A METHOD FOR DEVELOPING CLOSED FUNCTIONAL REPRESENTATIONS OF SERVICE RATES AND ARRIVAL RATES IN THE SIMULATION OF A NONSTATIONARY QUEUE

Peter M. Ellis, Ph.D.

Utah State University

ABSTRACT

A Fourier series and stepwise multiple repression analysis can be utilized in the approximation of the time dependency of the average arrival rates of a queuing simulation. The procedure is not difficult, the approximation is monitored and the final model is easy to incorporate into the simulation.

BACKGROUND

Monte Carlo simulation has been widely applied to queuing problems, both in the standard operations research textbooks and in the research literature. An important reason for this is that underlying probability distributions of arrivals and service times are often not amenable to a tractable or close-form mathematical analysis, whereas utilization of random number generators in a simulation is typically not difficult.

Many replications of a queuing simulation may be executed to obtain excellent numerical results relating to the qualitative nature of the queue. These results are more likely to be understood by operating managers than are mathematical derivations, state equations and generating functions. And, of course, the inherent mathematical difficulty of the queuing analysis may preclude obtaining exact results, leaving simulation as the only quantitative tool reasonably applicable to the study.

NONSTATIONARY QUEUES

In a stationary queuing study the average customer arrival rates and expected service time are constant over time. There are many situations where the assumption of stationarity is untenable. Many retail stores, restaurants and emergency services face customer arrival rates that vary greatly through the hours of the day. It is possible that service rates vary over time because of learning or operator fatigue.

The average queue structure is known as nonstationary or transient if either the average arrival rate or average service rate is permitted to vary over time. Following convention and noting the time dependence of nonstationarity we denote the average rate at time t as $\lambda(t)$ and the average service rate at time t as $\mu(t)$. The subject of interest is the development of continuous functions which closely approximate the actual values of $\lambda(t)$ and $\mu(t)$.

There is not a vast body of literature devoted to simulation or other approaches to nonstationary queues. Leese and Body (1966) developed a numerical technique to evaluate a nonstationary single server queue. They also compared their results and results of the numerical studies of other researchers to the performance of a Monte Carlo simulation model. Kolesar, et al, (1975) used numerical integration of a truncated subset of the state differential-difference equations to estimate the queue characteristics over time. An excellent review of this field is presented by Rothkopf and Oren (1979). As well as providing the overview, they also developed approximations of the characteristics of the M/M/S queue.

THE USE OF A FOURIER SERIES

Simulation of a queue over time will be facilitated through the use of a closed functional representation of $\lambda(t)$ and $\mu(t)$. If the actual average arrival and service rates are continuous, or at least piecewise continuous, functions of time' a Fourier Series representation can be used to approximate them as closely as desired. A Fourier Series can be used to represent any continuous or piecewise continuous function f(x) on the interval $[-\pi,\pi]$, as shown below.

$$f(x) = a_o + \Sigma_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

A queuing simulation over time deals with a time domain [0,T], whereas the Fourier Series of

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(1) is defined on $[-\pi, \pi]$. Therefore, the transformation $x = \pi(2t-T)/T$ is needed. The unknown functions $\lambda(t)$ and $\mu(t)$ may thereby be written as

$$\left.\begin{array}{l}\lambda(t)\\ \mu(t)\end{array}\right\} = a_o + \Sigma_{n=1}^{\infty} ((a_n \cos (n\pi(2t-T)/T) + (b_n \sin n\pi(2t-T)/T)) \quad (2)$$

Although the equality of (2) requires an infinite member of terms in the series, a satisfactory approximation can be made by truncating to an adequate finite series. It is clear that the values of a_o, a_n and b_n must be determined in order to complete the approximations for $\lambda(t)$ and $\mu(t)$. Stepwise multiple regression analysis is used to supply these. It is necessary to collect data on arrival and service rate experience at specified time points over many time cycles. These values are supplied to the stepwise multiple regression as dependent variable values. The independent variables are the various sine and cosine functions of time. Stepwise multiple regression is used to include as many of these sine and cosine terms as desired. For example, the stepwise regression may be continued until R^2 reaches a specified level. This procedure has the advantage of having the model choose which sine and cosine terms to include, freeing the user from the task of making arbitrary choices. Another attractive feature of this procedure is that the various sine and cosine terms are orthogonal. This means that the correlation matrix of all the independent variables will have zeroes everywhere off the main diagonal. The problem of multicollinearity is completely removed because of this property.

AN ACTUAL NUMERICAL EXAMPLE

A corporation which owns and operates several hospitals throughout the region constructed a new hospital with an emergency treatment section that has the flexibility of being able to modify or alter the number and type of treatment stations, depending upon need. A queuing analysis was carried out to investigate the expected patient load throughout the day. Two observations became quickly obvious. First, patient arrival rates vary significantly through the day. Second, weekend and holiday arrival rates are different from weekday arrival rates. An analysis of variance test was used to confirm that average patient stay times in emergency treatment do not vary throughout the day. The data on patient stay times suggest that average treatment time is $1/\mu = 42.3$ minutes. Therefore, no Fourier Series computation is needed for $\mu(t)$.

Hospital records were used to note the time of day at which 1816 patients arrived at the emergency ward over the weekdays of 90 day period. A twenty-four hour day was divided into 96 fifteen minute intervals. The 1816 arrival times were entered into the appropriate fifteen minute interval so that an average arrival rate could be obtained for each time block. For this purpose of carrying out the regression analysis the rate for each block of time was assigned to the moment at the end of the block of time. Therefore, the data utilized in the regression study consisted of 96 arrival rates from each of the consecutive 15 minute periods of the day, and a corresponding set of sine and cosine function values for each of these time points.

Stepwise multiple regression analysis was used to bring in appropriate sine and cosine terms until R^2 exceeded .9. The resulting model is shown below. Figure 1 shows the graph of the average arrival rate data.

$$\begin{aligned}\lambda(t) &= .01940 + .003533 \cos (3\pi(2t-1440)/1440)\\ &+ .01444 \sin (\pi(2t-1440)/1440)\\ &- .00902007 \sin (2\pi(2t-1440)/1440) \quad (3)\\ &(R^2 = .90737)\end{aligned}$$

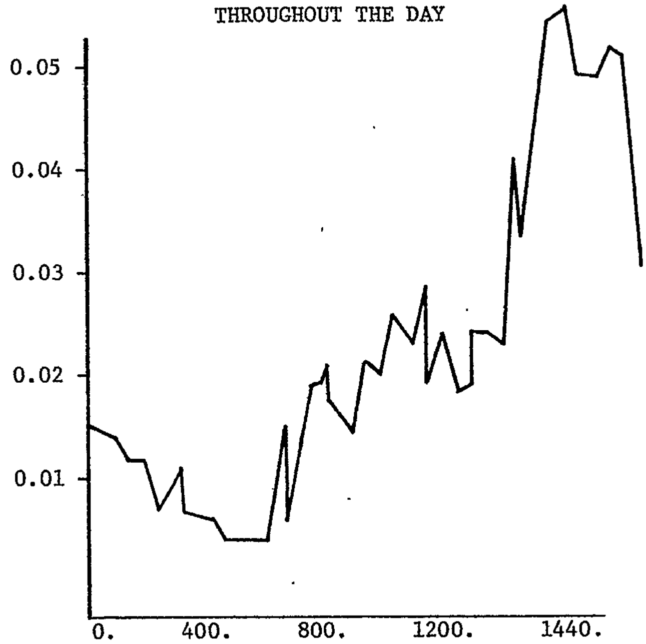
This method of derivation of a closed functional form for $\lambda(t)$ and of $\mu(t)$ provides the user with a straightforward equation that is easily incorporated into the simulation. For example, if the user is dealing with a nonstationary M/M/S queue, then a random number R $(0 < R < 1)$ is generated to determine the moment of arrival of the next customer, given a customer arrival at time 0. The interarrival time is the value of t at which

$$-\int_o^t \lambda(s)ds = \ln(1-r).$$

CONCLUSION

Queuing simulations dealing with either arrival or service rates which vary over time can be modeled with a Fourier Series and stepwise multiple regression approach. This method is attractive for several reasons. The terms are orthogonal. The user can specify the level of precision (as measured by R^2) that is desired. The model itself chooses which terms to include, removing any need for the user to subjectively choose independent variables. A simple and closed-form functional representation is obtained. This facilitates implementation in the simulation.

FIGURE 1
AVERAGE PATIENT ARRIVAL RATE
THROUGHOUT THE DAY



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