THE PROBABILITY OF STABILITY
- AN EMPIRICAL DETERMINATION -

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1 - INTRODUCTION

This study will try to develop a probability function which can predict an important aspect of a dynamic system's behavior : its ability to be stable over a long period of time.

The data used by the study were taken to represent systems in general, and it is hoped that the results can be used by any discipline dealing with systems, in accordance with General System Theory's principles (von Bertalanffy, pp. 1 - 10, 1956), even though it was intended primarily for the behavioral sciences, where the need for dynamic considerations of systems behavior is much greater than in the natural sciences.

2(a) - DEFINITION OF A SYSTEM

A system can be defined as a vector differential or difference equation of the form (Hall, pp. 18 - 28) :

$$(1)\quad \begin{cases} 1.(a)\quad \dfrac{dX_i}{dt} = f^j\,(X_i) & i = 1,2,\ldots\ldots,n \quad j = 1,2,\ldots\ldots,m \\ 1.(b)\quad X_{t+1} = (A + I)\, X_t^j & j = 1,2,\ldots\ldots,m \end{cases}$$

In dealing with large systems we cannot trace their behavior over time (trajectory) and it is extremely difficult in many cases, if not impossible, to know the state of their variables at the equilibrium position of the system (von Bertalanffy, pp. 20, 1968). However, it is often possible to discover if a system is stable (converges) or unstable (diverges) over time.

2(b) - DEFINITION OF STABILITY

A system is stable if :

$$\lim \left| X_i\,(t) - \bar{X}_i \right| = 0 \quad \text{as } t \to \infty \qquad i = 1,2,\ldots\ldots,n$$

where $X_i(t)$ is the state of the system at period t, and $\bar{X}_i$ is the equilibrium state of the system, that is $f(\bar{X}_i) = 0$

3 - TESTING FOR SYSTEM STABILITY

There are several tests to determine the stability of a given system. It is known that if the real parts of the eigen values (λ_i) of the matrix of coefficients A are negative in (1a) or smaller than one in (1b), then the system will be stable.

In the case of non-linearity the same stability conditions exists as if (1) is linear ; but, instead of global, we test for local stability. This can be seen by linearizing a non-linear function through the Taylor's series expansion, in which case we can see that the stability is not influence by the higher degree terms (Lotka, pp. 60, 1956).

Testing the stability of large systems by the method of eigen values can become computationally by itself an extremely difficult task. Since the latter part of the nineteenth century (Routh 1877)[1] methods have been developed to determine a system's stability without the need of finding its characteristic roots. These methods applicable for linear systems were developed by Routh - Hurwits (Samuelson, pp. 430, 1967) and Nyquist (Ashby, pp. 256, 1960). They are much simpler than the testing of eigen values but they involve a lot of computations too, and are based on the negativity of the eigen values. Their main disadvantage is that they do not disclose anything about classes of systems, but rather test the stability of specific systems.

The Lyapunov's "second method" can also be used to test for stability, if the form of differential equation is known, and a function V(X), fulfilling certain properties can be found (Kalman, pp. 371 - 373, 1960). In specific problems, however, "it may be quite hard to hit on a function (V(X)) displaying the required properties" (Newman, pp. 26, 1961).

In the field of Economics, qualitative (pattern of signs, -, 0, +) (Samuelson, 1967), (Quirk, pp. 311 - 326, 1965) and dominal diagonal conditions were considered which, when applicable, it was assured by theorems that the system was stable. Thus, classes of systems, like certain types of markets, can be tested for stability in a way which does not involve much computational effort. An excellent discussion and summary of these stability conditions is given in two papers by Newman (pp. 1 - 8, 1959), (pp. 12 - 29, 1961).

4 - DIRECT CALCULATION OF PROBABILITY OF STABILITY

The qualitative and dominal diagonal have certain advantages over the testing of eigen values methods ; however, their applicability is limited to systems whose properties are very well known. Nor is there a method which can tell us for example, classes of systems which can be stable 97%, 98% or even 99,99% of the time. These classes will have to be labelled as unstable ; even though very few of the individual systems composing the class really are. Furthermore, there is no method which treats stability as an explicit function of time.

Ashby (pp. 260, 1960) was the first to attempt to determine the probability of stability as Z, the system size, increases. He did this for a limited number of uniform distributed matrices. He found that the probability of stability decreases at $1/2^Z$ rate (when the parameters are a = -10, b = +10), where Z is the system size. In which case only a system of size one will be stable.

Since it is felt that an explicit function should be known relating stability to both system size and the properties of the system, this study moves directly into calculating empirically the probability of stability of systems of size 1,2,... drawn at random from a certain type of distribution with given parameters. In this way, the chances of stability of systems of any size, distribution and parameters can be determined. Furthermore, it is hoped that the knowledge of the probability of stability can be used for optimizing system designs[2] an area demanding a high concentration of research (Simon, pp. 55 - 56, 1969).

5(b) - COEFFICIENT OF VARIATION

To control the sampling error, the sample size of matrices generated, n, was chosen in each case so that the coefficient of variation γ ($\gamma = \frac{\sigma p}{p}$) would be .05. Then, the n matrices representing a given type of system were tested and the percentage of stable matrices was found in each case (see table 1).

6 - EMPIRICAL DETERMINATION OF STABILITY FUNCTIONS

a) Classes of systems : The resulting percentages of each type of system (table 1) were regressed as a function of system size Z and the estimated values of the parameters as well as the functional form that best fit the data were found (see table 2).

The general form of the estimated equations was :

$$(2)\quad p(s) = e^{\hat{a} - \hat{b}Z}$$

where p(s) is the probability of stability, Z the system size and $\hat{a}$ and $\hat{b}$ are the estimated values of the true theoretical values of a and b.

Thus, if we have a system of size Z of the same distribution and parameters as in (2) then p(s) will give us the probability of stability of this system. It is obvious that p(s) is at least one, if Z is equal or smaller than a/b. In this case, the system will always be stable. If Z is greater than a/b, then not only do we know that it is unstable, but we also know what is the exact probability. (This probability will refer to a large number of systems of the same characteristics). Furthermore, assuming we can extrapolate, the probability of systems of any size can be found, thus removing the extremely difficult computational task of testing for the stability of large systems.

(2) can be treated as a p.d.f. whenever the values Z can take permit it. Such a treatment can facilitate the use of (2) in analytical manipulations. (2) is a p.d.f. because :

$$(3)\quad \int_{\frac{a + \log_e b}{b}}^{\infty} e^{a - bz}\, dz = 1$$

the c.d.f. of (2) is

$$(4)'\quad \begin{cases} \dfrac{1 - e^{a - bz}}{b} & \text{for } \dfrac{a + \log_e b}{b} < Z < \infty \\ 0 & \text{otherwise} \end{cases}$$

b) Uniform systems : We should note that (2) or (3) and (4) have only limited applicability because they deal with systems of specific distributions and parameters (see table 2). However if all the uniform data are combined and regressed, then the functional form and the parameters of a generalized function can be found. The resulting equation, (5) expresses the probability function of any kind of uniform system where Z_1, Z_2, Z_3 differentiate the probabilities of different kinds (parameters) of uniform system.

$$(5)\quad p(s_u) = .1860 - .1092Z + .2607Z - .0938Z_2 + .056Z_3$$
$$(2.2)\quad (-19.5)\quad (-2.4)\quad (-21.7)\quad (4.9)^*$$

where Z is the order of the matrix

Z_1 is the ratio of diagonal over the range of non diagonal elements

Z_2 is the mean of diagonal elements

Z_3 is the mean of the non diagonal elements

(5) has an R^2 of .93
and an F-test of 204.7

It can be seen from (5) that the coefficient of Z_1 is positive which means that the $p(s_u)$ will increase as the ratio of Z_1 increases. This in accord with dominal diagonal theorems which are directly quantified in this way. The coefficient of Z_2 is negative which again agrees with diagonal theorems because as Z_2 decreases, other things remaining equal, then the matrix becomes more diagonal and $p(s_u)$ increases. As is expected, the sign of Z_3 is the opposite of Z_2 and $p(s_u)$ will decrease as Z_3 increases.

(5), like (2), can be treated as a probability function and the chances that it is at least one can be found, for certain values of Z. That is :

$$(6)\quad Z \leq \frac{.1860 + .2670Z_1 - .0938Z_2 + .0563Z_3 - 1}{.1092}$$

in this case we will have to treat Z_1, Z_2 and Z_3 as known, which is true if we are dealing with some specific system or classes of systems. Thus, we do not need to know the difference or differential equation of the system in order to determine the system's stability. But only Z_1, Z_2 and Z_3 which are easily found. As with (2), the values of Z greater than (6) will determine the probability of stability of all systems.

$$(7)\ p(s_a) = .3401 - .1061Z + .1737Z_1 - .0794Z_2 + .0454Z_3$$
$$(4.9)\quad (-14.9)\quad (2.9)\quad (-16.2)\quad (5.4)$$

the R^2 of (7) is .85
and the F-test is 117.1
where Z, Z_1, Z_2 and Z_3 are defined the same as in (5). A system, then will be stable if :

$$(8)\ Z \leq \frac{.3401 + .1737Z_1 - .0794Z_2 + .0454Z_3 - 1}{.1061}$$

d) Stability functions of existing and expanding systems :

The functions developed so far deal with new systems ; that is the probability of stability of a brand new system of size Z is sought. However, in real life situations this is not always the case. In both living and organizational systems (this is a distinctly different case than when a new system is examined) the growth is successive. Thus we have a system of size Z - K which eventually can grow to be of size Z. It is obvious that the Z - K size system will be stable and the probability that the expanding Z - K to Z size will be required where K = 1,2,3,.... (the p(Z/Z - K is stable). To obtain data, stable systems of size say 2 x 2, were expanded to 3 x 3, 4 x 4.... or stable systems of 3 x 3 were expanded to 4 x 4, 5 x 5.... and the probability of stability of the expanding systems found (see table 3 and 4). As with new systems the results were regressed and the functional form and parameters of the probability of stability found. The number of different types of systems generated was kept only to two because of the tremendous amount of computations required (about 4 computer hours of a 360/65 to obtain the data of table 3 and 4). However, it is obvious that one can collect data for any desired number of system types. As with the new systems, generalized equations of expanding systems can be found. (This was not done in this study because not enough data were available).

The two equations obtained were for uniform distributions. The first one of all elements of distribution can take values between - 10 and + 10 (a = -10, b = + 10).

The estimated equation is :

$$(9)\ p(s_u / p(s_n) = 1) = \exp\ (1.73 - 1.24Z - .115Z_4)$$
$$(5.0)\quad (-21.3)\quad (-15.8)$$

with R^2 = 88 and F-test = 262.9
where Z_4 is the sum of the eigen values and $p(s_u/p(s_n) = 1$ is the probability of stability of a uniform system with a = -10, b = + 10, given that the original system before the expansion is stable.

The second equation occurs when the distribution is again uniform but with diagonal values ranging between zero and minus ten, and the non-diagonal between minus ten and plus ten. Then, the estimated equation is :

$$(10)\ p(s_u/p(s_n) = 1) = \exp\ (1.51 - .75Z - .049Z_4)$$
$$(6.9)\quad (-19.1)\quad (10.1)$$

with R^2 = 86 and F-test = 190

It is interesting to compare the coefficients of (9) and (10). We can see that the coefficient of Z is greater in (10) which means that the $(ps_u/p(s_n) = 1)$ will decrease much faster as the system size Z increases, in (9) than in (10). In addition the coefficient of Z_4 is larger in (10) than in (9), but since Z_4 is negative (for stable systems) then its effect will be contrary to that of Z, that is, as the negativity of the diagonal increases (Z_4 decreases) this will have a greater effect on the system described by (9). This observation can be a useful one for a system designer.

It can be seen that the probabilities in (9) and (10) other things being equal, are higher than those in (2) since a stable system's sum of eigen values will always be less than zero. We can also see that $p(s_u/p(s_n) = 1)$ will be higher, the smaller the sum of eigen values.

(9) and (10), as (2) and (7) can be looked as p.d.f. or c.p.d.

On the average Z_4 will be $\bar{\lambda}Z$ where $\bar{\lambda}$ is the mean eigen value thus, (9) or (10) become :

$$(11)\quad p(s_u/p(s_n) = 1) = e^{a-(b + \bar{\lambda} b_4)Z}$$

and the value of Z which makes the system stable is

$$Z \leq \frac{a}{b + \bar{\lambda} b_4}$$

and since $\bar{\lambda}$ is negative (for stable systems) then Z will be greater than in the case of new systems.

7 - CONCLUSIONS

The probability functions estimated are highly significant (high F-tests) with the independent variables explaining about ninety percent of the variation of the probability of stability. It is obvious that the R^2 and F-test can be increased if the coefficient of variation of the generated systems is reduced below the five percent level used by this study. This is simply a computational matter. Furthermore, there were no restricting assumptions concerning the type of the system, its parameters, its functional relation, or its difference or differential form. Thus, the findings enable us to estimate a probability function applicable to systems in general capable of the following :

a - to determine if a system or class of systems is stable or unstable

b - to know what is the probability for an individual member, or a class of systems of being stable

c - to express stability explicitly as a function of the system size Z

d - to test the stability or know the probability (by extrapolation) of very large systems e - to test the stability of a system whose difference or differential equation is not known, as long as the mean and the range of diagonal and non-diagonal elements is known

f - to test the stability of non-stationary (whose matrix of coefficients A changes over time) systems as long as the changes of coefficients of A has some known pattern or the mean and the ranges of the elements can be determined. Such systems can be considered to behave exactly as new systems of a given class and their stability can be tested.

g - to distinguish between completely new systems or those already in existence in terms of their probabilities of stability which is substantially higher in the latter category

h - finally, the availability of a p.d.f. or c.p.d. (if the values of Z are within the required limits) can be of importance since it is easier to use in analytical manipulations.

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* The numbers in parentheses are the t-tests of the regression coefficients, which test the hypothesis that the coefficients are significantly different from zero.

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TABLE 1

Raw Data of Probabilities of Stability (as function of System size Z)

	Distrib.	a_{ij} $i \neq j$	a_{ij} $i = j$	SYSTEM SIZE Z											
				1	2	3	4	5	6	7	8	9	10	11	12
1	Uniform	-10 to +10	-10 to +10	.50	.2590	.1062	.0415	.0062	.0030	.0012	.0004				
2	Uniform	0 to -10	0 to -10	1.0	.50	.3009	.1182	.0325	.0131	.0040	.0020				
3	Uniform	-10 to +10	0 to -10	1.0	.75	.469	.3227	.170	.0940	.0351	.0117				
4	Uniform	-10 to +10	-2 to -12	1.	.83	.54	.48	.2733	.1683	.089	.032	.0091			
5	Uniform	0 to -10	-1 to -11	1.	.60	.31	.1567	.0950	.0360	.0110	.0029	.0008			
6	Uniform	0 to -10	-4 to -14	1.	.82	.68	.65	.4667	.2667	.2250	.1000	.0854			
7	Uniform	0 to -10	-3 to -13	1.	.8571	.60	.525	.32	.1625	.12	.045	.014	.002		
8	Uniform	0 to -10	-6 to -16	1.	1.0	.8667	.85	.78	.70	.58	.43	.27	.09	.025	.003
9	Uniform	0 to -10	-2 to -12	1.	.64	.49	.3067	.1867	.1050	.0410	.021				
10	Uniform	0 to -10	-7 to -17	1.0	1.0	1.0	1.0	.8333	.7667	.6731	.575	.45	.31	.145	.087
11	Normal	$\mu = 0$ $\sigma = 10$	$-\mu = 0$ $\sigma = -10$	1.0	.80	.4867	.3578	.1731	.0673	.0356	.0133				
12	Normal	$\mu = 0$ $\sigma = 1, 3$ or 10	$\mu = 0$ $\sigma = 1, 3$ or 10	.50	.333	.0885	.0351	.0124	.0015	.0008	.0003				
13	Normal	$\mu = 0$ $\sigma = 1.0$	$\mu = -3.3$ $\sigma = 1.0$	.9999	.6500	.2035	.0863	.0263	.0065	.0014	.0003				
14	Normal	$-\left\lvert \mu = 0,\ \sigma = 10 \right\rvert$	$-\left\lvert \mu = 0,\ \sigma = 10 \right\rvert$	1.0	.45	.1858	.399	.0263	.0113	.0034	.0010				
15	Poisson	$\lambda = 5$ or $\lambda = 50$	$\lambda = -5$ or $\lambda = -50$	1.	.50	.1239	.0319	.0031	.00	.00	.00				
16	Uniform	0 to -10	-10 to -20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.95	.925	.27	.08	.016
17	Uniform	-10 to +10	-1 to -11	1.0	.80	.56	.4467	.2967	.1630	.092	.031				

TABLE 2

Regression Results (All systems)

	Distrib.	aij $i \neq j$	aij $i = j$	$\hat{a}$	$\hat{b}$	F-test	R^2
1	Uniform	-10 to +10	-10 to +10	1.023	-1.11	1298	.99
2	Uniform	-10 to +10	0 to -10	1.057	-.65	134.9	.95
3	Uniform	0 to -10	0 to -10	1.074	-.98	595.4	.99
4	Uniform	0 to -10	-2 to -12	1.021	-.58	175.6	.96
5	Uniform	0 to -10	-1 to -11	1.073	-.86	166.2	.96
6	Uniform	0 to -10	-4 to -14	.988	-.33	48.8	.88
7	Uniform	0 to -10	-3 to -13	1.022	-.47	104.0	.94
8	Uniform	0 to -10	-6 to -16	.951	-.13	49.8	.89
9	Uniform	10 to -10	-2 to -12	1.029	-.52	93.2	.93
10	Uniform	0 to -10	-7 to -17	.946	-.08	22.5	.78
11	Normal	$\mu = 0$ $\sigma = 1,3$ or 10	$\mu = 0$ $\sigma = 1,3$ or 10	1.05	-1.2	379.3	.98
12	Normal	$\mu = 0$ $\sigma = 1.0$	$\mu = -3.3$ $\sigma = 1.0$	1.159	-1.27	529.3	.99
13	Normal	$-\|\mu = 0$ $\sigma = 10\|$	$-\|\mu = 0$ $\sigma = 10\|$	1.03	-.98	409.4	.98
14	Poisson	$\lambda = 5$ or 50	$\lambda = -5$ or -50	1.76	-3.6	42.8	.87
15	Normal	$\mu = 0$ $\sigma = 10$	$-\|\mu = 0$ $\sigma = 10\|$	1.063	-.68	219.8	.97

TABLE 3

Raw data (Probabilities) for Expanding Systems, Uniform Distribution, Parameters, Non diagonal a = -10, b = +10, Diagonal a =-10, b =0

	System size	Probability of Stability	Sum of eigen values		System size	Probability of Stability	Sum of eigen values
1	2	.80	1	36	4	.62	27.96
2	3	.54	1	37	5	.41	27.96
3	4	.40	1	38	6	.21	27.96
4	5	.25	1	39	7	.11	27.96
5	6	.13	1	40	8	.03	27.96
6	7	.05	1	41	5	.28	11.00
7	8	.01	1	42	6	.14	11.00
8	3	.41	4.19	43	7	.05	11.00
9	4	.25	4.19	44	8	.02	11.00
10	5	.13	4.19	45	5	.64	22.70
11	6	.06	4.19	46	6	.33	22.70
12	7	.02	4.19	47	7	.13	22.70
13	8	.01	4.19	48	8	.05	22.70
14	3	.71	10.24	49	5	.36	32.10
15	4	.45	10.24	50	6	.17	32.10
16	5	.19	10.24	51	7	.07	32.10
17	6	.11	10.24	52	8	.02	32.10
18	7	.05	10.24	53	6	.40	16.54
19	8	.01	10.24	54	7	.13	16.54
20	3	.66	16.60	55	8	.04	16.54
21	4	.38	16.60	56	6	.26	24.31
22	5	.17	16.60	57	7	.08	24.31
23	6	.10	16.60	58	8	.02	24.31
24	7	.05	16.60	59	6	.54	43.12
25	8	.02	16.60	60	7	.23	43.12
26	4	.36	8.11	61	8	.10	43.12
27	5	.14	8.11	62	7	.29	22.18
28	6	.07	8.11	63	8	.07	22.18
29	7	.02	8.11	64	7	.33	37.78
30	8	.01	8.11	65	8	.10	37.78
31	4	.62	18.21	66	7	.36	54.32
32	5	.41	18.21	67	8	.14	54.32
33	6	.16	18.21	68	8	.23	28.35
34	7	.07	18.21	69	8	.60	42.10
35	8	.02	18.21	70	8	2.28	56.40

TABLE 4

Raw Data (Probabilities) for Expanding Systems, Uniform Distribution, Parameters (both diagonal and non diagonal elements) a = -10, b = +10

	System size	Probability of Stability	Sum of eigen values		System size	Probability of Stability	Sum of eigen values
1	2	.2700	1	36	4	.33	20.79
2	3	.1062	1	37	5	.12	20.79
3	4	.0415	1	38	6	.0336	20.79
4	5	.0094	1	39	7	.0048	20.79
5	6	.0029	1	40	8	.0018	20.79
6	7	.0012	1	41	5	.0422	8.50
7	8	.0004	1	42	6	.0040	8.50
8	3	.1375	1.99	43	7	.0012	8.50
9	4	.0550	1.99	44	8	.0002	8.50
10	5	.0156	1.99	45	5	.2333	18.20
11	6	.0032	1.99	46	6	.0528	18.20
12	7	.0009	1.99	47	7	.0108	18.20
13	8	.0002	1.99	48	8	.0016	18.20
14	3	.2125	8.38	49	5	.30	33.66
15	4	.065	8.38	50	6	.072	33.66
16	5	.0178	8.38	51	7	.024	33.66
17	6	.0048	8.38	52	8	.0042	33.66
18	7	.0012	8.38	53	6	.0792	15.12
19	8	.0003	8.38	54	7	.0128	15.12
20	3	.50	13.43	55	8	.0018	15.12
21	4	.19	13.43	56	6	.168	29.44
22	5	.0439	13.43	57	7	.0268	29.44
23	6	.0176	13.43	58	8	.0026	29.44
24	7	.0028	13.43	59	6	.1976	42.36
25	8	.0006	13.43	60	7	.048	42.36
26	4	.145	2.69	61	8	.0088	42.36
27	5	.0311	2.69	62	7	.0196	11.65
28	6	.0048	2.69	63	8	.0022	11.65
29	7	.0014	2.69	64	7	.078	25.12
30	8	.0004	2.69	65	8	.0086	25.12
31	4	.2844	11.68	66	7	.2776	43.14
32	5	.0711	11.68	67	8	.0492	43.14
33	6	.0208	11.68	68	8	.01	24.15
34	7	.0044	11.68	69	8	.16	37.32
35	8	.0006	11.68	70	8	.0874	44.94