# Fitting a Distribution to Data Using an Alternative to Moments

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Abstract

A four-parameter probability distribution that is useful in fitting data is considered. This distribution can approximate many well-known distributions and provides a simple and effective algorithm for generating random variates. Its utility, however, is limited by one's ability to determine its parameters. The method of moments has been suggested as a means of selecting the four parameter values when the first four moments are specified or can be estimated. Sample moments, however, are sensitive to extreme observations and are subject to large sampling variability. Hence, the method of moments is generalized to allow the use of surrogate measures of location, scale, symmetry, and tailweight. A new procedure using two statistics that are functions of order statistics is developed and is compared with the method of moments by means of an example and a Monte Carlo study.

KEY WORDS: Data Fitting, Moments, Monte Carlo, Order Statistics, Systems of Probability Distributions

## INTRODUCTION

Recently, a versatile four-parameter probability distribution whose density function can assume a wide variety of curve shapes, was proposed by Ramberg, Dudewicz, Tadikamalla, and Mykytka [12]. Originally developed by Ramberg and Schmeiser [13,14] to provide an effective algorithm for generating random variates in Monte Carlo simulation studies, the distribution is useful in itself for modeling probability distributions and representing data when the underlying distribution is unknown. It can be used to approximate many well-known distributions and its simple form makes it analytically as well as computationally tractable.

Particular applications of the distribution include its use as a tool for solving certain production-inventory problems. Kottas and Lau [8], for example, suggest its use in modeling lead time demands. Silver [17] demonstrates how the distribution can be utilized in estimating the safety factor in an inventory control model.

The distribution is also useful in Monte Carlo studies of the robustness of statistical procedures and for sensitivity analyses in simulation experiments. For instance, in a Monte Carlo study of linear rank statistics, O'Meara [11] found it to be useful not only because it allowed a broad class of distributions to be generated, but also because it offered an easy means of computing the scores of simple linear rank statistics. Fisher [3] used the distribution in a simulation study of the properties of two statistics that describe the tailweight and symmetry of a distribution and found it convenient for deriving general asymptotic results for these statistics. In a robustness study of adaptive estimation procedures, Moberg, Ramberg, and Randles [9] used it because it provided a single family of dis-

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tributions that was not only computationally tractable, but extensive enough to cover a wide range of symmetric and asymmetric distributions.

As an illustration of the use of this distribution in fitting data, consider the histogram for 70 observations of the yield, in pounds, of a chemical process (obtained from Bury [1, p. 6]) given in Figure 1. The two superimposed distributions were fit by methods described in this paper.

Despite its versatility and simple form, the usefulness of this distribution is, however, limited by one's ability to determine its parameters. The following two sections give a review of some properties of this distribution and outline the use of the method of moments for parameter estimation. In the subsequent section, an alternative to the method of moments is presented which is based on two statistics that provide alternative measures of the symmetry and tailweight of a distribution. The two procedures are then compared by means of an example, followed by a Monte Carlo study which examines the effects of sampling variability on them. Finally, a computer program for the alternate procedure and some theoretical results are given in the appendices.

## PROPERTIES OF THE DISTRIBUTION

Although most continuous probability distributions are defined in terms of their density functions, f(x), or distribution functions, F(x), it is equally valid to define a distribution by its quantile function, if that quantile function exists. The quantile (or percentile) function, R(p), is simply the inverse of the distribution function.

The ability to express a random variable in terms of its quantile function is quite useful in Monte Carlo simulation studies. In particular, it is well known that if R is the quantile function of a continuous probability distribution and if the random variable U is uniformly distributed on the (0,1) interval, then the transformation X = R(U) yields a continuous random variable with quantile function R. Thus, since sources of uniform (0,1) pseudo-random variates are readily available, this transformation yields a simple method for generating pseudo-random variates from distributions whose quantile functions are known and are computationally tractable.

This technique was utilized by Ramberg and Schmeiser [13,14], who generalized Tukey's lambda function [18] to a four-parameter distribution in order to facilitate the modeling of both symmetric and asymmetric distributions. This distribution is defined by the quantile function

$$R(p) = \lambda_1 + [p^{\lambda_3} - (1 - p)^{\lambda_4}] / \lambda_2 \qquad (0 \leq p \leq 1). \tag{1}$$

The parameters $\lambda_1$ and $\lambda_2$ are location and scale parameters, respectively, while $\lambda_3$ and $\lambda_4$ jointly determine the shape of the distribution. When $\lambda_3 = \lambda_4$, the resulting density is symmetric.

The density function corresponding to the percentile function (1) is

$$f(x) = f[R(p)]$$

$$= \lambda_2[\lambda_3 p^{\lambda_3 - 1} + \lambda_4(1-p)^{\lambda_4 - 1}]^{-1} \qquad (0 \leq p \leq 1). \tag{2}$$

It should be noted that although $\lambda_1$ does not appear explicitly in this expression, f(x) is implicitly a function of $\lambda_1$ since it is defined in terms of R(p) which does depend on $\lambda_1$.

The distribution function does not, in general, exist in "simple closed form." It is, however, simple to obtain a plot of the distribution function by plotting p on the y-axis versus R(p) on the x-axis. Similarly, a plot of the density function is easily obtained by plotting, for p ranging from zero to one, f[R(p)] on the y-axis against R(p) on the x-axis.

This four-parameter distribution includes a wide range of curve-shapes as demonstrated by the density plots given by Ramberg, Dudewicz, Tadikamalla, and Mykytka [12]. The density function can assume the usual (symmetric or asymmetric) unimodal shape, J- and reverse J-shapes, and even some U-shapes. Many truncated densities can also be obtained.

The distribution can also provide a good approximation to many well-known distributions. For example, Schmeiser [16] has shown that setting $\lambda_1 = 0$, $\lambda_2 = 0.1975$ and $\lambda_3 = \lambda_4 = 0.1349$ results in an approximation to the standard normal distribution for which $\max_x | \Phi(x) - R^{-1}(x) | \approx 0.001$, where $\Phi(x)$ is the normal distribution function. Similarly, as depicted in Table 1, a good approximation to the exponential density results when $\lambda_1 = \lambda_2 = \lambda_4 = 0.0004$ and $\lambda_3 = 0.0$. (This result should not be surprising since Schmeiser [15] has shown that the limiting distribution as $\lambda_4 \to 0$ is exponential with parameter $\theta$ when $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = \lambda_4 / \theta$.)

## PARAMETER SELECTION USING MOMENTS

Although the moments of a probability distribution do not determine that distribution uniquely, they do convey useful information about the distribution. The relationship suggests that one method for determining the values of the four lambda parameters is to choose $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ so that the resulting distribution has a specified mean, variance, skewness, and kurtosis. (The skewness and kurtosis are, of course, the standardized third and fourth moments as defined by

$$\alpha_3 = E(X - \mu)^3 / \sigma^3$$

and

$$\alpha_4 = E(X - \mu)^4 / \sigma^4,$$

respectively, where $\mu$ and $\sigma^2$ are the mean and variance of the distribution.)

Ramberg and Schmeiser [14] have derived expressions for the mean, variance, skewness, and kurtosis of the proposed distribution as functions of the four lambda parameters. (See also [12].) From these expressions, it can be shown that legitimate densities whose first four moments exist occur only when $\lambda_3$ and $\lambda_4$ have the same sign and $\mathrm{Min}(\lambda_3,\lambda_4) \geq -1/4$.

It can also be shown that the skewness and kurtosis are functions of $\lambda_3$ and $\lambda_4$ alone. Thus, if the values of $\alpha_3$ and $\alpha_4$ are known (or can be estimated), the corresponding values of $\lambda_3$ and $\lambda_4$ can be determined (or estimated) by solving the system of simultaneous non-linear equations

$$\begin{cases} \alpha_3 = \alpha_3(\lambda_3,\lambda_4) \\ \alpha_4 = \alpha_4(\lambda_3,\lambda_4), \end{cases}$$

where $\alpha_3(\lambda_3,\lambda_4)$ and $\alpha_4(\lambda_3,\lambda_4)$ are the values of skewness and kurtosis calculated as functions of $\lambda_3$ and $\lambda_4$. Alternately, one can apply techniques of function minimization to search for values of $\lambda_3$ and $\lambda_4$ which minimize an objective function such as

$$f(\lambda_3,\lambda_4) = (\alpha_3 - \alpha_3(\lambda_3,\lambda_4))^2 + (\alpha_4 - \alpha_4(\lambda_3,\lambda_4))^2,$$

subject to the constraint that $\lambda_3$ and $\lambda_4$ have the same sign.

The moment formulas of Ramberg and Schmeiser further indicate that the variance depends on the shape parameter $\lambda_2$ as well as on $\lambda_3$ and $\lambda_4$, and that the mean is a function of all four parameters. Hence, once one has determined values for $\lambda_3$ and $\lambda_4$, one can then consecutively solve for the values of $\lambda_2$ and $\lambda_1$ which correspond to specified values of $\sigma^2$ and $\mu$.

Either of these methods requires computerized numerical procedures since $\alpha_3(\lambda_3,\lambda_4)$ and $\alpha_4(\lambda_3,\lambda_4)$ are highly non-linear. To simplify this process, a table has been constructed by Ramberg, Dudewicz, Tadikamalla and Mykytka [12] which permits the immediate selection of the lambda parameters for given values of the first four moments. (The complete table, along with a discussion of its construction and accuracy, can also be found in Mykytka [10].)

If $\mu$, $\sigma^2$, $\alpha_3$, and $\alpha_4$ are unknown, they must be estimated from sample data. The preceding procedure is then used to estimate the lambda parameters with $\mu$, $\sigma^2$, $\alpha_3$, and $\alpha_4$ replaced by their usual sample estimates, which we shall denote as $\bar{x}$, $\hat{\sigma}^2$, $\hat{\alpha}_3$, and $\hat{\alpha}_4$, respectively. It should be noted, however, that the third and fourth sample moments are quite sensitive to extreme observations and the variability of these sample moments can be large.

## AN ALTERNATE METHOD FOR PARAMETER ESTIMATION

The first four moments can provide a convenient procedure for parameter selection since their population values are often known as a result of experience or from theoretical considerations. On the other hand, the use of sample moments can reasonably be questioned because of the great deal of variability associated with the third and fourth sample moments. It would thus seem reasonable to seek a method for estimating the lambda parameters based on a less variable set of sample statistics which describe, as fully as possible, the distribution being sampled.

Consider, for example, the most common alternate measures of location and scale, the median and the interquartile range (IQR). Each can easily be expressed as a function of the lambda parameters, as follows:

$$\text{median} = R(0.50) = \lambda_1 + [(0.50)^{\lambda_3} - (0.50)^{\lambda_4}] / \lambda_2 \tag{3}$$

and

$$\text{IQR} = R(0.75) - R(0.25) = [(0.75)^{\lambda_3}-(0.25)^{\lambda_4}-(0.25)^{\lambda_3}+(0.75)^{\lambda_4}]/\lambda_2. \tag{4}$$

Note that the median is analogous to the mean in that it is a function of all four parameters. Similarly, the IQR depends only on $\lambda_2$, $\lambda_3$, and $\lambda_4$ as does the variance.

Alternate measures of symmetry and tailweight (surrogates for the skewness and kurtosis) are not as well-known. Hogg, Fisher, and Randles [6] have proposed an attractive pair of statistics which are functions of linear combinations of the order statistics. In particular, let $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ be the order statistics of a random sample of size n from a continuous distribution. The skewness indicator, $Q_3$, is then defined by

$$Q_3 = \frac{\bar{U}(0.05) - \bar{M}(0.50)}{\bar{M}(0.50) - \bar{L}(0.05)}, \tag{5}$$

where $\bar{U}(0.05)$, $\bar{M}(0.50)$ and $\bar{L}(0.05)$ are, respectively, the averages of the largest (uppermost) five percent of the Y's, the middle 50 percent of the Y's, and the smallest (lowest) five percent of the Y's. Fractional quantities are used in the computation of these averages; for example, if n = 50, then $\bar{L}(0.05) = (Y_1 + Y_2 + (Y_3/2))/2.5$, since $(0.05)n = 2.5$. Similarly, $Q_4$, the tailweight descriptor, is defined as

$$Q_4 = \frac{\bar{U}(0.05) - \bar{L}(0.05)}{\bar{U}(0.50) - \bar{L}(0.50)}. \tag{6}$$

The particular notation used for these statistics, i.e., $Q_3$ and $Q_4$, is that of Moberg, Ramberg, and Randles [9] and is used because it suggests their use as alternatives to $\alpha_3$ and $\alpha_4$.

Hogg, Fisher, and Randles suggest that if $Q_3$ is large, say greater than two, then there is an indication that the distribution is skewed to the right, i.e., positively skewed. Similarly, if $Q_3 < 1/2$, the indication is that it might be negatively skewed. For symmetric distributions, this ratio should be nearly one. In a like manner, $Q_4 > 7$ suggests a very heavy-tailed distribution while if $Q_4 \leq 2$, a light-tailed distribution is suggested.

Fisher [3] has shown that, asymptotically,

$$E(Q_3) = \frac{\mu_{\bar{U}(0.05)} - \mu_{\bar{M}(0.50)}}{\mu_{\bar{M}(0.50)} - \mu_{\bar{L}(0.05)}} \tag{7}$$

and

$$E(Q_4) = \frac{\mu_{\bar{U}(0.05)} - \mu_{\bar{L}(0.05)}}{\mu_{\bar{U}(0.50)} - \mu_{\bar{L}(0.50)}}. \tag{8}$$

For our particular four-parameter distribution,

$$\mu_{\bar{U}}(\alpha) = \frac{1}{\alpha\lambda_2}\left[\frac{1-(1-\alpha)^{\lambda_3+1}}{\lambda_3+1} - \frac{\alpha^{\lambda_4+1}}{\lambda_4+1}\right] + \lambda_1,$$

$$\mu_{\bar{L}}(\alpha) = \frac{1}{\alpha\lambda_2}\left[\frac{\alpha^{\lambda_3+1}}{\lambda_3+1} + \frac{(1-\alpha)^{\lambda_4+1} - 1}{\lambda_4+1}\right] + \lambda_1,$$

and

$$\mu_{\bar{M}}(\alpha) = \frac{1}{\alpha\lambda_2}\left[\frac{(\frac{1+\alpha}{2})^{\lambda_3+1} - (\frac{1-\alpha}{2})^{\lambda_3+1}}{\lambda_3+1} + \frac{(\frac{1-\alpha}{2})^{\lambda_4+1} - (\frac{1+\alpha}{2})^{\lambda_4+1}}{\lambda_4+1}\right] + \lambda_1.$$

Upon substituting these last expressions into (7) and (8), it should be clear that $E(Q_3)$ and $E(Q_4)$ are functions of $\lambda_3$ and $\lambda_4$ alone.

Although (7) and (8) are asymptotic results, they do provide one with a method of estimating the lambda parameters analogous to the method of moments. In particular, let $Q_3(\lambda_3,\lambda_4) = E(Q_3)$ and $Q_4(\lambda_3,\lambda_4) = E(Q_4)$ and let $Q_3$ and $Q_4$ denote the sample statistics calculated by (3) and (4). Then, for example, one can estimate the lambda parameters by first searching for $\lambda_3$ and $\lambda_4$ to minimize the objective function

$$f(\lambda_3,\lambda_4) = (Q_3 - Q_3(\lambda_3,\lambda_4))^2 + (Q_4 - Q_4(\lambda_3,\lambda_4))^2 \quad (9)$$

subject to the constraint that $\lambda_1$ and $\lambda_2$ have the same sign. The parameters $\lambda_1$ and $\lambda_2$ can then be determined from the expressions for the mean and variance of the distribution, or from the expressions for the median and interquartile range. (A FORTRAN program which minimizes the objective function (9) in order to determine the lambda parameters corresponding to specified values for the mean, variance, $Q_3$ and $Q_4$, is described in Appendix A and is listed in Figure 8.)

## EXAMPLE

In order to compare this alternate procedure with the method of moments, consider once again the data represented by the histogram in Figure 1. The ordered data are presented for reference in Table 2. The data are taken from Bury [1, p.6] and are summarized by the sample statistics

$\bar{x} = 24.186, \quad \hat{\sigma}^2 = 14.494,$

$\hat{\alpha}_3 = 0.67, \quad \hat{\alpha}_4 = 3.69.$

Using the method of moments (following the procedure outlined in [12]), the corresponding parameter estimates are

$\hat{\lambda}_1 = 22.122, \quad \hat{\lambda}_2 = 0.0349,$

$\hat{\lambda}_3 = 0.0435, \quad \hat{\lambda}_4 = 0.1283.$

The statistics $Q_3$ and $Q_4$ are calculated from the data (as illustrated in Table 2) to be

$Q_3 = 1.5901$ and $Q_4 = 2.8607.$

Solving for $\lambda_3$ and $\lambda_4$ by minimizing (9) (using the program listed in Figure 8), the lambda parameters are estimated to be

$\hat{\lambda}_1 = 22.706, \quad \hat{\lambda}_2 = 0.0006184,$

$\hat{\lambda}_3 = 0.0008252, \quad \hat{\lambda}_4 = 0.001742.$

Although these estimates of the parameter values appear to differ considerably from those obtained using sample moments, the differences between the corresponding density plots are not nearly as radical as one might expect. Figure 1 shows both probability densities superimposed over the relative frequency histogram. The density from the method of moments appears to be slightly more skewed; however, both seem to fit the data well. (Corresponding chi-square goodness-of-fit test statistics are calculated in Table 3.) One might consider the $(Q_3,Q_4)$ density as the better fit, since it best reflects the peakedness of the data.

## A MONTE CARLO STUDY

The example in the previous section suggests that the $(Q_3,Q_4)$ method of estimating the lambda parameters results in a slightly better fit to the data than does the method of moments. Despite this, one does not know which of the two methods best estimates the shape of the true underlying density function.

Initially, one might be interested in determining which pair of statistics, $(\hat{\alpha}_3,\hat{\alpha}_4)$ or $(Q_3,Q_4)$, is least sensitive to sampling variability. A start has been made in this direction with the results given in Appendix B. In particular, expressions are derived for the moments of the order statistics of the proposed distribution as well as for the covariance between any pair of order statistics. These are potentially useful since $Q_3$ and $Q_4$ are functions of order statistics. Furthermore, these results enable one to approximate the moments of various theoretical distributions that can be approximated with the proposed distribution.

A Monte Carlo simulation study was performed to compare the sampling variability effects of the two methods. Thirty random samples of size n = 50 and n = 100 were generated from three particular distributions and both methods were used to estimate the lambda parameters from these samples. The sampling variability effects were then compared by plotting -- for each method, distribution, and sample size -- the estimated density functions for each of the 30 replications on the same coordinate axes. The results are shown in Figures 2 through 7.

The random samples were generated using the quantile function (1) to transform uniform (0,1) random variates into variates from the specified distribution by the method outlined previously. The specific random number generator employed was Marsaglia's SUPER-DUPER (available from and documented in reference [2]) and, for the same sample sizes, the same sequences were employed to generate the 30 samples from each distribution.

The parameters for the distributions were chosen so that each had a zero mean and unit variance. The distribution considered in Figures 2 and 3 is the approximation to the normal which has a zero skewness and a kurtosis of three. Equivalently, the (asymptotically) expected values of $Q_3$ and $Q_4$ are 1.0 and 2.5959 respectively. The second distribution (Figures 4 and 5) is also symmetric and has a kurtosis of six. Roughly equivalent to a Student's t distribution with six degrees of freedom, this distribution has $E(Q_3) = 1.0$ and $E(Q_4) = 3.0604$. Figures 6 and 7 depict a skewed distribution with $\alpha_3 = 1$ and $\alpha_4 = 5$; the corresponding values of $E(Q_3)$ and $E(Q_4)$ are 1.835 and 2.8119, respectively.

From each sample, the statistics $\hat{\alpha}_3$ and $\hat{\alpha}_4$ as well $Q_3$ and $Q_4$ were calculated and then used to estimate the lambda parameters by each of the two methods. Since the intent of this study was to compare only the shapes of the resulting densities, in both cases $\lambda_1$ and $\lambda_2$ were estimated using the sample mean and variance.

The sampling variability of the moments presented itself even before the estimated densities were plotted, producing some combinations of (sample) skewness and kurtosis for which corresponding values of $\lambda_3$ and $\lambda_4$ do not exist. This phenomenon occurred primarily for a sample size of n = 50, although a few incidents were observed for the larger sample size. This phenomenon did not occur even once using the $Q_3$ and $Q_4$ statistics.

The plots in Figures 2 through 7 show that the curve shapes of the densities estimated by the method of moments are more varied than the shapes of the densities estimated by the $(Q_3,Q_4)$ method. The curve shapes for the latter method conform more closely to the form of the specified distributions (shown by the dotted curves). In contrast, the method of moments sometimes estimated these generally "bell-shaped" curves with J-shapes and often was unable to estimate the density at all. It is thus concluded that the sampling variability effects of $Q_3$ and $Q_4$ are considerably less than those of the sample moments.

## APPENDIX A

The FORTRAN program listed in Figure 8 searches for the values of the four lambda parameters corresponding to specified values of mean, variance, $Q_3$, and $Q_4$. It employs Powell's algorithm for unconstrained function minimization (to minimize the objective function (9)) and is designed to interface with the version of the algorithm presented as SUBROUTINE BOTM by Keuster and Mize [7]. (This subroutine is <u>not</u> listed in Figure 8.)

The program requires the user to input the desired values of $\mu$, $\sigma^2$, $Q_3$, and $Q_4$ and starting values for $\lambda_3$ and $\lambda_4$. Two useful pairs of starting values are (0.1, 0.1) and (-0.05, -0.05). In addition, the user must supply values for some parameters used by Powell's algorithm. Recommended values for these are MAXIT = 100, ESCALE = 1000, and

E(1) = E(2) = 0.00001. A complete discussion of the definition and significance of these can be found in Keuster and Mize [7].

This program has been run and tested on an IBM 360 computer at the University of Iowa, and requires the built-in double precision functions DABS, DSQRT, DSIGN, and DGAMMA, supplied by IBM. The first three are generally available on most FORTRAN facilities, however the gamma function DGAMMA may not be readily available. If a substitute cannot be obtained, the user then may wish to alter the program to determine $\lambda_2$ and $\lambda_1$ using the expressions for the interquartile range (2) and the median (1), instead of the variance and mean. (The gamma function is required only in the computation of the variance as a function of the lambda parameters.)

The following is a list of variable names used in the program and their definitions.

| | |
|---|---|
| LAMBDA | Vector containing the four lambda parameters. |
| MU | Desired mean (input). |
| SIGMA2 | Desired variance (input). |
| Q3 | Desired value of $Q_3$ (input). |
| Q4 | Desired value of $Q_4$ (input). |
| MEAN | Calculated mean. |
| VAR | Calculated variance. |
| Q3CALC | Calculated value of $Q_3$. |
| Q4CALC | Calculated value of $Q_4$. |
| START | Vector of starting values for $\lambda_3$ and $\lambda_4$ (input). |
| MAXIT | Maximum number of iterations for Powell's algorithm (input). |
| ESCALE | Maximum step-size multiplier -- X(I) will not be incremented by more than Escale*E(I) (input). |
| E | Convergence criteria for Powell's algorithm -- Convergence assumed when values for parameters between successive iterations differ by less than these values (input). |
| N | Number of parameters determined via minimization routine (N = 2). |
| IFON | Number of function evaluations required in minimization procedure. |
| TOL | Tolerance value -- if optimal value of objective function exceeds TOL a warning message is printed. |
| IPRINT | Controls printing in BOTM (not used). |
| X | Parameter values -- initially contains the starting values for $\lambda_3$ and $\lambda_4$ -- finally contains the optimal values of $\lambda_3$ and $\lambda_4$. |
| EF | Optimal value of objective function. |
| W | Working vector area for BOTM. |
| NW | Dimension of W (NW = N*(N+3)). |
| NI | Card reader unit number. |
| NO | Printer unit number. |

## APPENDIX B

In this section, a generalization is given for a result stated by O'Meara [11] for the expected value of the $j^{th}$ order statistic of a random sample of size n from a member of the proposed distribution. In particular, expressions are given for the $k^{th}$ moment of the $j^{th}$ order statistic and the covariance between any pair of order statistics.

<u>RESULT</u>: For $\lambda_1 = 0$, the $k^{th}$ moment of the $j^{th}$ order statistic of a random sample of size n from a distribution having quantile function (1) is given by

$$E(Y_j^k) = \lambda_2^{-k} \frac{n!}{(j-1)!\ (n-j)!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \cdot \beta(\lambda_3(k-i)+j, \lambda_4 i+n-j+1), \qquad (10)$$

where, by the definition of the beta distribution (see, for example, Hogg and Craig [5, p.134])

$$\beta(a,b) = \int_0^1 u^{a-1} (1-u)^{b-1}\, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} .$$

The proof of this result is given by Mykytka [10] and can be sketched as follows. By definition, Fitting A Distribution To Data (continued)

$$E(Y_j^k) = \int_{-\infty}^{\infty} x^k g_j(x)\, dx$$

where $g_j(y_j)$ is the density function for the $j^{th}$ order statistic of a random sample from a distribution with distribution function F(x) and density function f(x). In particular, as given by Hogg and Craig [5, p.150],

$$g_j(y_j) = \frac{n!}{(j-1)!(n-j)!} [F(y_j)]^{j-1} [1-F(y_j)]^{n-j} f(y_j).$$

Making the change of variable u = F(x) so that

$$x = R(u) = [u^{\lambda_3} - (1-u)^{\lambda_4}]\lambda_2^{-1}$$

one then has

$$E(Y_j^k) = \lambda_2^{-k} \frac{n!}{(j-1)!(n-j)!} \int_0^1 [u^{\lambda_3} - (1-u)^{\lambda_4}]^k \cdot u^{j-1} (1-u)^{n-j}\, du.$$

Expanding $[u^{\lambda_3} - (1-u)^{\lambda_4}]^k$ in a binomial series and interchanging the order of summation and integration yields

$$E(Y_j^k) = \lambda_2^{-k} \frac{n!}{(j-1)!(n-j)!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \cdot \int_0^1 u^{\lambda_3(k-i)+j-1} (1-u)^{\lambda_4 i+n-j}\, du.$$

Recognizing that the integral has the form of the beta function, the result follows directly.

When k = 1, the result (10) reduces to the expression for the mean of the $j^{th}$ order statistic given by O'Meara. Similarly, setting k = 2 yields an expression for $E(Y_j^2)$. Then, the variance of the $j^{th}$ order statistic of a random sample from a distribution with quantile function (1) is given by

$$Var(Y_j) = E(Y_j^2) - [E(Y_j)]^2.$$

In order to calculate the variance of the statistics $\bar{U}(\alpha)$, $\bar{L}(\alpha)$, or $\bar{M}(\alpha)$, which are linear combinations of the order statistics, one needs to compute the covariance between pairs of the order statistics. The following result enables one to determine these covariances.

RESULT: For $\lambda_1 = 0$ and any two order statistics $Y_i$ and $Y_j$ such that $Y_i \le Y_j$, of a random sample of size n from a distribution with quantile function (1),

$$E(Y_iY_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} (-1)^k \cdot [\beta(\lambda_3+j-i-k, n-j+1) - \beta(j-i-k, \lambda_4+n-j+1)] \cdot [\frac{1}{\lambda_3+k+i} - \beta(k+i, \lambda_4+1)]. \quad (11)$$

The proof of this result is also given by Mykytka [10] and proceeds in a similar manner.

Given the two results, (10) and (11), the covariance of any two order statistics, $Y_i \le Y_j$, is then given by

$$Cov(Y_iY_j) = E(Y_iY_j) - E(Y_i)E(Y_j).$$

The variances of the statistics $\bar{U}(\alpha)$, $\bar{L}(\alpha)$, and $\bar{M}(\alpha)$ can then be determined using the usual formula for the variance of a sum of dependent random variables.

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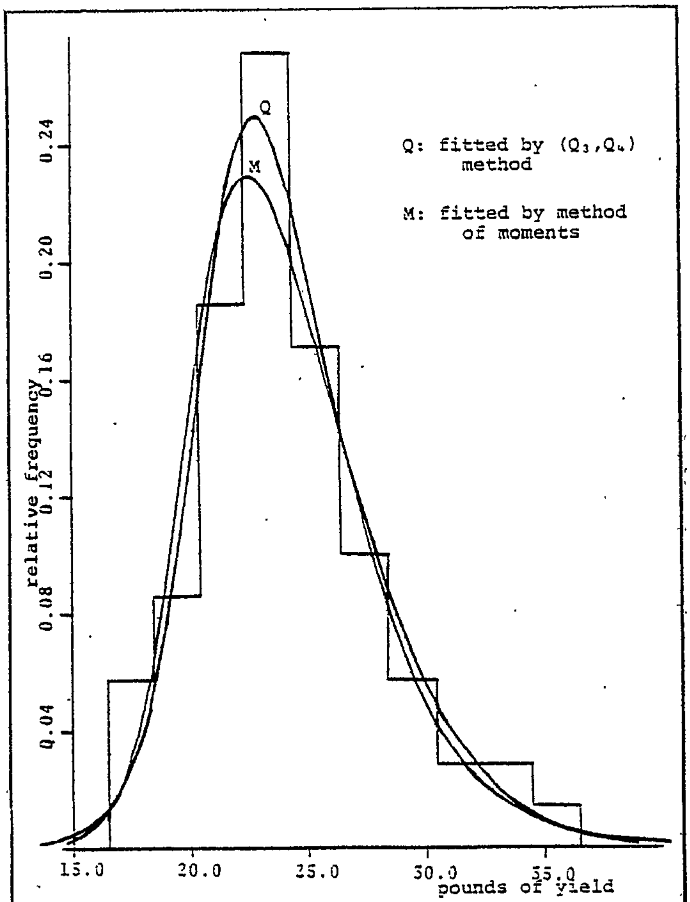



FIGURE 1. Relative Frequency Histogram and the Fitted Distributions

TABLE 1. Comparison of the Actual and Approximated Percentiles of the Exponential Distribution

| p | Actual Percentile $R(p)=-\ln(1-p)$ | Approximated Percentile $\hat{R}(p)=x_p$ | $F(x_p)=1-e^{-x_p}$ | Distance Criterion* $\lvert p-F(x_p)\rvert$ |
|---|---|---|---|---|
| 0.01 | 0.0100 | 0.0104 | 0.0103 | 0.0003 |
| 0.05 | 0.0513 | 0.0517 | 0.0504 | 0.0004 |
| 0.10 | 0.1054 | 0.1058 | 0.1004 | 0.0004 |
| 0.50 | 0.6931 | 0.6935 | 0.5002 | 0.0002 |
| 0.75 | 1.3863 | 1.3863 | 0.7500 | 0.0000 |
| 0.90 | 2.3052 | 2.3019 | 0.8999 | 0.0001 |
| 0.95 | 2.9957 | 2.9943 | 0.9499 | 0.0001 |
| 0.99 | 4.6052 | 4.6012 | 0.9900 | 0.0000 |
| 0.995 | 5.2983 | 5.2931 | 0.9950 | 0.0000 |

*The distance criterion, $\lvert p-F(x_p)\rvert$, is the absolute difference between the specified probability, p, and the value of the c.d.f. for the exponential distribution at the approximated percentile.

TABLE 2. Ordered Data and Calculation of $Q_3$ and $Q_4$

Ordered Data

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 17 | 20 | 22 | 23 | 23 | 24 | 25 | 26 | 27 | 29 |
| 17 | 20 | 22 | 23 | 23 | 24 | 25 | 26 | 28 | 29 |
| 18 | 20 | 22 | 23 | 23 | 24 | 25 | 26 | 28 | 31 |
| --18-- | 21 | --22-- | 23 | 23 | 25 | 25 | --26-- | 28 | --31-- |
| 19 | 21 | 22 | 23 | 23 | 24 | 25 | 26 | 28 | 33 |
| 19 | 21 | 22 | 23 | 23 | 24 | 25 | 27 | 29 | 34 |
| 19 | 21 | 22 | 23 | 24 | 24 | 25 | 27 | 29 | 36 |

Dashes (--) bracketing an observation indicate that fractional portions of the observation were used in calculating $Q_3$ and $Q_4$.

Calculation of the $Q_3$ and $Q_4$ Statistics

$\bar{U}(.05) = ((31/2)+33+34+36)/3.5 = 33.8571$

$\bar{U}(.50) = (24+24+\cdots+34+36)/35 = 27.0571$

$\bar{L}(.05) = (17+17+18+(18/2))/3.5 = 17.4286$

$\bar{L}(.50) = (17+17+\cdots+23+24)/35 = 21.3143$

$\bar{M}(.50) = ((22/2)+22+\cdots+26+(26/2))/35 = 23.7714$

$Q_3 = (33.8571-23.7714)/(23.7714-17.4286) = 1.5901$

$Q_4 = (33.8571-17.4286)/(27.0571-21.3143) = 2.8607$

TABLE 3. Chi-Square Goodness-of-Fit Test Statistics* for the Fitted Distributions

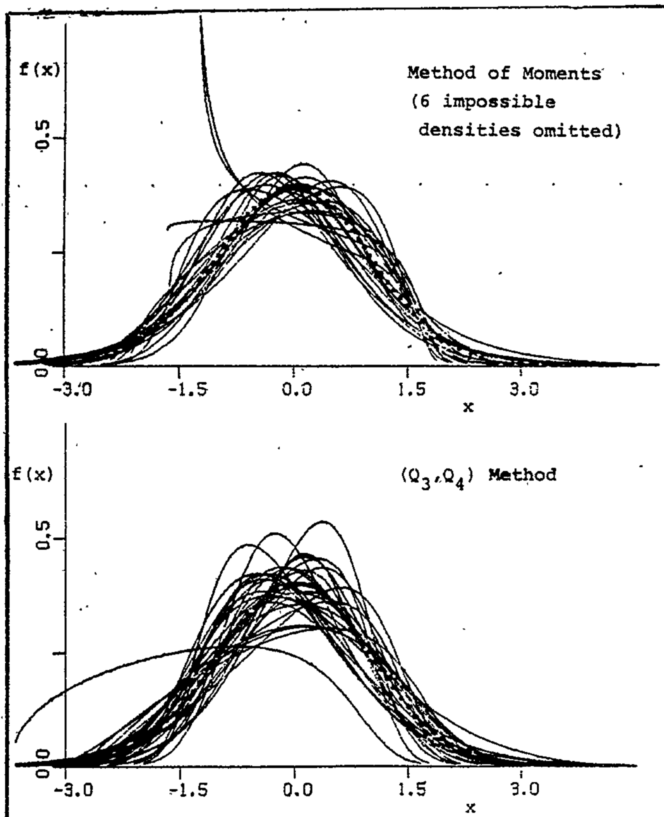
| Yield (interval) | Observed Frequency | | Expected Frequencies: Density Fit Using Moments | Expected Frequencies: Density Fit Using $Q_3$, $Q_4$ |
|---|---|---|---|---|
| less than 18.5 | 4 | 10 | 10.84 | 9.75 |
| 18.5 - 20.5 | 6 | | | |
| 20.5 - 22.5 | 13 | | 14.59 | 14.59 |
| 22.5 - 24.5 | 19 | | 15.39 | 16.86 |
| 24.5 - 26.5 | 12 | | 12.01 | 12.73 |
| 26.5 - 28.5 | 7 | | 7.92 | 7.67 |
| 28.5 - 30.5 | 4 | | 4.58 | 4.15 |
| 30.5 - 32.5 | 2 | 5 | 4.57 | 4.25 |
| 32.5 - 34.5 | 2 | | | |
| more than 34.5 | 1 | | | |

test statistic values: $\chi^2 = 1.33$ $\chi^2 = 0.69$

$P[\chi^2(2) < 1.39] = 0.50$; $P[\chi^2(2) < 0.575] = 0.25$

Comparison of the values of the test statistics with the above tabulated values of $\chi^2$ (with (7-4-1)=2 degrees of freedom) indicates that either model fits the data well.

* The $\chi^2$ test statistics are computed according to the method outlined by Hahn and Shapiro [6, p. 219].



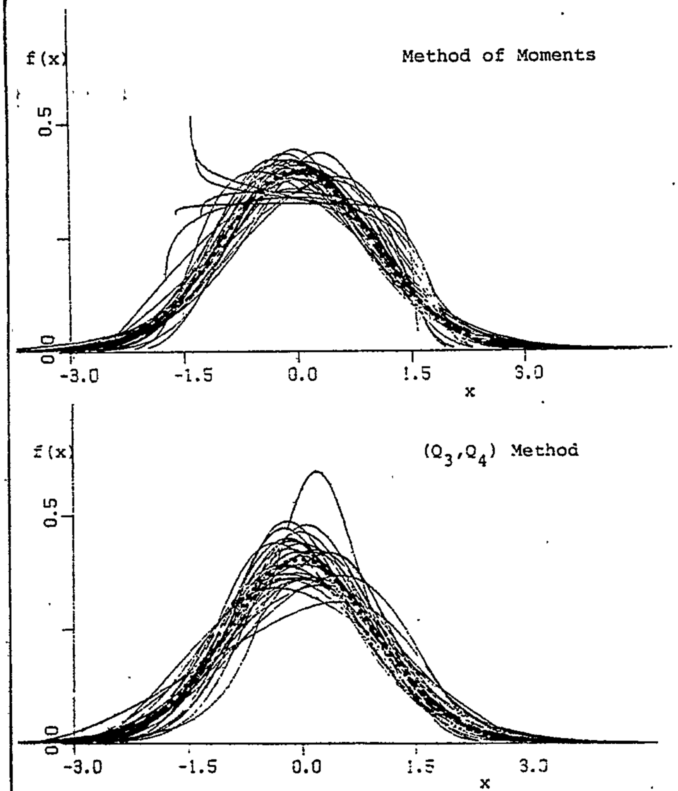

Behavior of the Sample Statistics in the 30 Replications for $\alpha_3 = 0$, $\alpha_4 = 3$ and $n = 50$

| Statistic | Minimum | Maximum | Average | Standard Deviation | Expected Value |
|---|---|---|---|---|---|
| $\bar{x}$ | -0.2545 | 0.2766 | -0.0413 | 0.123 | 0 |
| $\hat{\sigma}^2$ | 0.6181 | 1.1345 | 0.9960 | 0.223 | 1 |
| $\hat{\alpha}_3$ | -0.6153 | 0.8234 | 0.0221 | 1.260 | 0 |
| $\alpha_4$ | 1.1471 | 6.0460 | 2.8668 | 1.260 | 3 |
| $\bar{L}(.05)$ | -2.8284 | -1.3893 | -2.0034 | 0.347 | -2.0677 |
| $\bar{L}(.50)$ | -1.1415 | -0.5493 | -0.8312 | 0.159 | -0.7965 |
| $\bar{U}(.05)$ | 1.3012 | 2.5837 | 1.9221 | 0.314 | 2.0677 |
| $\bar{U}(.50)$ | 0.5159 | 1.1102 | 0.7486 | 0.154 | 0.7965 |
| $\bar{M}(.50)$ | -0.2456 | 0.2002 | -0.0509 | 0.137 | 0 |
| $Q_3$ | 0.6194 | 1.5090 | 1.0429 | 0.256 | 1 |
| $Q_4$ | 2.1342 | 3.0141 | 2.4926 | 0.203 | 2.5959 |

Observed Combinations of $\hat{\alpha}_3$ and $\hat{\alpha}_4$ For Which $\lambda_3$ and $\lambda_4$ Do Not Exist

(0.0487, 1.3132)
(0.2428, 1.6360)
(0.0795, 1.4148)
(-0.2126, 1.2616)
(-0.1860, 1.1471)
(-0.0566, 1.5082)

FIGURE 2. Monte Carlo Results -- $\alpha_3 = 0$, $\alpha_4 = 3$, and $n = 50$



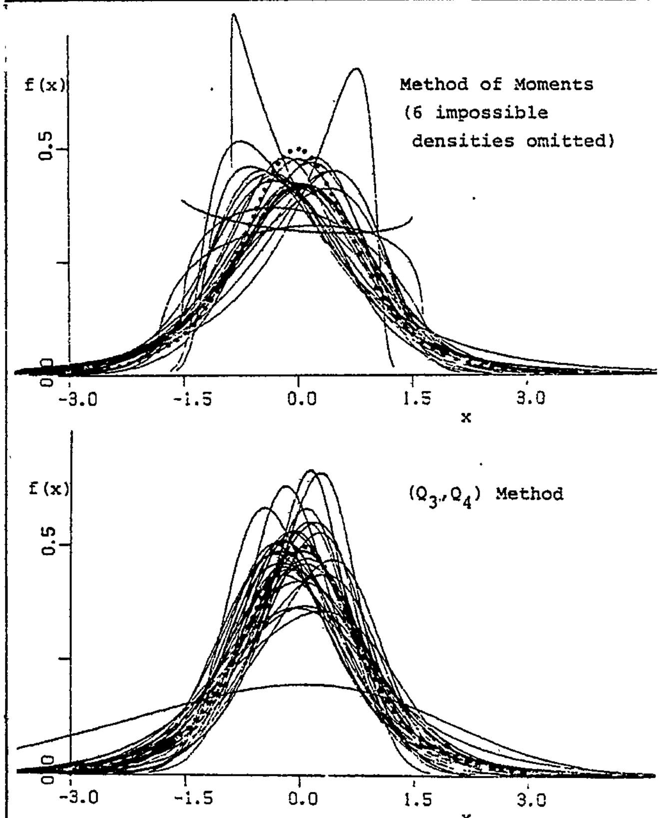

Behavior of the Sample Statistics
in the 30 Replications for
$\alpha_3 = 0$, $\alpha_4 = 3$ and $n = 100$

| Statistic | Minimum | Maximum | Average | Standard Deviation | Expected Value |
|---|---|---|---|---|---|
| $\bar{x}$ | -0.2061 | 0.2179 | 0.0209 | 0.102 | 0 |
| $\hat{\sigma}^2$ | 0.7328 | 1.4770 | 1.0006 | 0.161 | 1 |
| $\hat{\alpha}_3$ | -0.5183 | 0.4876 | 0.0688 | 0.239 | 0 |
| $\hat{\alpha}_4$ | 1.8193 | 6.1141 | 3.0748 | 1.003 | 3 |
| $\bar{L}(.05)$ | -2.5663 | -1.5239 | -1.9973 | 0.253 | -2.0677 |
| $\bar{L}(.50)$ | -1.0119 | -0.4489 | -0.7699 | 0.133 | -0.7965 |
| $\bar{U}(.05)$ | 1.6834 | 2.6757 | 2.0922 | 0.243 | 2.0677 |
| $\bar{U}(.50)$ | 0.5997 | 1.0374 | 0.8118 | 0.117 | 0.7965 |
| $\bar{M}(.50)$ | -0.2165 | 0.2575 | 0.0073 | 0.111 | 0 |
| $Q_3$ | 0.6943 | 1.3994 | 1.0609 | 0.185 | 1 |
| $Q_4$ | 2.3404 | 3.1195 | 2.5973 | 0.183 | 2.5959 |

Observed Combinations of $\hat{\alpha}_3$ and $\hat{\alpha}_4$
For Which $\lambda_3$ and $\lambda_4$ Do Not Exist

none

FIGURE 3. Monte Carlo Results -- $\alpha_3 = 0$, $\alpha_4 = 3$, and $n = 100$



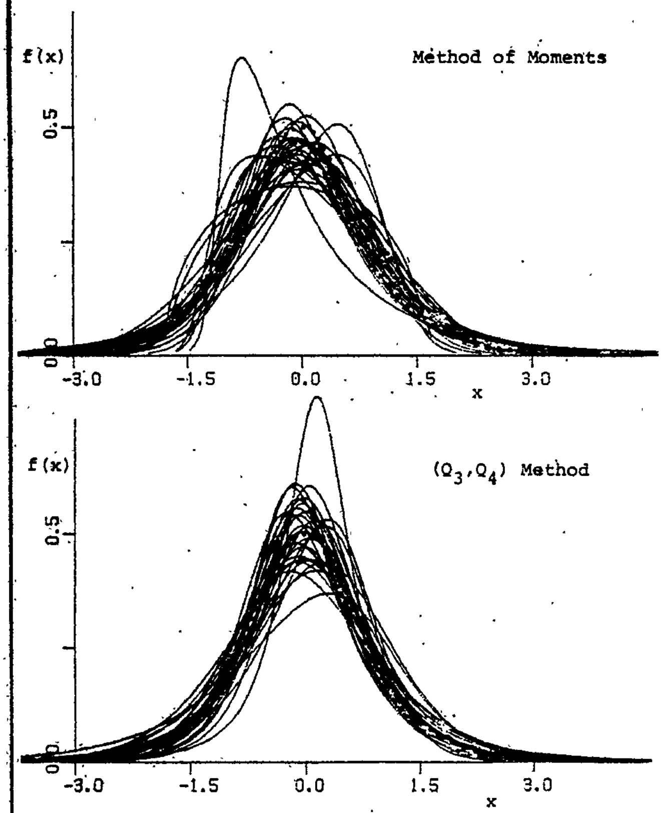

Behavior of the Sample Statistics
in the 30 Replications for
$\alpha_3 = 0$, $\alpha_4 = 6$ and $n = 50$

| Statistic | Minimum | Maximum | Average | Standard Deviation | Expected Value |
|---|---|---|---|---|---|
| $\bar{x}$ | -0.2678 | 0.3384 | -0.0396 | 0.125 | 0 |
| $\hat{\sigma}^2$ | 0.5301 | 1.6635 | 0.9786 | 0.304 | 1 |
| $\hat{\alpha}_3$ | -2.5322 | 2.1240 | -0.0453 | 0.799 | 0 |
| $\hat{\alpha}_4$ | 1.0138 | 17.6625 | 4.4964 | 3.821 | 6 |
| $\bar{L}(.05)$ | -3.8817 | -1.2839 | -2.1728 | 0.595 | -2.2563 |
| $\bar{L}(.50)$ | -1.1037 | -0.4731 | -0.7717 | 0.168 | -0.7372 |
| $\bar{U}(.05)$ | 1.1761 | 3.4497 | 2.0343 | 0.521 | 2.2563 |
| $\bar{U}(.50)$ | 0.3939 | 1.1668 | 0.6925 | 0.162 | 0.7372 |
| $\bar{M}(.50)$ | -0.2374 | 0.1653 | -0.0422 | 0.113 | 0 |
| $Q_3$ | 0.5223 | 2.1348 | 1.0439 | 0.385 | 1 |
| $Q_4$ | 2.4035 | 3.7639 | 2.8708 | 0.308 | 3.0604 |

Observed Combinations of $\hat{\alpha}_3$ and $\hat{\alpha}_4$
For Which $\lambda_3$ and $\lambda_4$ Do Not Exist

(-0.0178, 1.0139)
(0.3873, 1.7385)
(0.1072, 1.1206)
(-0.2777, 1.3161)
(-0.2208, 0.9514)
(-0.1441, 1.3227)

FIGURE 4. Monte Carlo Results -- $\alpha_3 = 0$, $\alpha_4 = 6$, and $n = 50$



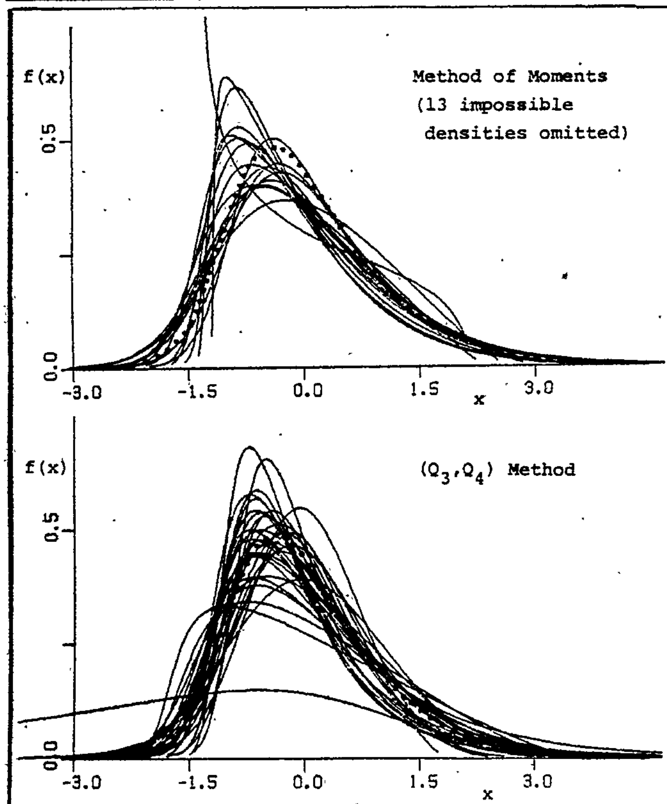

Behavior of the Sample Statistics in the 30 Replications for $\alpha_3 = 0$, $\alpha_4 = 6$ and $n = 100$

| Statistic | Minimum | Maximum | Average | Standard Deviation | Expected Value |
|---|---|---|---|---|---|
| $\bar{x}$ | -0.2055 | 0.2481 | 0.0253 | 0.102 | 0 |
| $\hat{\sigma}^2$ | 0.7135 | 1.7589 | 1.0177 | 0.241 | 1 |
| $\hat{\alpha}_3$ | -1.6700 | 3.5123 | 0.2759 | 0.864 | 0 |
| $\hat{\alpha}_4$ | 2.0243 | 34.4756 | 6.1313 | 5.855 | 6 |
| $\bar{L}(.05)$ | -3.2040 | -1.5634 | -2.1522 | 0.424 | -2.2563 |
| $\bar{L}(.50)$ | -0.9905 | -0.4096 | -0.7105 | 0.136 | -0.7372 |
| $\bar{U}(.05)$ | 1.5923 | 3.7806 | 2.3398 | 0.435 | 2.2563 |
| $\bar{U}(.50)$ | 0.5337 | 0.9993 | 0.7611 | 0.124 | 0.7372 |
| $\bar{M}(.50)$ | -0.1792 | 0.2159 | 0.0061 | 0.092 | 0 |
| $Q_3$ | 0.6018 | 1.5696 | 1.1205 | 0.282 | 1 |
| $Q_4$ | 2.7139 | 3.7799 | 3.0565 | 0.264 | 3.0604 |

Observed Combinations of $\hat{\alpha}_3$ and $\hat{\alpha}_4$ For Which $\lambda_3$ and $\lambda_4$ Do Not Exist

none

FIGURE 5. Monte Carlo Results -- $\alpha_3 = 0$, $\alpha_4 = 6$, and $n = 100$



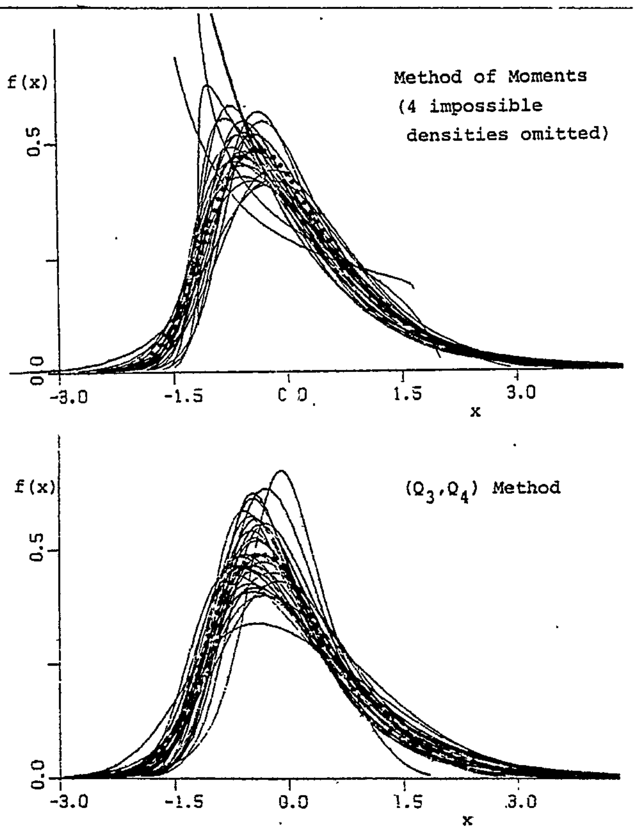

Behavior of the Sample Statistics in the 30 Replications for $\alpha_3 = 1$, $\alpha_4 = 5$ and $n = 50$

| Statistic | Minimum | Maximum | Average | Standard Deviation | Expected Value |
|---|---|---|---|---|---|
| $\bar{x}$ | -0.2481 | 0.3434 | 0.0341 | 0.123 | 0 |
| $\hat{\sigma}^2$ | 0.5212 | 1.8083 | 0.9649 | 0.290 | 1 |
| $\hat{\alpha}_3$ | 0.1059 | 3.4517 | 0.8691 | 0.790 | 1 |
| $\hat{\alpha}_4$ | 0.6749 | 16.9619 | 4.0721 | 3.565 | 5 |
| $\bar{L}(.05)$ | -2.2217 | -1.1551 | -1.5712 | 0.250 | -1.6277 |
| $\bar{L}(.50)$ | -0.9895 | -0.5675 | -0.7719 | 0.114 | -0.7548 |
| $\bar{U}(.05)$ | 1.3359 | 3.8521 | 2.3618 | 0.574 | 2.6149 |
| $\bar{U}(.50)$ | 0.3402 | 1.2542 | 0.7037 | 0.197 | 0.7541 |
| $\bar{M}(.50)$ | -0.3799 | 0.0465 | -0.1752 | 0.121 | -0.1313 |
| $Q_3$ | 1.1490 | 3.0847 | 1.8779 | 0.544 | 1.8349 |
| $Q_4$ | 2.2443 | 3.1935 | 2.6710 | 0.279 | 2.8118 |

Observed Combinations of $\hat{\alpha}_3$ and $\hat{\alpha}_4$ For Which $\lambda_3$ and $\lambda_4$ Do Not Exist

| | |
|---|---|
| (0.5572, 2.0429) | (0.0733, 0.6749) |
| (3.1295, 16.9619) | (3.4517, 14.4515) |
| (1.0338, 1.7667) | (0.3814, 1.7880) |
| (0.9107, 0.3097) | (2.2734, 8.7950) |
| (0.4514, 1.6498) | (0.2459, 1.1087) |
| (0.1059, 0.8689) | (0.2499, 1.6776) |

FIGURE 6. Monte Carlo Results -- $\alpha_3 = 1$, $\alpha_4 = 5$, and $n = 50$




Behavior of the Sample Statistics in the 30 Replications for $\alpha_3 = 1$, $\alpha_4 = 5$ and $n = 100$

| Statistic | Minimum | Maximum | Average | Standard Deviation | Expected Value |
|---|---|---|---|---|---|
| $\bar{x}$ | -0.1940 | 0.2422 | 0.0272 | 0.101 | 0 |
| $\hat{\sigma}^2$ | 0.7299 | 1.4443 | 1.0388 | 0.214 | 1 |
| $\hat{\alpha}_3$ | 0.3727 | 4.2466 | 1.2060 | 0.770 | 1 |
| $\hat{\alpha}_4$ | 1.6667 | 27.2252 | 5.9566 | 4.631 | 5 |
| $\bar{L}(.05)$ | -1.9892 | -1.3061 | -1.5650 | 0.180 | -1.6277 |
| $\bar{L}(.50)$ | -0.9007 | -0.4906 | -0.7286 | 0.096 | -0.7548 |
| $\bar{U}(.05)$ | 1.8594 | 3.5431 | 2.6896 | 0.469 | 2.6149 |
| $\bar{U}(.50)$ | 0.5126 | 1.0646 | 0.7831 | 0.151 | 0.7541 |
| $\bar{M}(.50)$ | -0.3211 | 0.1152 | -0.1266 | 0.100 | -0.1313 |
| $Q_3$ | 1.2729 | 2.7176 | 1.9929 | 0.401 | 1.8349 |
| $Q_4$ | 2.4721 | 3.3816 | 2.8216 | 0.249 | 2.8118 |

Observed Combinations of $\hat{\alpha}_3$ and $\hat{\alpha}_4$ For Which $\lambda_3$ and $\lambda_4$ Do Not Exist

(4.2466, 27.2252)
(0.6159, 2.3576)
(2.3762, 9.9702)
(0.1925, 1.6667)

FIGURE 7. Monte Carlo Results -- $\alpha_3 = 1$, $\alpha_4 = 5$, and $n = 100$

```
C ****                  PROGRAM QPOW                         ****
C
C **** PROGRAM SEARCHES FOR THE 4 PARAMETERS OF THE          ****
C **** RAMBERG-SCHMEISER-TUKEY DISTN. CORRESPONDING          ****
C **** TO SPECIFIED MEAN, VARIANCE, Q3 (A MEASURE OF         ****
C **** SYMMETRY), AND Q4 (A MEASURE OF TAILWEIGHT)           ****
C
C **** PARAMETERS ARE DETERMINED BY SEARCHING FIRST          ****
C **** FOR LAMBDA(3) AND LAMBDA(4) FOR WHICH MINIMIZE ****
C **** THE SUM OF SQUARED DEVIATIONS BETWEEN THE             ****
C **** SPECIFIED AND CALCULATED VALUES OF Q3 AND Q4          ****
C
C **** SEARCH METHOD USED IS POWELL'S ALGORITHM FOR          ****
C **** UNCONSTRAINED FUNCTION MINIMIZATION                   ****
C
C **** LAMBDA(1) AND LAMBDA(2) ARE THEN DETERMINED           ****
C **** FROM LAMBDA(3) AND LAMBDA(4) USING FORMULAS           ****
C **** FOR THE MEAN AND VARIANCE                             ****
C
      IMPLICIT DOUBLE PRECISION(A-H,L,M,O-Z)
      DOUBLE PRECISION MEAN,MU
      DIMENSION X(10),E(10),W(130),LAMBDA(4),START(2)
C
      COMMON Q3,Q3CALC,Q4,Q4CALC,MU,SIGMA2,IFUN
C
      NI=5
      NO=6
C
      READ(5,605) MAXIT,ESCALE,(E(I),I=1,2)
 605  FORMAT(I10,3D10.0)
C
      WRITE(6,701) MAXIT,ESCALE,(E(I),I=1,2)
 701  FORMAT(35H1PARAMETERS FOR POWELL'S ALGORITHM:,/,
     A5X,9H MAXIT = ,I4,12H   ESCALE = ,F5.0,
     B10H   E(1) = ,D10.3,10H   E(2) = ,D10.3,/////)
C
      IPRINT=3
      N=2
      NW=N*(N+3)
      TOL=5.D-4
C
 105  READ(5,610) MU,SIGMA2,Q3,Q4
 610  FORMAT(4D10.0)
      IF(Q4.LE.0.D0) STOP
C
      IFUN=0
C
      READ(5,615) (X(I),I=1,2)
 615  FORMAT(2D10.0)
C
      START(1)=X(1)
      START(2)=X(2)
C
 140  CALL BOTH(X,E,N,EF,ESCALE,IPRINT,MAXIT,W,NI,NO,NW)
C
C    CALL FUNCTION TO INSURE THAT Q3 AND Q4 CORRESPOND
C    TO OPTIMAL VALUES OF LAMBDA(3) AND LAMBDA(4) FROM
C    POWELL'S ALGORITHM
C
      CALL CALCFX(N,X,EF)
      LAMBDA(3)=X(1)
      LAMBDA(4)=X(2)
C
C    CALCULATE LAMBDA(1) AND LAMBDA(2) FROM LAMBDA(3)
C    AND LAMBDA(4)
C
      CALL FN2(LAMBDA)
C
      WRITE(6,710) (J,LAMBDA(J),J=1,4)
 710  FORMAT(///,2(8X,2(7HLAMBDA(,I1,3H): ,D14.7,4X),/))
C
C    COMPUTE MEAN AND VARIANCE FROM LAMBDA VALUES
C
      CALL MVCHCK(LAMBDA,MEAN,VAR)
C
      WRITE(6,715)
 715  FORMAT(16X,34HMEAN      VARIANCE      Q3          Q4,/)
C
      WRITE(6,720) MU,SIGMA2,Q3,Q4
 720  FORMAT(14H     DESIRED:  ,4(F8.4,2X))
      WRITE(6,725) MEAN,VAR,Q3CALC,Q4CALC
 725  FORMAT(14H CALCULATED:  ,4(F8.4,2X),//)
```

FIGURE 8. FORTRAN Program to Determine Lambda Parameters Corresponding to Specified Mean, Variance, $Q_3$, and $Q_4$

```
C
      WRITE(6,730) EF
  730 FORMAT(30H MINIMUM VALUE OF FUNCTION IS ,D15.7,//)
C
      WRITE(6,732) (START(I),I=1,2)
  732 FORMAT(10X,34HSTARTING VALUES USED -- START(1): ,
     ZD14.7,/,34X,10HSTART(2): ,D14.7,//)
C
      IFUN=IFUN-1
      WRITE(6,735) IFUN
  735 FORMAT(28H POWELL'S ALGORITHM REQUIRED,I6,
     A21H FUNCTION EVALUATIONS,//)
C
      IF(EF.LT.TOL) GO TO 105
      WRITE(6,770)
  770 FORMAT(1X,36H*** THIS VALUE OF OBJECTIVE FUNCTION,
     B17H MAY BE TOO LARGE,/,10X,18H--- SUGGEST USE OF,
     C30H DIFFERENT STARTING VALUES ***,//)
C
      GO TO 105
C
      END


      SUBROUTINE CALCFX(N,L,FN)
C
C **** SUBROUTINE EVALUATES OBJECTIVE FUNCTION (9)     ****
C **** THE SUM OF THE SQUARED DIFFERENCES BETWEEN       ****
C **** THE SPECIFIED VALUES OF Q3 AND Q4 AND THOSE      ****
C **** CALCULATED AS FUNCTIONS OF LAMBDA(3) AND         ****
C **** LAMBDA(4)                                        ****
C
      IMPLICIT DOUBLE PRECISION (A-H,L,M,O-Z)
      DOUBLE PRECISION MU
      DIMENSION L(2)
      COMMON Q3,Q3CALC,Q4,Q4CALC,MU,SIGMA2,IFUN
C
      IFUN=IFUN+1
C
      L3=L(1)+1.D0
      L4=L(2)+1.D0
C
C   NOTE THAT L(1) IS ACTUALLY LAMBDA(3)+1 AND
C             L(2) IS ACTUALLY LAMBDA(4)+1
C
C   INSURE THAT L(1) AND L(2) HAVE SAME SIGN
C
      IF(L(1)*L(2).LT.0.D0) GO TO 999
C
      U05=20.D0*(((1.D0-(0.95D0**L3))/L3)-((0.05D0**L4)/L4))
      U50=2.D0*(((1.D0-(0.5D0**L3))/L3)-((0.5D0**L4)/L4))
C
      L05=20.D0*(((0.05D0**L3)/L3)+(((0.95D0**L4)-1.D0)/L4))
      L50=2.D0*(((0.5D0**L3)/L3)+(((0.5D0**L4)-1.D0)/L4))
C
      M50=((.75D0**L3)-(.25D0**L3))/L3
      M50=2.D0*(M50+(((.25D0**L4)-(.75D0**L4))/L4))
C
      Q3CALC=(U05-M50)/(M50-L05)
      Q4CALC=(U05-L05)/(U50-L50)
C
      T3=Q3-Q3CALC
      T4=Q4-Q4CALC
C
C   FUNCTION VALUE IS SQUARED DIFFERENCE BETWEEN
C   CALCULATED AND DESIRED VALUES OF Q3 AND Q4
C
      FN=(T3*T3)+(T4*T4)
      RETURN
C
C
  999 FN=1.D3
      RETURN
      END


      SUBROUTINE FN2(LAMBDA)
C
C **** SUBROUTINE CALCULATES LAMBDA(1) AND LAMBDA(2) ****
C **** AS FUNCTIONS OF LAMBDA(3) AND LAMBDA(4)       ****
C
      IMPLICIT DOUBLE PRECISION(A-H,L,O-Z)
      DOUBLE PRECISION MU
      DIMENSION LAMBDA(4)
      COMMON A3,A3S,A4,A4S,MU,SIGMA2,IFUN
C
C
   10 A=(1.D0/(1.D0+LAMBDA(3)))-(1.D0/(1.D0+LAMBDA(4)))
      B20=1.D0/(1.D0+(2.D0*LAMBDA(3)))
      B02=1.D0/(1.D0+(2.D0*LAMBDA(4)))
      B11=BETA(LAMBDA(3)+1.D0,LAMBDA(4)+1.D0)
      B=B20+B02-(2.D0*B11)
C
      LAMBDA(2)=DSQRT(B-(A*A))*DSIGN(1.D0,LAMBDA(3))
      LAMBDA(2)=LAMBDA(2)/DSQRT(SIGMA2)
      LAMBDA(1)=MU-A/LAMBDA(2)
C
      RETURN
C
      END


      SUBROUTINE MVCHCK(LAMBDA,MEAN,VAR)
C
C **** SUBROUTINE CALCULATES THE MEAN AND VARIANCE ****
C **** AS FUNCTIONS OF THE FOUR LAMBDA PARAMETERS ****
C
      IMPLICIT DOUBLE PRECISION(A-H,L,O-Z)
      DOUBLE PRECISION MEAN
      DIMENSION LAMBDA(4)
C
      A=(1.D0/(1.D0+LAMBDA(3)))-(1.D0/(1.D0+LAMBDA(4)))
      B20=1.D0/(1.D0+(2.D0*LAMBDA(3)))
      B02=1.D0/(1.D0+(2.D0*LAMBDA(4)))
      B11=BETA(LAMBDA(3)+1.D0,LAMBDA(4)+1.D0)
      B=B20+B02-(2.D0*B11)
C
      MEAN=LAMBDA(1)+(A/LAMBDA(2))
      VAR=(B-(A*A))/(LAMBDA(2)*LAMBDA(2))
C
      RETURN
      END


      DOUBLE PRECISION FUNCTION BETA(X,Y)
C
      DOUBLE PRECISION X,Y,DGAMMA,BETA
C
      BETA=DGAMMA(X)*DGAMMA(Y)/DGAMMA(X+Y)
      RETURN
      END


      SUBROUTINE BOTH (X,E,N,EF,ESCALE,IPRINT,MAXIT,W,
     1NI,NO,NW)
C
C **** SUBROUTINE PERFORMS POWELL'S ALGORITHM FOR      ****
C **** UNCONSTRAINED FUNCTION MINIMIZATION             ****
C **** DEVELOPED BY M.J.D.POWELL                       ****
C
C **** SOURCE:                                         ****
C ****     "OPTIMIZATION TECHNIQUES WITH FORTRAN"      ****
C ****       BY J.L.KEUSTER AND J.H.MIZE               ****
C ****       (NEW YORK, MCGRAW-HILL, 1973)             ****
C
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FIGURE 8. (Continued)