TOLERANCES IN DYNAMIC SYSTEMS

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ABSTRACT

Parameter variation is a critical consideration in the design of most dynamic systems. A case in point arises when the system components are random variables selected from known distributions; the designer wishes to find a distribution for the response enabling a probabilistic statement concerning the design specifications. Several general techniques are discussed for solution of this problem. The thrust of the paper aims at using a quadrature method coupled with a digital simulation as an efficient computational approach for solving this tolerancing problem.

INTRODUCTION

In the engineering design of dynamic systems, the tolerances to which each of the component parts is made affects the performance of the composite system. In a resonating RLC circuit for example, the natural frequency and the damping ratio depend upon the resistance, capacitance and the inductance. If each of these parameters is known with certainty, then the dynamic response is known with certainty. If however, the electrical components are randomly selected from elements whose statistical distributions are known, then the dynamic response is a random variable. Generally the system response is a function of either time or frequency; while dependence on these extrinsic parameters can be considered, the response can often be characterized by one or more performance indices, such as peak overshoot, settling time, integral squared error, phase margin, resonant amplitude peak, etc. Using such a measure of system performance greatly simplifies the presentation and interpretation of the random character of the response induced through parameter uncertainty.

The dynamic response index J depends upon a set of system parameters λ_k.

$$J = F(\lambda_1,\lambda_2,\ldots,\lambda_N) \tag{1}$$

Here the function F could be a single algebraic expression relating the λ_k to J, or more generally it may be a rule for uniquely computing J given values for the parameters; such a rule might consist of a set of equations which involve non-algebraic operations, numerical integration, a successive substitution solution or a simulation of the system response. The problem at hand is to find the statistical distribution of J if the parameters are randomly selected from known, statistically independent distributions. Let the probability density function for J be represented by w(J); we desire to compute w(J) given the density function for the parameters $w_k(\lambda)$ $k = 1,2,\ldots,N$. In a deterministic analysis, specific numerical values for the parameters lead to a numerical value for the response index: for example, a resistance of 10KΩ and a capacitance of 1μf produce an R-C circuit which has a time constant of 10 m-sec. When the parameters are known statistically calculation of the probability density function w(J) allows the formulation of statements such as: the probability that the time constant exceeds 15 m-sec is .05. Several different approaches are possible for the solution of the problem posed above. These include the linear propagation of errors method which is based upon the first order terms in the Taylor series expansion of expression (1), the extended Taylor series method, Monte Carlo methods, and the quadrature or numerical integration method. The linear propagation of errors attack is the best known and most widely used. We will discuss it at some length and, in this context, present tolerancing as we view it in this paper. We will then proceed to discuss the other approaches but focus principally on the quadrature method, an efficient computational approach for solving the tolerancing problem.

Let us consider the case in which (1) is either linear or can be linearized.

$$J = a_1\lambda_1+a_2\lambda_2+\ldots+ a_N\lambda_N \tag{2}$$

The a_k are static sensitivity coefficients which reflect the impact that parameter variations have upon the index of performance. Since by assumption the λ_k are statistically independent random variables it is well known that

$$\begin{aligned} \text{ave } J &= a_1\text{ave}\lambda_1 + a_2\text{ave}\lambda_2 + \ldots + a_N\text{ave}\lambda_N \\ \text{var } J &= a_1^2\text{var}\lambda_1 + a_2^2\text{var}\lambda_2 + \ldots + a_N^2\text{var}\lambda_N \end{aligned} \tag{3}$$

One can usually find justification for assuming that the index J is normally distributed; thus the mean and variance computed from equation (3) can be used to make quantitative statements concerning the probability that J lies within some specified interval. Such an analysis is important in manufacturing; here the λ_k are component values (e.g., resistance, capacitance, etc.) and J is a measure of the response of a system fabricated from these components. In order to justify such probability statements concerning the response, the component tolerances must be given as distributions: e.g., λ_k is uniformly distributed in (a_k,b_k) or λ_k is normally distributed with mean μ_k and standard deviation σ_k, etc. It is very risky to make probability statements when the tolerances are given in the more usual high/low or go/no-go format. The classical linear propagation of errors technique (based upon retention of the first order terms in the Taylor series expansion for J) yields reasonable results; the results are perhaps better than one would expect from experience with linearization in deterministic engineering problems. The reader is referred to a recent three part paper by Evans on the state of the art of statistical tolerancing [1], [2], [3] for further discussion of these points. When a linear analysis is not good enough, three alternate approaches are possible. The non-linear propagation of errors technique based upon a Taylor series expansion through order five [2], [4], [5], [6] may be used. Although the method is powerful, it poses computational difficulties if the function F is not sufficiently tractable. Monte Carlo methods form a second conceptually simple alternative: the parameter values λ_k are randomly selected from known distributions and used to compute the response index J. The resulting set of values of J is used to compute the statistical nature of the distribution w(J) by ordinary statistical methods. The drawback of Monte Carlo method is the rate of convergence. The final procedure--a quadrature formula for numerical integration--forms the computational basis for the examples presented in this paper. The quadrature formula which we will discuss in a subsequent section uses the nonlinear equation relating J to the λ_k; it can be used if J can be evaluated for a given set of parameters. The method is straightforward and easily programmed on a digital computer. Evans [2] compares the methods enumerated above and describes the salient computational aspects of each.

THE QUADRATURE METHOD

The problem we have set forth involves finding the density function for the response, w(J), given the densities of the N independent random variables $w_k(\lambda_k)$. The quadrature method is an approximation technique based upon the fact that distributions are determined by their moments and that the moments are integrals; the integrals are further approximated by numerical integration and the densities are specified by their lower moments.

Let us introduce the following notation for the central moments of the system parameters, λ_k:

$$E[\lambda_k] = \mu_k$$

$$E[(\lambda_k-\mu_k)^2] = \sigma_k^2$$

$$E[(\lambda_k-\mu_k)^3] = \gamma_k\sigma_k^3$$

$$E[(\lambda_k-\mu_k)^4] = \Gamma_k\sigma_k^4$$

For the normal or gaussian distribution the mean of λ_k is μ_k and its variance is σ_k^2. Since the normal distribution is symmetric, its coefficient of skewness is $\gamma_k = 0$, while the normalized fourth central moment is $\Gamma_k = 3$. For a uniform distribution, i.e., any λ_k equally likely in $(a < \lambda_k < b)$, $\mu_k = (b-a)/2$, $\sigma_k^2 = (b-a)^2/3, \gamma_k = 0$, and $\Gamma_k = 9/5$.

The moments of the response J are given by

$$M_i' = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_i(\lambda_1,\ldots,\lambda_n) \prod_{k=1}^{N} w_k(\lambda_k)d\lambda_k \qquad (5)$$

where

$$h_1 = F(\mu_1,\mu_2,\ldots,\mu_N)$$

$$h_i = [F(\lambda_1,\ldots,\lambda_N)-h_1]^i$$

These are the moments around $J = h_1$; the central moments for J are obtained by application of the moment transfer formulas [7]. The integral for M_i' is approximated by a quadrature formula which is constructed using standard symmetry arguments of multidimensional numerical integration [8], [9], [10]. The quadrature approximation requires that the response index $J = F(\lambda_1,\lambda_2,\ldots,\lambda_N)$ be evaluated at $2N^2 + 1$ prescribed values of the parameter set; these prescribed values are specified in terms of the first four moments of the parameters. All four moments of the response J can be approximated from this single set of $2N^2 + 1$ values.

Since the quadrature technique is an approximation, two important points should be discussed: how precise is the approximation for the moments, and how good are the moments for making probability statements. It has been shown that the precision of the estimates of the moments is $O(\sigma^5)$, in general, and in the special case when the parameter distributions are all symmetric it is $O(\sigma^6)$, where σ is a representative standard deviation from the set $\{\sigma_k\}$; this aspect of the error problem has been discussed [9]. The use of the first four moments to graduate a distribution is a widely used approximation device in statistics; its acceptance is based more on experience than on theory since all of the moments would be required to obtain exact results. The Pearson system [11] is perhaps the oldest and most widely used such device; the system encompasses a wide class of distributions and contains the standard ones, e.g., the normal, the beta, and the gamma or type III distributions. Tables are available [12]. Pearson's third and fourth moments $\sqrt{\beta_1}$ and β_2 are defined exactly as γ and Γ. The reader is cautioned against putting too much reliance on the value for β_2 obtained by quadrature [9] in using these tables.

SIMULATION AND TOLERANCING

If the dynamic system under consideration is sufficiently simple, it may be possible to construct an explicit formula for the system response index. For example the settling time observed in the dynamic response of a linear second order differential equation is given by $T_s = 4/\xi\omega_n$. It is important to realize that the damping ratio ξ and the natural frequency ω_n are generally not independent random variables, but each depends upon the set of system parameters λ_k. A second important point should be emphasized; although the system may be described by a linear differential equation, the settling time, percent overshoot, etc. are generally non-linear functions of the system parameters. For a simple system such as the one described above, the linear propagation of errors method or the extended Taylor series method could be easily applied. When a dynamic system is of third order or higher or when non-linear elements are present, exact expressions are generally not available to compute measures of the system performance. In this case dynamic simulation provides the means of evaluating $J = F(\lambda_1,\lambda_2\ldots\lambda_N)$ and hence the use of the quadrature method of statistical toler ncing.

Let us consider the dynamic response of the position control system pictured in figure 1. The forward loop gain, and the location of the poles and zeros of the transfer function depend upon nine parameters. These are the resistance(R) and inductance(L) of the armature, the motor constant (K_m), the electrical parameters of the lead compensation network (R_1,R_2,C), the controller gain (K), and the mechanical load parameters--the effective moment of inertia (J) and the viscous damping coefficient (b). Parameter variation in a control system affects the dynamic response and hence the system performance. There are several sources of parameter variation which impact the control system design:

(1) The designer may be uncertain of the process parameters. If he bases his design on the best estimates for the parameters, there may be a non-zero probability that his performance specifications will not be met due to his uncertain knowledge about the system.

(2) The control system may be mass produced; each component has a variability which is inherent in the manufacturing process; the performance of the set of systems produced is thus a random variable.

(3) The system parameters may change with time. This could be due to natural aging of the components, or it may arise from the fact that the system is subjected to different loads at different times; such variations in load manifest themselves by altering the viscous damping coefficient or the moment of inertia of the mechanical load which the servo drives. For the analysis presented in this paper such variations are assumed to occur on a time scale which is much longer than the characteristic response time of the system. In other words, the parameters do not vary during the transient response of the system.

For the position servo of figure 1 each of the parameters was assumed to be a random variable. In order to apply the quadrature method, the distributions for each of these parameters must be established. We have chosen all parameters except the inertia (J) and the viscous damping coefficient (b) to be normally distributed. The assumption that these distributions are gaussian is not necessary to the quadrature method, but the assumption seems reasonable. In order to establish the standard deviation, a 3σ tolerance interval was assumed for each parameter. Such a specification assures us that 99.7% of all observations of that parameter will fall within a tolerance band of $\mu \pm 3\sigma$. Having established a mean value and a standard deviation for each of these parameters, we must specify the third and fourth central moments in order to apply the quadrature method. Since by assumption the distributions are gaussian, $\gamma = 0$ and $\Gamma = 3$ (see section 2). We have purposefully chosen the distributions for the moment of inertia J and the viscous damping coefficient b to be non-normal. In the case of the viscous damping coefficient, the chance of an extreme deviation below the mean value is patently less than that of the same deviation above the mean. Thus, the distribution is non-symmetric. We have chosen to represent the fluctuations in these two parameters by the use of a beta distribution of the first kind [7], and to adapt the methodology used in PERT analysis [13] to find the distribution. In PERT the random nature of the time to complete an activity is characterized by three estimates: the optimistic time, the pessimistic time, and the most likely time. Here, we use three estimates for a parameter--minλ, maxλ and modeλ. Consider the moment of inertia: the minimum value of the moment of inertia (this corresponds to the optimistic estimate for an activity time in PERT) is the value that the system designer thinks under normal circumstances will correspond to the smallest load which the servo system will drive. The maximum value (this corresponds to the pessimistic estimate for a PERT activity time) is the largest value for the moment of inertia that the designer expects the system to drive. Finally, the mode is defined as the most likely value for the parameter; it is the value of the moment of inertia which will most likely load the system. Of course min$\lambda \le$ mode$\lambda \le$ maxλ, and modeλ will not be the average value of λ unless the distribution is symmetric.

The procedure for determining the moments is straightforward. A beta distribution of the first kind is fitted to these estimates. It is assumed that $6\sqrt{\mathrm{var}\lambda} = (\max\lambda - \min\lambda)$ and that the mode of the distribution is the estimate for mode λ. The first assumption means that one is virtually certain that the parameter will be within the 6-sigma limits; the second assumption is merely a definition of what is meant by most likely. See [10] for the mathematical details of the procedure. A summary of the moments of the distributions for the parameter values is given in table 1.

	$b[\mathrm{kg\text{-}m^2/sec}]$	$J[\mathrm{kg\text{-}m^2}]$
min	.04	.04
mode	.05	.05
max	.08	.07
μ	.0529	.0513
σ	.0067	.0050
γ	.4795	.3094
Γ	2.7020	2.5565

Table 1a: Statistical Tolerances for Mechanical Parameters

	μ	$100\%(3\sigma/\mu)$	σ	γ	Γ
R [Ω]	1.620	2%	.0108	0.	3.
L [mh]	30	2%	.20	0.	3.
K_m[nt-m/amp]	.905	2%	.006	0.	3.
R_1[KΩ]	20.0	1%	.070	0.	3.
R_2[KΩ]	50.0	1%	.070	0.	3.
C [μf]	1.0	5%	.0167	0.	3.

Table 1b: Statistical Tolerances for Electrical Parameters

In order to solve the tolerancing problem for the response of a dynamic system, we have developed a quadrature program which computes the specified set of $2N^2+1$ parameter values. We use these values as input to an IBM 1130 CSMP simulation package. The maximum peak of the response is detected and this response data is processed by the quadrature program to find the moments of the response. For the example problem we have posed, there are 9 parameters; this requires that 163 response evaluations be performed to apply the quadrature procedure.

RESULTS

A summary of the moments of the distribution of the maximum peak is shown in table 2.

μ	1.167
σ	0.044
γ	0.050
Γ	2.504

Table 2: Moments for Response c_{max}/c_{ss}

The mean value for the maximum peak is $c_{max}/c_{ss} = 1.167$, yielding slightly greater than 16% overshoot. Suppose that the designer had chosen 20% as the maximum permissible value for the peak overshoot. Given the four moments of the distribution of the response, we can compute the probability that this design constraint is exceeded. Since the resulting distribution approaches a normal distribution ($\gamma=0.050\approx0$ and $\Gamma=2.504\approx3.0$) tables for the standard normal curve can be used to approximate this probability. We compute the probability that the design constraint is violated is .23, or in 23% of the cases the peak overshoot will exceed the specified constraint. If we are not willing to accept the approximation by a normal distribution, more accurate estimates can be obtained by using the Pearson tables [12].

SUMMARY

Often parameter variation or uncertainty is a reality with which the designer must somehow come to grips. Standard sensitivity analysis is often used to discuss the effect of such parameter variation. The drawback of sensitivity analysis is that it is a linear analysis; often the assumption of linearity is unwarranted. The quadrature method provides a computationally efficient method for solving the tolerancing problem and provides the designer with a tool for making probabilistic statements concerning the indices of the dynamic response.

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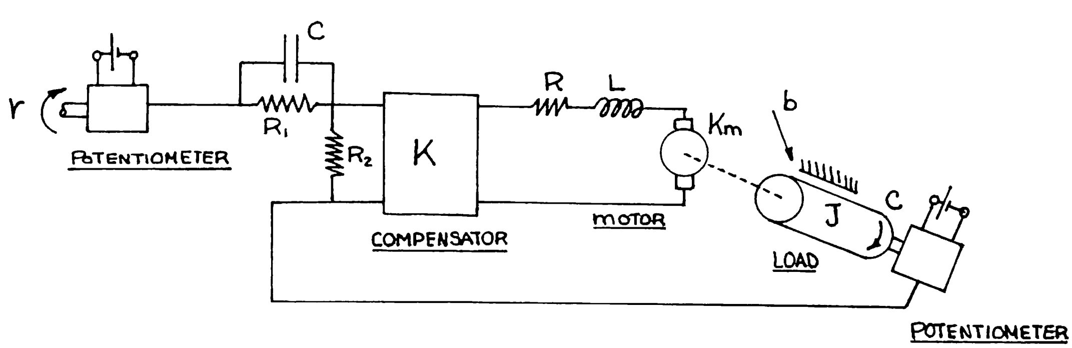

Figure 1: Control System Example