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EXTERNAL CONTROL VARIANCE REDUCTION FOR NONSTATIONARY SIMULATION

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A statistically efficient method for performing simulation experimentation of nonstationary queueing models is outlined. The method utilizes a nonstationary queueing approximation as an external control variate system. A simple nonstationary tandem queueing model serves as an example of this variance reduction method.

1. INTRODUCTION

Efficient Monte-Carlo simulation experimentation of nonstationary stochastic service systems is the topic of this presentation. Realistic models of stochastic service systems often require explicit consideration of nonstationary components. Mean arrival rates that vary with the time of day, mean service rates or number of servers that are time-dependent are often more appropriate modeling assumptions than the typical stationarity assumptions about the arrival and service processes.

If Monte-Carlo simulation experimentation is the analysis method of a nonstationary system, then a large number of replications of the experiment are required. One simulation run of a nonstationary system model yields just one observation point per time unit. If the performance measure being estimated in the experiment is a mean or variance, for instance, a reasonably sized sample or set of independent replications of the simulation is required. Further, if properties of the estimators of the measures of performance are of interest, say a confidence interval for E(N(t)) (the estimate of the expected number in the system at time t), then several sets of replications or runs are required. Small confidence intervals for estimates of means, for example, can result in a number of simulation runs that is prohibitively costly.

In one relatively simple simulation of a multi-echelon inventory system, which the authors performed, it was determined that tight confidence intervals for the performance measure of interest required two million two hundred fifty thousand simulation runs. Clearly efficient methods to perform simulation experimentation with nonstationary stochastic systems is needed. Statistically efficient methods for simulation experimentation of stationary queueing systems, especially networks of queues has been the subject of much recent research (see [Iglehart, Shedler, 1980] or [Iglehart, Shedler, 1981] for instance) however, statistically efficient methods for simulating any sort of nonstationary systems have not been widely reported in the literature [Fishman, 1979].

Of course analytically deriving performance measures, or at least theoretically exact numerical algorithms, result in estimates with zero variance. In general analytic and numerical approaches to modeling nonstationary system are unavailable. For the special case of stochastic system models driven by Nonstationary Poisson Processes and random variables with exponential distributions with nonstationary means, a theoretically exact numerical solution is available. Numerical integration of the Kolmogorov forward equations is a simple and well known method of generating time-dependent measures of performance, of stochastic system models such as E(N(t)) for a M(t)/M(t)/1 queueing system. The problem quickly encountered when using the numerical approach is the combinatorial probability and dimensionality problem. Multivariate stochastic service systems have state-spaces that become unmanageably large very quickly as the complexity of the system increases. Numerically integrating hundreds of Kolmogorov difference-differential equations is quite easily accomplished, but when the number increases to tens of thousands or more the computational demands are too great for practical modeling use.

For example a M(t)/M(t)/1/k queue has one hundred and one states (thus Kolmogorov equations) for k (capacity) of one hundred. Two M(t)/M(t)/1/k queues linked in a tandem manner have ten thousand two hundred one states/equations for $k_1 = k_2$ = one hundred. A single M(t)/M(t)/1/k queueing system that recognizes five priority classes of entities has approximately four hundred sixty million states/equations for k of one hundred. Clearly the theoretically exact numerical solution is only practical for the simplest cases.

External Control Variates used in Monte-Carlo simulation experimentation is a method that can combine the analytic and simulation approach. The method of analysis for nonstationary stochastic systems being presented here is the use of a nonstationary stochastic system approximation, whose exact performance measure values are easily computable, as an external control variate. Both problems outlined above (the large number of replications of the simulation approach and the problem of impractical computational burdens of the numerical approach) are directly attacked and eased in the analysis method.

One purpose of a control variate in simulation experimentation is to reduce the number of replications required to compute small confidence intervals for estimated performance measures (e.g. E(N(t))). Control variates that perform well are random variables that react in a similar manner to random variation, as the random variables being estimated (i.e. correlated). External control variates are random variables in a model that is similar to the model of the actual system of interest, or at least in a model that is believed to be correlated in some respects with the model of the actual system. In the case of External Control Variates for a simulated nonstationary queueing model, the external control model, of course, must be not only correlated with the real model but also must have available an exact solution. The exact solution for the control system can be the time-dependent performance measures generated by numerically integrating the associated Kolmogorov forward equations.

The key issue in developing external control variates for nonstationary simulation is the development of control systems whose solution can be conveniently generated by numerically integrating a small number of Kolmogorov difference-differential equations. Such a control system for the sample tandem queue will be described next.

2. THE SIMPLE TANDEM QUEUE

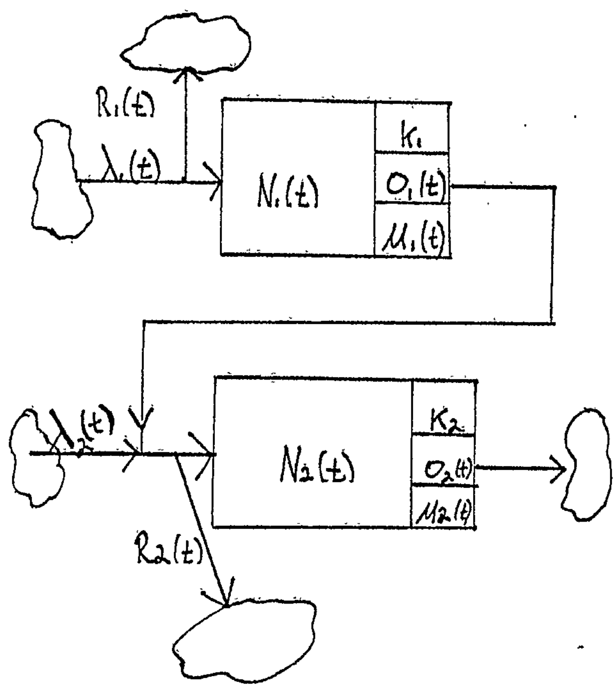
Figure one describes a simple tandem queueing network.

The number of states/Kolmogorov equations required to describe the time-dependent behavior of this system is $(k_1 + 1) \cdot (k_1 + 1)$.

An approximation for tandem queues developed by Taaffe and Clark [1982] can also serve as a control system for tandem queueing models. As described in [Taaffe, 1982] the approximation is an approximate solution to the exact problem described above. Essentially if the following approximation is made

$$Pr(N_1(t)=i, N_2(t)=0) = Pr(N_1(t)=i) Pr(N_2(t)=0) \qquad (1)$$

Then the departure process from node one is a Nonstationary Poisson Process; thus the system can be analyzed via $(k_1 + 1) + (k_2 + 1)$ Kolmogorov equations. Note (1) is not as strong an assumption as an independence assumption. The approximation provided excellent results for cases studied. For more details see [Taaffe, 1982].



TANDEM QUEUE
Figure 1

where $N_i(t)$ ≡ number of entities at node i at time t

K_i ≡ system capacity (queue and service) at node i

$\lambda_i(t)$ ≡ mean rate of the time dependent Poisson arrival process at node i at time t

$\mu_i(t)$ ≡ mean rate of service at node i at time t for the time dependent exponential service times

$O_i(t)$ ≡ mean departure rate from node i at time t

$R_i(t)$ ≡ mean rate arriving entities at node i are lost to the system due to lack of queue space at time t.

So for the tandem queue control system two hundred two differential equations instead of ten thousand two hundred one for the case $k_1 = k_2 = 100$ are required.

Given a control system, the variance reduction method for the quantity of interest, say $E(N_2(t))$ for various values of t, is quite straight forward. An excellent review of control variates and their use in simulation experimentation is available in Lavenberg and Welch [1981].

In particular, using the notation of Lavenberg and Welch:

Let $\mu \equiv E(N_2(t))$ - quantity of interest

$y \equiv 1/k \sum_{i=1}^{k} N_2(t)$ - random variable representing the average number at node 2 at time t over k runs

So $E(Y) = E(1/k \sum_{i=1}^{k} N_{2_i}(t) = \mu$

$C \equiv 1/k \sum_{i=1}^{k} N_{2_i}^{C}(t)$ - The control variable

Where $N_{2_i}^{C}(t) \equiv$ number of entities at node 2 at time t during run i of the control system

So $E(C) = E(1/k \sum_{i=1}^{k} N_{2_i}^{C}(t)) \equiv \mu_c$

Note: $E(N_{2_i}^{C}(t)) \equiv \mu_c$ is available via numerical integration of the Kolmogorov forward equations associated with the control system

So if $y(b) = y - b(C-\mu_c)$, $b \in \mathbb{R}$ Then Y(b) is a family of random variables with the property

$$E(Y(b)) = \mu - b\mu_C + b\mu_C = \mu \qquad (2)$$

i.e. the expectations of the random variables Y(b) are the same as the expectation of Y, the variable of interest. Also

$$Var(Y(b)) = Var(Y) + b^2 Var(C) - 2b\ Cov(Y,C) \qquad (3)$$

so if Cov (Y,C) is large, then Var (Y,(b)) is smaller than Var (Y) and a variance reduction is achieved. Details of implementation of this external control variate system include the estimation of optimal values for b (i.e. $\hat{b}$), as well as estimates of the indicated variances and covariances. These details are made clear in [Lavenberg, Welch, 1981] and [Law, Kelton, 1982].

3. SUMMARY

In the simple nonstationary tandem queueing network, a simple variance reduction scheme making use of a control system has been outlined. The control used is a nonstationary tandem queueing model approximation. Significant variance reductions, thus significant simulation experimentation efficiency increases, can be expected for implementations of the outlined method. In general excellent variance reduction can be achieved for nonlinear model parameter estimation using control variates [Swain, Schmeiser, 1982]. The simple nonstationary tandem queueing system discussed in this presentation, can easily be extended to larger more complex systems and nonstationary system approximations used as control variate systems. The resulting increase in efficiency of simulation experimentation of nonstationary systems may be quite substantial.

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