APPLICATION OF MONTE CARLO SIMULATION TO A CIRCLE PACKING PROBLEM

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OBJECTIVE

Frequently, there is a requirement to pack cylindrical objects of varying radii in a larger cylindrical container. Several methods of packing the objects have been investigated in [1] and [2]. This paper describes a method to pack the circles using Monte Carlo simulation and then analyzes the resulting data to determine the relationship among several selected variables.

A basic assumption is that the cylindrical objects are of equal length. Under this assumption, the problem reduces to an investigation of the cross section of the cylindrical container, or equivalently to an investigation of small circles packed within a larger circle. We specifically investigate how packing density (ratio of the total area of the inner circles to the area of the outer circle) varies as the number and radii of the inner circles vary.

METHOD

As the initial step in the Monte Carlo process, ten pairs of uniformly distributed random numbers between 0 and 1 were generated. The pairs of random numbers were converted to polar coordinates with the center of the outer circle acting as the center of the coordinate system. The first number of a pair, when multiplied by 100 units (i.e., the radius of the outer circle), became the magnitude; the second number, when multiplied by 2π became the angle. The polar coordinates, themselves, designated points within the outer circle which would be centers for ten inner circles. Once the centers were established, the polar coordinates were converted to rectangular coordinates to facilitate the checking of constraints.

The first constraint is that the inner circles cannot overlap the outer circle. This constraint is satisfied if

\[ r_1 \leq a_i = 100 - \sqrt{x_1^2 + y_1^2} \]

for \( i = 1, 2, \ldots, 10 \),

where \( r_i \) is the radius of the \( i \)th circle, \( x_1 \) and \( y_1 \) are the rectangular coordinates of the center of the \( i \)th circle, and 100 is the radius of the outer circle.

The second constraint is that the inner circles do not overlap each other. This constraint is satisfied if \( r_i \)

\[ r_i + r_j \leq b_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \]

for all \( i \neq j \).

In other words, the sum of the radii of any two inner circles must be less than or equal to the distance between their centers.

In order to preclude the clustering of circles in one section of the outer circle, a restriction was placed on the acceptable values of \( a_i \) and \( b_{ij} \). If for any \( i \) and \( j \) the constraints \( a_i \) or \( b_{ij} \) assumed values less than 25 units, the corresponding centers were rejected and new centers were generated. If necessary, the rejection and generation process was repeated five times. If ten points could not be found to conform to the clustering restriction, the minimum value of 25 units was cut in half and the process was repeated. If necessary, the routine would have helped the minimum value again; but in practice only one reduction in the constraint was necessary.

Once ten acceptable centers were established and the constraints on the radii were derived, linear programming was utilized to determine the radii. The linear program may be stated as:

\[
\begin{align*}
& \text{maximize} & & \sum_{i=1}^{10} r_i \\
& \text{subject to} & & r_i \leq a_i \\
& & & r_i + r_j \leq b_{ij} \text{ for all } i \neq j.
\end{align*}
\]

Once the radii were determined for the first set of ten centers, the procedure was repeated for a total of 20 times. Then the number of inner circles was increased to 15 and the procedure was repeated 20 times. The same was performed for 20, 25, and 30 inner circles although for 30 circles only 5 sets of centers were generated because of the excessive amount of time required to find 30 centers which met the constraints. The results for the 85 solution sets are listed under Figure 1.

ANALYSIS OF DATA

The primary tool in the analysis was the Statistical Analysis System (SAS), developed by the Department of Statistics, North Carolina State University, and described in [3]. As input to the regression procedure of SAS we chose as independent variables the number of inner circles, the mean radius of the circles, and the variance of the radii. The packing density was chosen to be the dependent variable. For the 85 sets of data the regression procedure produced the following expression:

\[
D = .0938 - .0172N - .0178R + .0008R^2 + .0026NR + .0100N*VAR
\]

where \( D = \) predicted packing density
\( N = \) number of inner circles
\( R = \) mean radius
\( VAR = \) variance of the radii

In order to determine which variables had the greatest effect on packing density, the stepwise procedure of SAS was used. This procedure finds the first single-variable model which produces the largest R-square (square of the multiple correlation coefficient) statistic. In our case the product N*VAR produced the largest value of .1767. By
adding the variable N to the model, the R-square statistic rose to .3423. With the addition of the product NVAR and the variable N², the R-square value approached .9952. The effect of the variable R was barely perceptible.

In order to ascertain how packing density varies with the number of circles, we evaluated the partial derivative of D with respect to N, or

\[
\frac{\partial D}{\partial N} = -0.0172 + 0.0026R + 0.0100VAR
\]

Since the minimum value of R is 11.39 and the minimum value of VAR is 3.48, the minimum value of the derivative is .067. Since this derivative assumes positive values over the domain of the variables, we conclude that the packing density is an increasing function with respect to the number of circles.

Differentiating the same expression with respect to R, we get

\[
\frac{\partial D}{\partial R} = -0.0178 + 0.0016R + 0.0026N
\]

Since the minimum value for N is 10, the minimum value for this derivative is .0026. Consequently, the packing density is also an increasing function with respect to the mean radius.

By observation one can see that the packing density increases as the variance of the radii increases.

**CONCLUSIONS**

Randomly generating the centers of the inner circles and maximizing the sum of their radii yield results which are statistically significant. In particular, one discovers that packing density is affected the most by the product NVAR, followed by the variable N, the product N²VAR, followed by the variable R, and the variable R². Furthermore, the packing density is an increasing function of the variables N, R, and VAR when each is treated separately.

**REFERENCES**

