak, and Hörmann (2014) extend the CV+CMC method of Dingeç and Hörmann (2013) to more general models than GBM. Moreover, they obtain the PD estimators of delta and gamma by using the derivatives of the smoothed price estimator. In this section, we derive the formulas for the GBM case. The PD estimators of Dingeç, Sak, and Hörmann (2014) for delta and gamma are

$$e^{-rT}\frac{\partial \mathrm{E}[Y|Z]}{\partial S(0)} + \frac{\partial LB}{\partial S(0)} \quad \text{and} \quad e^{-rT}\frac{\partial^2 \mathrm{E}[Y|Z]}{\partial S(0)^2} + \frac{\partial^2 LB}{\partial S(0)^2},$$

respectively. For the general model setting of Dingeç, Sak, and Hörmann (2014), it is possible to evaluate $LB$ and its derivatives only by numerical inversion of the characteristic functions. However, for the GBM case, we have a simple closed form solution for $LB$ given in (1). Furthermore, by using Leibniz rule, we obtain the derivatives of $LB = e^{-rT}\int_k^\infty (h(x)-K)\phi(x)dx$ as

$$\frac{\partial LB}{\partial S(0)} = e^{-rT}\left(\frac{1}{d}\sum_{i=1}^{d} e^{ri\Delta t}\,\Phi(-k+a_i) + \frac{(h(k)-K)\phi(k)}{S(0)\sigma_{\tilde{s}}}\right), \tag{5}$$

$$\frac{\partial^2 LB}{\partial S(0)^2} = \frac{e^{-rT}\phi(k)}{S(0)^2\sigma_{\tilde{s}}^2}[2h(k)\sigma_{\tilde{s}} - h'(k) - (h(k)-K)(\sigma_{\tilde{s}}-k)]. \tag{6}$$

Similar expressions were presented by Nielsen and Sandmann (2003) and Vanmaele et al. (2006) as approximation formulas for the greeks of Asian options under GBM.

The formulas for the derivatives of $\mathrm{E}[Y|Z]$ are given by

$$\frac{\partial \mathrm{E}[Y|Z]}{\partial S(0)} = \frac{1}{S(0)}\left[\frac{1}{d}\sum_{i=1}^{d} s_i\, e^{a_i^2/2}\,[\Phi(k-a_i)-\Phi(b-a_i)] - (A(k)-K)\phi(k)/\sigma_{\tilde{s}}\right] \tag{7}$$

$$\frac{\partial^2 \mathrm{E}[Y|Z]}{\partial S(0)^2} = \frac{1}{S(0)^2\sigma_{\tilde{s}}^2}\left(K^2\sigma_{\tilde{s}}^2\phi(b)/A'(b) + \phi(k)\left[A'(k) - 2A(k)\sigma_{\tilde{s}} + (A(k)-K)(\sigma_{\tilde{s}}-k)\right]\right), \tag{8}$$

where $A(x) \equiv A(x,z)$ and $A'(x) \equiv \partial A(x,z)/\partial x$.

## 3 IMPROVING THE PD METHOD

In this section, we introduce our improvements for the method explained in Section 2. We first suggest a new multiple CV method in Section 3.1 by using the derivatives of the quadratic CVs given in Section 2.3. Then in Section 3.2, we propose to use RQMC sampling. Since the estimators are smoothed by CMC, we expect a superior performance for RQMC. However, we observed in our experiments that smoothing by CMC is not enough to obtain a higher rate of convergence. It is also necessary to reduce the effective dimension of the smoothed integrands by using some transformation method. In Section 3.2.1, we explain the application of PCA sampling, which is one of the standard dimension reduction techniques in the literature.

It is a common observation in the literature that combination of RQMC with variance reduction techniques does not bring as much improvement as its use for naive simulation. So, it is interesting to investigate the performance of RQMC when it is combined with the newly proposed CVs. In Section 3.2.2, we present the combination of RQMC sampling with our new CVs.

### 3.1 New CVs

We realized that it is possible to reduce the variance of the estimators by using also the derivatives of the quadratic CVs of Dingeç and Hörmann (2013) as CVs. Our CV estimators for price, delta, and gamma are

$$
\begin{aligned}
Y_P &= e^{-rT}\left(\mathrm{E}[Y|Z] + c_P^\top(\Upsilon - \mathrm{E}[\Upsilon])\right) + LB \\
Y_D &= e^{-rT}\left(\frac{\partial \mathrm{E}[Y|Z]}{\partial S(0)} + c_D^\top(\Upsilon - \mathrm{E}[\Upsilon])\right) + \frac{\partial LB}{\partial S(0)} \\
Y_G &= e^{-rT}\left(\frac{\partial^2 \mathrm{E}[Y|Z]}{\partial S(0)^2} + c_G^\top(\Upsilon - \mathrm{E}[\Upsilon])\right) + \frac{\partial^2 LB}{\partial S(0)^2},
\end{aligned}
$$

where

$$
\Upsilon = \left(\Psi, \Psi^2, \frac{\partial \Psi}{\partial S(0)}, \frac{\partial^2 \Psi}{\partial S(0)^2}, \frac{\partial \Psi^2}{\partial S(0)}, \frac{\partial^2 \Psi^2}{\partial S(0)^2}\right)^\top, \tag{9}
$$

is the CV vector and $c_P, c_D$, and $c_G$ are the CV coefficient vectors for price, delta, and gamma, respectively. All of the above CVs contain some information about the variability of the smoothed estimators of the price and greeks. So, instead of using only two CVs, i.e. $\{\Psi, \Psi^2\}, \{\frac{\partial \Psi}{\partial S(0)}, \frac{\partial \Psi^2}{\partial S(0)}\}$, and $\{\frac{\partial^2 \Psi}{\partial S(0)^2}, \frac{\partial^2 \Psi^2}{\partial S(0)^2}\}$ for price, delta, and gamma, respectively, we use all possible CVs together. In our experiments, we observed that using all six CVs together yields significantly larger variance reduction than using only two of them. Also, using all CVs does not increase the computational time significantly.

The derivatives of $\Psi$ are given by

$$
\begin{aligned}
\frac{\partial \Psi}{\partial S(0)} =& \frac{1}{S(0)}\left[\frac{1}{d}\sum_{i=1}^{d} s_i e^{a_i^2/2}\left[\Phi(k-a_i) - \Phi(b^*-a_i)\right] - (A(k)-K)\phi(k)/\sigma_{\tilde{s}} + (A(b^*)-K)\phi(b^*)K/h'(b^*)\right] \\
\frac{\partial^2 \Psi}{\partial S(0)^2} =& \frac{\phi(k)}{S(0)^2\sigma_{\tilde{s}}^2}\left[A'(k) - 2A(k)\sigma_{\tilde{s}} + (A(k)-K)(\sigma_{\tilde{s}}-k)\right] \\
&+ \frac{\phi(b^*)K}{S(0)^2}\left[\frac{2A(b^*)}{h'(b^*)} - \frac{KA'(b^*)}{h'(b^*)^2} - (A(b^*)-K)\left(\frac{1}{h'(b^*)}\left(2 - \frac{Kh''(b^*)}{h'(b^*)^2}\right) - \frac{b^*K}{h'(b^*)^2}\right)\right].
\end{aligned}
$$

The derivatives of $\Psi^2$ are simply

$$
\frac{\partial \Psi^2}{\partial S(0)} = 2\Psi\frac{\partial \Psi}{\partial S(0)} \qquad \text{and} \qquad \frac{\partial^2 \Psi^2}{\partial S(0)^2} = 2\left[\left(\frac{\partial \Psi}{\partial S(0)}\right)^2 + \Psi\frac{\partial^2 \Psi}{\partial S(0)^2}\right].
$$

The expectation vector $\mathrm{E}[\Upsilon]$ is also available in closed form. Since $\Psi$ is a smooth function of $S(0)$, the order of expectation and differentiation can be interchanged. Therefore, by taking the derivatives of $\mathrm{E}[\Psi]$ and $\mathrm{E}[\Psi^2]$, we obtain the the expectations of the CVs as

$$
\begin{aligned}
\mathrm{E}\left[\frac{\partial \Psi}{\partial S(0)}\right] &= \frac{\partial \mathrm{E}[\Psi]}{\partial S(0)} = \frac{1}{d}\sum_{i=1}^{d} e^{ri\Delta t}\left[\Phi(k-a_i) - \Phi(b^*-a_i)\right] - \frac{(h(k)-K)\phi(k)}{S(0)\sigma_{\tilde{s}}} \\
\mathrm{E}\left[\frac{\partial^2 \Psi}{\partial S(0)^2}\right] &= \frac{\partial^2 \mathrm{E}[\Psi]}{\partial S(0)^2} = \frac{1}{S(0)^2\sigma_{\tilde{s}}^2}\left(K^2\sigma_{\tilde{s}}^2\phi(b^*)/h'(b^*) + \phi(k)\left[h'(k) - 2h(k)\sigma_{\tilde{s}} + (h(k)-K)(\sigma_{\tilde{s}}-k)\right]\right) \\
\mathrm{E}\left[\frac{\partial \Psi^2}{\partial S(0)}\right] &= \frac{\partial \mathrm{E}[\Psi^2]}{\partial S(0)} = \sum_{i=1}^{d}\sum_{j=1}^{d} M_{ij}\frac{\partial (S(0)^2\gamma_i\gamma_j)}{\partial S(0)} - 2K\sum_{i=1}^{d} H_i\frac{\partial (S(0)\gamma_i\eta)}{\partial S(0)} + \frac{\partial (K\eta)^2}{\partial S(0)} \\
\mathrm{E}\left[\frac{\partial^2 \Psi^2}{\partial S(0)^2}\right] &= \frac{\partial^2 \mathrm{E}[\Psi^2]}{\partial S(0)^2} = \sum_{i=1}^{d}\sum_{j=1}^{d} M_{ij}\frac{\partial^2 (S(0)^2\gamma_i\gamma_j)}{\partial S(0)^2} - 2K\sum_{i=1}^{d} H_i\frac{\partial^2 (S(0)\gamma_i\eta)}{\partial S(0)^2} + \frac{\partial^2 (K\eta)^2}{\partial S(0)^2},
\end{aligned}
$$

where

$$\frac{\partial(S(0)^2\gamma_i\gamma_j)}{\partial S(0)} = 2S(0)\gamma_i\gamma_j + S(0)^2(\gamma_{i0}\gamma_j + \gamma_i\gamma_{j0}),$$

$$\frac{\partial(S(0)\gamma_i\eta)}{\partial S(0)} = \gamma_i\eta + S(0)(\gamma_{i0}\eta + \gamma_i\eta_0), \qquad \frac{\partial(K\eta)^2}{\partial S(0)} = 2K^2\eta\eta_0,$$

$$\frac{\partial^2(S(0)^2\gamma_i\gamma_j)}{\partial S(0)^2} = 2\gamma_i\gamma_j + 4S(0)(\gamma_{i0}\gamma_j + \gamma_i\gamma_{j0}) + S(0)^2(\gamma_{i00}\gamma_j + 2\gamma_{i0}\gamma_{j0} + \gamma_i\gamma_{j00}),$$

$$\frac{\partial^2(S(0)\gamma_i\eta)}{\partial S(0)^2} = 2(\gamma_{i0}\eta + \gamma_i\eta_0) + S(0)(\gamma_{i00}\eta + 2\gamma_{i0}\eta_0 + \gamma_i\eta_{00}), \qquad \frac{\partial^2(K\eta)^2}{\partial S(0)^2} = 2K^2(\eta_0^2 + \eta\eta_{00}).$$

Here, $\gamma_{i0}, \gamma_{i00}$, $\eta_0$, and $\eta_{00}$ denote the first and second derivatives of $\gamma_i$ and $\eta$ with respect to $S(0)$. They are given by

$$\gamma_{i0} = \frac{\partial\gamma_i}{\partial S(0)} = \frac{e^{a_i^2/2}}{d}\left[\phi(k-a_i)k_0 - \phi(b^*-a_i)b_0^*\right],$$

$$\gamma_{i00} = \frac{\partial^2\gamma_i}{\partial S(0)^2} = \frac{e^{a_i^2/2}}{d}\left[\phi(k-a_i)(k_{00} - k_0^2(k-a_i)) - \phi(b^*-a_i)(b_{00}^* - (b_0^*)^2(b^*-a_i))\right],$$

$$\eta_0 = \frac{\partial\eta}{\partial S(0)} = \phi(k)k_0 - \phi(b^*)b_0^*, \qquad \text{and} \qquad \eta_{00} = \frac{\partial^2\eta}{\partial S(0)^2} = \phi(k)(k_{00} - k_0^2 k) - \phi(b^*)(b_{00}^* - (b_0^*)^2 b^*),$$

where $k_0 = \partial k/\partial S(0) = -1/(S(0)\sigma_{\bar{s}})$, $k_{00} = \partial^2 k/\partial S(0)^2 = 1/(S(0)^2\sigma_{\bar{s}})$,

$$b_0^* = \frac{\partial b^*}{\partial S(0)} = -\frac{K}{S(0)h'(b^*)}, \qquad \text{and} \qquad b_{00}^* = \frac{\partial^2 b^*}{\partial S(0)^2} = \frac{K}{S(0)^2 h'(b^*)}\left(2 - \frac{Kh''(b^*)}{h'(b^*)^2}\right).$$

### 3.2 RQMC

Another way of improving a simulation method is to employ RQMC sampling. The price ($P$), delta ($\Delta$), and gamma ($\Gamma$) can be written in the following integral form

$$(P,\Delta,\Gamma)^\top = e^{-rT}(I(f_P), I(f_D), I(f_G))^\top + \left(LB, \frac{\partial LB}{\partial S(0)}, \frac{\partial^2 LB}{\partial S(0)^2}\right)^\top,$$

where $(I(f_P), I(f_D), I(f_G))^\top = \left(\int_{[0,1)^d} f_P(u)du, \int_{[0,1)^d} f_D(u)du, \int_{[0,1)^d} f_G(u)du\right)^\top$ is the vector of the integrals of the functions $f_P(u) = \mathrm{E}[Y|Z = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))], f_D(u) = \partial f_P(u)/\partial S(0)$, and $f_G(u) = \partial^2 f_P(u)/\partial S(0)^2$, which are given by (3), (7), and (8), respectively. In MC simulation, we randomly sample uniform numbers from $[0,1)^d$ to approximate the integrals. When using RQMC, these uniform numbers are replaced by randomized independent copies of a low-discrepancy point set $P_n = \{u_1 \ldots, u_n\}$ where each $u_i \in [0,1)^d$. The randomization guarantees that each $u_i \sim U([0,1)^d)$, and therefore the estimators based on $P_n$ are unbiased. We are also able to obtain error bounds for the estimates by using $m$ independent copies of $P_n$. Lattices can be randomized by using random shifts, whereas for the randomization of digital nets, digital shift and scrambling methods are preferred, since the random shift does not preserve the low-discrepancy properties of digital nets.

The RQMC estimators of $(I(f_P), I(f_D), I(f_G))^\top$ are given by

$$(\hat{\mu}_P, \hat{\mu}_D, \hat{\mu}_G)^\top = \left(\frac{1}{m}\sum_{l=1}^m \hat{Y}_{P,l}, \frac{1}{m}\sum_{l=1}^m \hat{Y}_{D,l}, \frac{1}{m}\sum_{l=1}^m \hat{Y}_{G,l}\right)^\top,$$

where $(\hat{Y}_{P,l},\hat{Y}_{D,l},\hat{Y}_{G,l})^\top, l = 1,\ldots,m$, are

$$(\hat{Y}_{P,l},\hat{Y}_{D,l},\hat{Y}_{G,l})^\top = \left(\frac{1}{n}\sum_{i=1}^{n} f_P(\tilde{u}_{i,l}), \frac{1}{n}\sum_{i=1}^{n} f_D(\tilde{u}_{i,l}), \frac{1}{n}\sum_{i=1}^{n} f_G(\tilde{u}_{i,l})\right)^\top, \tag{10}$$

and $\{\tilde{u}_{1,l},\ldots,\tilde{u}_{n,l}\}, l = 1,\ldots,m$, are independent randomized copies of the point set $P_n = \{u_1,\ldots,u_n\}$.

It is a well known fact that the smoothness of the integrand $f(u)$ has a significant impact on the performance of RQMC. In fact, Owen (1997) shows that for smooth integrands, the variance of the RQMC estimators using scrambled digital nets is of the order $O(n^{-3}\log^{d-1} n)$. As the functions, $f_P(u), f_D(u)$, and $f_G(u)$, in our simulation are smoothed by means of CMC, we expect a good performance. However, even for smooth integrands, RQMC has a smaller error than MC only for very large sample sizes, unless the effective dimension of the integrand is reduced by using special transformation techniques. In Section 3.2.1, we show that it is possible to reduce the effective dimension of the integrands $\{f_P(u), f_D(u), f_G(u)\}$ by principal component analysis (PCA). Then, in Section 3.2.2, we present the combination of the RQMC sampling with the new CV method suggested in Section 3.1.

### 3.2.1 PCA Sampling

Reducing the effective dimension of an integrand $f$ in truncation sense corresponds to the maximization of the variance explained by the first few uniforms. This improves the variance reduction of QMC sampling, as the first low dimensional projections of the point sets have better uniformity properties. In our algorithm, the CMC method is the first step of dimension reduction, since it transforms the integrand by smoothing out the most important variable. Indeed all variability coming from the geometric average is removed by means of CMC. As second step we try to reduce the effective dimension of this smoothed integrand.

The simulated random variates in our estimators (3), (7), and (8) are the $s_i$ values, which are given by the following function of the standard normal vector: $s_i(Z) = S(0)\exp\left((r-\sigma^2/2)i\Delta t + \sigma\sqrt{\Delta t}\,\xi_i\right)$, where $\xi = L(I_d - \upsilon\upsilon^\top)Z$ with $Z \sim N(0, I_d)$ and $L$ is a lower triangular matrix of which the entries on and below the diagonal are all equal to one. Our aim is to replace $L(I_d - \upsilon\upsilon^\top)$ by another matrix $Q$ so that the variance explained by $Z_1$ is maximized, then conditional on $Z_1$, the variance contribution of $Z_2$ is maximized, and so on. Under such a simulation scheme, the first coordinates of $Z$ explain most of the variance of the estimator. Therefore, if the $Z_i$'s are generated by inversion $Z_i = \Phi^{-1}(u_i)$, the effective dimension of the integrand is reduced.

To select the matrix $Q$, the two standard dimension reduction methods in the literature are Brownian bridge sampling and PCA sampling. The former is not suitable for our problem, as we do not simulate Brownian paths due to the use of CMC. Therefore, we employ the latter technique. Let $\Sigma$ denote the variance-covariance matrix of $\xi$. It is given by $\Sigma = \mathrm{Var}(\xi) = L(I_d - \upsilon\upsilon^\top)(L(I_d - \upsilon\upsilon^\top))^\top = L(I_d - \upsilon\upsilon^\top)L^\top$, since the matrix $B = I_d - \upsilon\upsilon^\top$ has the property that $BB^\top = B$. Our aim is to find a matrix $Q$ such that $QQ^\top = \Sigma$ and the effective dimension is reduced as much as possible. If we use PCA, we obtain $Q = VD^{1/2}$ where $D$ is a diagonal matrix that contains the eigenvalues $\{\lambda_1,\ldots,\lambda_d\}$ of the covariance matrix $\Sigma$ in decreasing order and $V$ is an orthogonal matrix whose columns are formed by the corresponding eigenvectors of $\Sigma$. Note that the rank of the covariance matrix $\Sigma = L(I_d - \upsilon\upsilon^\top)L^\top$ in our problem is $d-1$, and so the last eigenvalue is zero, $\lambda_d = 0$.

We also implemented the linear transformation (LT) method of Imai and Tan (2006) and observed that it slightly improves the variance reduction for the price estimates, whereas for delta and gamma PCA sampling performs better. Therefore, since the implementation of the LT method is more difficult and it does not yield a large improvement, we prefer to use PCA sampling in our algorithm.

By PCA sampling, we reduce the effective dimension, but we also increase the computational complexity from $O(nd)$ to $O(nd^2)$. Also, the setup for computing all eigenvalues and eigenvectors requires $O(d^3)$ operations. Still in our experiments we observed that even for large dimensions (like $d = 250$), the variance reduction obtained by PCA sampling is much larger than the resulting slow-down.

### 3.2.2 Combination of RQMC with CVs

It is possible to combine the CVs of Section 3.1 with the RQMC sampling. But the optimal CV coefficients of RQMC are different than those of MC, as noted by Hickernell, Lemieux, and Owen (2005). Our three RQMC estimators combined with the new CVs have the form

$$(\hat{P},\hat{\Delta},\hat{\Gamma})^\top = e^{-rT}\hat{\mu}_{cv} + \left(LB, \frac{\partial LB}{\partial S(0)}, \frac{\partial^2 LB}{\partial S(0)^2}\right)^\top,$$

where $\hat{\mu}_{cv} = (\hat{\mu}_{cv,P},\hat{\mu}_{cv,D},\hat{\mu}_{cv,G})^\top = (\frac{1}{m}\sum_{l=1}^m \hat{Y}_{cv,P,l}, \frac{1}{m}\sum_{l=1}^m \hat{Y}_{cv,D,l}, \frac{1}{m}\sum_{l=1}^m \hat{Y}_{cv,G,l})^\top$ is the vector of the sample means of the estimators $\hat{Y}_{cv,l} = (\hat{Y}_{cv,P,l},\hat{Y}_{cv,D,l},\hat{Y}_{cv,G,l})^\top, l = 1,\ldots,m$, which are given by

$$\hat{Y}_{cv,l} = \hat{Y}_l - C_{rqmc}^\top(\hat{\Upsilon}_l - \mathrm{E}[\Upsilon]), \tag{11}$$

where $\hat{Y}_l = \left(\hat{Y}_{P,l},\hat{Y}_{D,l},\hat{Y}_{G,l}\right)^\top$ is given by (10) and $C_{rqmc}$ is a $6\times 3$ matrix whose columns are the CV coefficient vectors $\{c_{P,rqmc}, c_{D,rqmc}, c_{G,rqmc}\}$ of the price, delta, and gamma, respectively. The CV vector $\hat{\Upsilon}_l$ is

$$\hat{\Upsilon}_l = \left(\frac{1}{n}\sum_{i=1}^n f_{\Upsilon,1}(\tilde{u}_{i,l}),\ldots,\frac{1}{n}\sum_{i=1}^n f_{\Upsilon,6}(\tilde{u}_{i,l})\right)^\top, \tag{12}$$

where $\{\tilde{u}_{1,l},\ldots,\tilde{u}_{n,l}\}, l = 1,\ldots,m$, are independent randomized copies of the low-discrepancy point set $P_n = \{u_1,\ldots,u_n\}$ and $f_\Upsilon(u) = (f_{\Upsilon,1}(u),\ldots,f_{\Upsilon,6}(u))^\top$ is the vector of the integrands of $\mathrm{E}[\Upsilon] = \int_{[0,1)^d} f_\Upsilon(u)du$. These integrands are given by (9) with $Z = \Phi^{-1}(u)$.

The matrix of the optimal CV coefficients for RQMC is $C^*_{rqmc} = \mathrm{Var}(\hat{\Upsilon})^{-1}\mathrm{Cov}(\hat{\Upsilon},\hat{Y})$, where $\hat{Y} = (\hat{Y}_P,\hat{Y}_D,\hat{Y}_G)^\top$ is the random vector given by (10) and $\hat{\Upsilon}$ is the CV vector of RQMC given by (12), see Hickernell, Lemieux, and Owen (2005) (that is, columns of $C^*_{rqmc}$ are $c^*_{P,rqmc} = \mathrm{Var}(\hat{\Upsilon})^{-1}\mathrm{Cov}(\hat{\Upsilon},\hat{Y}_P)$, $c^*_{D,rqmc} = \mathrm{Var}(\hat{\Upsilon})^{-1}\mathrm{Cov}(\hat{\Upsilon},\hat{Y}_D)$, and $c^*_{G,rqmc} = \mathrm{Var}(\hat{\Upsilon})^{-1}\mathrm{Cov}(\hat{\Upsilon},\hat{Y}_G)$ for price, delta, and gamma, respectively). Therefore, the optimal coefficients depend on the point set $P_n$ used in (10) and (12). Also, the variance reduction factor of the CVs changes depending on the point set $P_n$ and its size $n$. This is in fact different to the situation for MC where the variance reduction brought by the CVs is independent of the sample size.

Like for MC simulation, $C^*_{rqmc}$ can be estimated by the least squares method using the sample $(\hat{Y}_1,\hat{\Upsilon}_1),\ldots,(\hat{Y}_m,\hat{\Upsilon}_m)$, but the least squares estimation in the setting of RQMC is more difficult than the case of MC simulation. The three possible approaches that we can follow are "using the same sample", "using pilot runs" or "splitting", see Nelson (1990). Although the CV remedies of Nelson (1990) are originally suggested in the context of MC, they are useful also for RQMC.

If we use the same sample for the least squares estimation, the resulting estimator is biased due to the dependence between $\hat{C}_{rqmc}$ and $\hat{\Upsilon}$ in (11). It is known that for large sample sizes the bias is negligible compared to the standard deviation of the estimator. But in the RQMC setting, the sample size $m$ is usually selected as a comparatively small number, say $m \le 50$, in order to allocate the large proportion of the computational budget to the QMC sampling by selecting a large $n$. If $(\hat{Y},\hat{\Upsilon})$ follows a multivariate normal distribution, then using the same sample does not introduce any bias. Loh (2003) shows that for large $n$, RQMC estimators of the smooth integrands using scrambled digital nets tend to be normally distributed. On the other hand, L'Ecuyer and Munger (2010) show that the asymptotic distribution of the estimators using randomly shifted lattices may be far from normal.

If we use pilot runs for the estimation of $C^*_{rqmc}$, the resulting estimator $\hat{Y}_{cv,l}$ is unbiased. But using pilot runs increases the computational time, since each repetition of RQMC in the pilot run requires $n$ evaluations of the integrands. In the splitting approach, we use a different CV coefficient matrix $\hat{C}_{rqmc,-l}$ for each $\hat{Y}_{cv,l}$. The CV coefficients matrix for the $l$-th sample $\hat{C}_{rqmc,-l}$ is estimated by the least squares method and omitting $(\hat{Y}_l, \hat{\Upsilon}_l)$ from the sample. That is, $\hat{C}_{rqmc,-l}$ is estimated by using the sample $(\hat{Y}_1, \hat{\Upsilon}_1), \ldots, (\hat{Y}_{l-1}, \hat{\Upsilon}_{l-1}), (\hat{Y}_{l+1}, \hat{\Upsilon}_{l+1}) \ldots, (\hat{Y}_m, \hat{\Upsilon}_m)$ (see Steps 17 and 18 of Algorithm 1). Therefore, since $\hat{C}_{rqmc,-l}$ and $\hat{\Upsilon}_l$ are independent, we get an unbiased estimator. Also, it is faster than using pilot runs, as $m$ least square estimations clearly take less time than the repetitions of RQMC in the pilot run. The complexity of the splitting approach is $O(m^2)$, since $m$ linear regressions are performed for the samples of sizes $m-1$. As $m$ is small compared to $n$, splitting does not increase the overall computational time significantly. But, since we introduce dependence among the samples $\hat{Y}_{cv,l}$, the standard sample variance formula underestimates the true variance of the estimator. The bias of the variance estimate may be significant for small sample sizes. In our experiments, we observed that $m \geq 50$ is enough to get sufficiently reliable variance estimates. Therefore, as the splitting approach yields unbiased estimators $\hat{Y}_{cv,l}$ in small computational times, we decided to use this method in our algorithm.

### 3.2.3 Algorithm

Algorithm 1 presents the details our new method (RQMC+CV) for Asian call options.

**Algorithm 1** RQMC+CV Algorithm for Price, Delta, and Gamma of Asian Call Options

**Require:** Low-discrepancy point set $P_n$, number of repetitions $m$, maturity $T$, number of control points $d$, volatility $\sigma$, initial stock price $S(0)$, strike price $K$, risk free interest rate $r$.

**Ensure:** Option price, delta, and gamma estimates and their $(1-\alpha)$ confidence intervals.

1: Compute the lower bound and its derivatives $\left(LB, \partial LB/\partial S(0), \partial^2 LB/\partial S(0)^2\right)^\top$ by (1), (5), and (6).
2: Compute the matrices of the eigenvalues $D$ and the eigenvectors $V$ of $\Sigma = L(I_d - \upsilon\upsilon^\top)L^\top$.
3: Compute the matrix $Q \leftarrow VD^{1/2}$.
4: Compute $\mathrm{E}[\Upsilon]$ by (4) and the formulas in Section 3.1.
5: **for** $l = 1$ to $m$ **do**
6:    Generate a randomized copy $\tilde{P}_n = \{\tilde{u}_{1,l}, \ldots, \tilde{u}_{n,l}\}$ of $P_n = \{u_1, \ldots, u_n\}$.
7:    **for** $i = 1$ to $n$ **do**
8:       Compute $Z_j \leftarrow \Phi^{-1}(\tilde{u}_{i,l,j})$, for $j = 1, \ldots, d$, and set $\xi \leftarrow QZ$.
9:       Set $s_j(Z) \leftarrow S(0)\exp\left((r - \sigma^2/2)j\Delta t + \sigma\sqrt{\Delta t}\,\xi_j\right)$, for $j = 1, \ldots, d$.
10:       Compute $\mathrm{E}[Y|Z], \frac{\partial \mathrm{E}[Y|Z]}{\partial S(0)}$, and $\frac{\partial^2 \mathrm{E}[Y|Z]}{\partial S(0)^2}$ by (3), (7), and (8).
11:       Set $f_P(\tilde{u}_{i,l}) \leftarrow \mathrm{E}[Y|Z], f_D(\tilde{u}_{i,l}) \leftarrow \frac{\partial \mathrm{E}[Y|Z]}{\partial S(0)}$, and $f_G(\tilde{u}_{i,l}) \leftarrow \frac{\partial^2 \mathrm{E}[Y|Z]}{\partial S(0)^2}$.
12:       Compute $\Upsilon$ by (9), and set $f_\Upsilon(\tilde{u}_{i,l}) \leftarrow \Upsilon$.
13:    **end for**
14:    Compute $\hat{Y}_l$ and $\hat{\Upsilon}_l$ by (10) and (12).
15: **end for**
16: **for** $l = 1$ to $m$ **do**
17:    Compute $\hat{C}_{rqmc,-l}$ by the least squares method and omitting $(\hat{Y}_l, \hat{\Upsilon}_l)$ from the sample.
18:    Set $\hat{Y}_{cv,l} \leftarrow \hat{Y}_l - \hat{C}_{rqmc,-l}^\top(\hat{\Upsilon}_l - \mathrm{E}[\Upsilon])$.
19: **end for**
20: Set $(\hat{P}, \hat{\Delta}, \hat{\Gamma})^\top \leftarrow e^{-rT}\left(\frac{1}{m}\sum_{l=1}^m \hat{Y}_{cv,P,l}, \frac{1}{m}\sum_{l=1}^m \hat{Y}_{cv,D,l}, \frac{1}{m}\sum_{l=1}^m \hat{Y}_{cv,G,l}\right)^\top + \left(LB, \frac{\partial LB}{\partial S(0)}, \frac{\partial^2 LB}{\partial S(0)^2}\right)^\top$.
21: **return** $(\hat{P}, \hat{\Delta}, \hat{\Gamma})^\top$ and $(1-\alpha)$ error bounds.

## 4 NUMERICAL RESULTS

We implemented our RQMC+CV algorithm in R to perform experiments (R Core Team 2013). See the R-package `OptionPricing` available on `http://cran.r-project.org/`. As randomized low-discrepancy point sets, we use randomly shifted Korobov lattices and scrambled Sobol nets. We apply the baker's transformation to the shifted Korobov point sets in order to periodize the integrand, see Hickernell (2002). The generators of the Korobov lattices are taken from L'Ecuyer and Lemieux (2000). For the Sobol nets, we use the implementation of Bratley and Fox (1988) used in the R-package `randtoolbox`. The three scrambling techniques that we consider are the Owen type, the Faure-Tezuka type, and the Owen+Faure-Tezuka type, see e.g. Faure and Tezuka (2002) and Hong and Hickernell (2003). In our experiments with $m = 50$ and $n = 1021$ for the Korobov lattices and $n = 1024$ for the Sobol nets, we observed that all of the above randomized point sets give considerable variance reductions compared to the MC sampling. The observed variance reduction factors are similar and depend, as the error estimates, on the used seed. We report our QMC results only for the Korobov lattices as they led to the smallest error bounds for the used $n$.

In Table 1, we report the price, delta, and gamma estimates with their 95% error bounds for $m = 50$ and $n = 1021$. For comparison of the methods, the error bounds of the RQMC+CV method ($EB_{QCV}$), the MC+CV method presented in Section 3.1 ($EB_{MCV}$), naive RQMC ($EB_{QN}$), and naive MC ($EB_{MN}$) are reported. As explained in Section 1, for naive MC simulation of delta and gamma, we use the PD and the PD+LR methods. For RQMC sampling of the naive estimators, PCA sampling is employed. We consider long and short time to maturity cases, $T = 30$ and $T = 1$, respectively. The former is relevant for pricing unit-linked life insurance contracts, see e.g. Schrager and Pelsser (2004).

From Table 1, we observe that the use of RQMC sampling clearly reduces the variance of the MC+CV method. $EB_{QCV}$ (for $n = 1021$ and $m = 50$) is smaller than $EB_{MCV}$ (for $n = 50 \times 1021$) with factors ranging between 2 and 30. Also, as expected, for larger sample sizes, improvement of RQMC is more pronounced. For the sample size $n = 16381$, $EB_{MCV}/EB_{QCV}$ is always larger than 5. In Table 1, we also see that the variance reduction over naive MC is very large. $EB_{MN}/EB_{QCV}$ is always larger than 200. Also, when we compare $EB_{QCV}$ and $EB_{QN}$, we see that the new RQMC+CV method brings substantial variance reduction over naive RQMC. In Table 1, $EB_{QCV}$ is at least 88 times smaller than $EB_{QN}$.

It is also interesting to compare the variance reductions brought by RQMC sampling under naive simulation and our new variance reduction method. In the literature, it is commonly observed that the use of RQMC sampling under variance reduction techniques shows less success than their use for naive simulation. In our numerical results, we can observe that behavior. In Table 1, $EB_{MN}/EB_{QN}$ is often larger than $EB_{MCV}/EB_{QCV}$. Although in some cases we observe the opposite (that is $EB_{MCV}/EB_{QCV}$ is larger than $EB_{MN}/EB_{QN}$), the difference in the ratios is not substantial.

In our implementation, the RQMC+CV method is a bit slower than the MC+CV method. The main reason for it is the use of PCA sampling. Under PCA sampling, the simulation requires $O(nd^2)$ operations, and there is a setup with complexity $O(d^3)$. Therefore, for large $d$ and small $n$, the slow-down due to PCA sampling is significant. For $n = 1021$ and $m = 50$, we observed that RQMC+CV is about 1.2, 1.5, 1.6, and 2.5 times slower than MC+CV for the dimensions of $d = 12$, 30, 250, and 1000, respectively. The slow-down factors of RQMC+CV compared to naive MC are 6.6, 8.5, 8.3, and 15.8 for $d = 12$, 30, 250, and 1000, respectively. These slow-downs are clearly negligible compared to the huge error reductions reported in Table 1.

Finally we consider the speed of convergence for the different point sets used. We observed in all our QMC experiments rates of convergence closer to $O(1/n)$ than the $O(1/\sqrt{n})$ rate of MC. To estimate the order of convergence $\beta$ in $O(n^{-\beta})$, we use the slope of the linear regression line, $\log \hat{\sigma}_{rqmc+cv} = \alpha - \beta \log n$, where $\hat{\sigma}_{rqmc+cv}$ denotes the estimate of the standard deviation of the RQMC+CV method. For the Korobov point sets, we use $n \in \{1021, 2039, 4093, 8191, 16381, 32749, 65521\}$ with the corresponding generators $\{306, 280, 1397, 7151, 5693, 8363, 944\}$, whereas for Sobol nets, we use $n = 2^q$ with $q = 10, \ldots, 16$. For the option parameters of Table 1 with $T = 1$ and $d = 12$, the estimate $\hat{\beta}$ of the Korobov point set is ranging between 0.60 and 0.94 depending on the seed used for the randomization. For the scrambled Sobol nets with different seeds the lowest and the highest $\hat{\beta}$'s we observed in our experiments were $(0.62, 0.98)$, $(0.74, 1.13)$, and $(0.53, 0.85)$ for the Owen type, the Faure-Tezuka type, and the Owen+Faure-Tezuka type of scramblings, respectively.

Table 1: Numerical results for Asian call options with parameters $S(0) = 100, \sigma = 0.5$, and $r = 0.05$. The Korobov point sets are used with sample size $n = 1021$, generator 306, $m = 50$ random shifts, and baker's transformation. Estimates of RQMC+CV (result) and the 95% error bounds of RQMC+CV ($EB_{QCV}$), MC+CV ($EB_{MCV}$), naive RQMC ($EB_{QN}$), and naive MC ($EB_{MN}$) are reported.

| $T$ | $d$ | $K$ | | result | $EB_{QCV}$ | $EB_{MCV}$ | $EB_{QN}$ | $EB_{MN}$ |
|---|---|---|---|---|---|---|---|---|
| 30 | 30 | 200 | price | 34.792197 | 9.1e−04 | 7.5e−03 | 1.3e+00 | 4.0e+00 |
| | | | delta | 0.429068 | 1.7e−05 | 3.6e−05 | 1.3e−02 | 4.0e−02 |
| | | | gamma | 0.000722 | 1.6e−07 | 5.9e−07 | 1.4e−05 | 3.6e−05 |
| | | 400 | price | 29.021773 | 1.1e−03 | 3.2e−02 | 4.1e+00 | 5.0e+00 |
| | | | delta | 0.374549 | 9.8e−06 | 1.1e−04 | 4.1e−02 | 5.1e−02 |
| | | | gamma | 0.000841 | 1.6e−07 | 1.7e−06 | 1.8e−05 | 5.9e−05 |
| | | 600 | price | 25.614258 | 1.1e−03 | 1.5e−02 | 1.5e+00 | 3.3e+00 |
| | | | delta | 0.339596 | 8.7e−06 | 6.3e−05 | 1.5e−02 | 3.4e−02 |
| | | | gamma | 0.000880 | 1.9e−07 | 1.1e−06 | 2.5e−05 | 6.2e−05 |
| 1 | 12 | 50 | price | 50.224309 | 9.6e−07 | 2.6e−06 | 5.6e−03 | 2.6e−01 |
| | | | delta | 0.972950 | 5.3e−08 | 1.3e−07 | 1.9e−04 | 2.8e−03 |
| | | | gamma | 0.000479 | 2.8e−09 | 7.3e−09 | 3.3e−05 | 2.7e−04 |
| | | 100 | price | 13.121994 | 3.1e−06 | 1.1e−05 | 1.1e−02 | 2.1e−01 |
| | | | delta | 0.573201 | 1.0e−07 | 4.7e−07 | 1.7e−03 | 5.7e−03 |
| | | | gamma | 0.012375 | 9.5e−09 | 4.0e−08 | 7.0e−05 | 4.2e−04 |
| | | 150 | price | 2.097908 | 9.9e−07 | 3.9e−06 | 9.8e−03 | 9.5e−02 |
| | | | delta | 0.141291 | 1.7e−07 | 4.6e−07 | 6.3e−04 | 4.4e−03 |
| | | | gamma | 0.006806 | 5.1e−09 | 1.9e−08 | 1.3e−04 | 3.2e−04 |

## 5 CONCLUSIONS

We developed new variance reduction methods to obtain the price, delta, and gamma of Asian options in a single simulation. In our method, we combine RQMC sampling with a very efficient variance reduction technique. Especially for the improvement of the pathwise derivative (PD) method used to simulate delta and gamma, we introduced new CVs. To increase the efficiency of RQMC, the discontinuous estimators of delta and gamma are also smoothed by CMC. Finally PCA sampling is employed to reduce the effective dimension of the smoothed integrands. Numerical results show that the new method yields significant variance reduction compared to naive MC and RQMC methods.

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