Proceedings of the 1984
Winter Simulation Conference
S. Sheppard, U. Pooch, D. Pegden (eds.)

ESTABLISHING FUNCTIONAL RELATIONSHIPS IN MULTIPLE RESPONSE SIMULATION: THE MULTIVARIATE GENERAL LINEAR METAMODEL*

Linda Weiser Friedman
Assistant Professor
Department of Statistics and Computer Information Systems
Baruch College of the City University of New York
New York, N.Y. 10010

ABSTRACT

The general linear metamodel has been found to be an effective tool in post-simulation analysis. This paper addresses the multiple response problem in simulation and proposes a multivariate general linear metamodel as an aid in understanding and interpreting simulation results. Such a metamodel was developed using multiple response data generated from an M/M/1 queueing system simulation. The results supported the author's contention as to the value and validity of this type of multivariate technique in simulation.

INTRODUCTION

The simulation model, although simpler than the real-world system, is still a very complex way of relating input to output. Since one of the aims of most simulations must be to gain an understanding of this relationship, an even simpler model may be used in addition to the simulation model. When a model is used as a device in order to better understand and explore a more complex model the simpler, auxilliary, model is frequently referred to as a metamodel. Obviously, this auxilliary model will not be merely a simplified version of the simulation model, since if the simulation model could have been simplified, the researcher would have done so.

Several authors have pointed out the need for an analytic auxilliary model, a metamodel, to aid in interpretation of the more detailed model: Geoffrion (5), whose concern was with mathematical programming models; Blanning (2,3,4), who championed the use of metamodels for all kinds of management science models; Lawless et al (13) and Rose and Harmsen (15), who made explicit use of metamodels for sensitivity analysis. As Pegels (14, p.203) so aptly demonstrated, use of a metamodel means that simulation analysis and analytic techniques do not have to be two divergent paths. The two techniques can work hand-in-hand to achieve the aim of the experimenter. Ignall et al (7) also advocate that one take advantage of the potential benefits of both simulation and analytic models.

*taken from the author's doctoral dissertation submitted to the Faculty of the Polytechnic Institute of New York in partial fulfillment of the requirements for the degree Doctor of Philosophy (Operations Research), 1983.

The use of a metamodel in post-simulation analysis has many benefits, some of which are: model simplification, enhanced exploration and interpretation of the model, the unraveling of a model's dynamics in order to gain a better understanding of the system's behavior, and generalization to models of other systems of the same type. In addition, many hypotheses regarding the system may be tested without the need for additional simulation runs to generate new values. Inverse questions are more easily answered, e.g., given a particular value for a response variable, what input value (factor level) is possible?

One simple metamodel favored by some simulation researchers, notably Kleijnen (8,9,11, 12), is the additive model of experimental design, often generalized as a "regression" model. The linear regression metamodel can provide additional information regarding the relative contribution of each input factor to the solution. In this paper, this model will be referred to as a general linear model (or, metamodel). In general, it may be represented as:

$$y_i = \beta_0 + \sum_j^k \beta_j x_{ij} + \varepsilon_i, \qquad i=1,\ldots,n$$

Figure 1 is a pictorial representation of the three levels of explanation of a system's dynamics. The first level, the real system itself, is unapproachable by the researcher, hence, the motivation to simulate. At the second level, the simulation model (flowchart or computer program) is "leaner" than the real system although it does attempt to replicate the real system both on a macro and a micro level. At the third level, the analytic (in this case, linear additive) metamodel attempts to approximate and aid in the interpretation of the simulation model and, ultimately, of the real-world system itself. In the current research, a multiple response metamodel developed from simular data is shown to be a good approximation to the relationships inherent not only in the simulation model but also in the real system itself (see Table 4).

Depending on the experimental layout, whether the factors are quantitative or qualitative, and the aim of the study, the general linear metamodel (linear in the parameters, not necessarily in the x's) in Figure 1 may be applied to regression analysis, analysis of variance, analysis of covariance, t-test, paired t-test, etc.

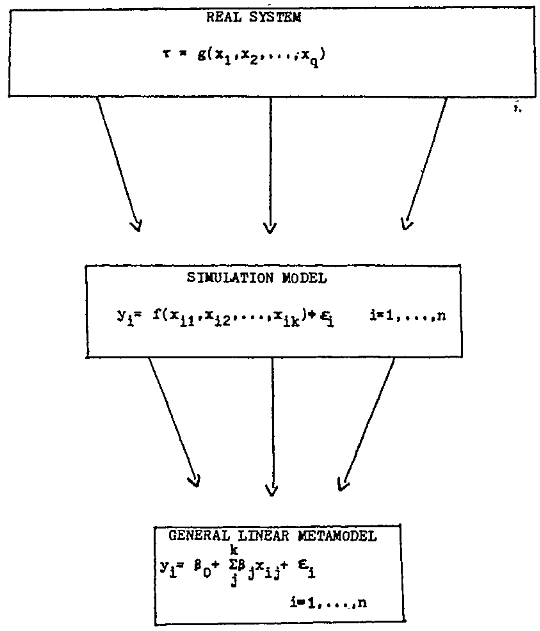

Figure 1: Single Response Simulation Analysis

Whether a researcher explicitly says so or not, designing simulation experiments which will be analyzed via one of these statistical tests implies the use of this general linear metamodel in one of its forms (see 16,17). And, in fact, the <u>explicit</u> use of a general linear metamodel enables one to interpret the simulated system more easily and more fully especially with regard to performing sensitivity analysis, answering what-if questions, and optimizing over "infinitely many" system configurations, within a specified range (10).

In a <u>multiple response</u> simular experiment we again see that there may be three levels of explanation of the system's dynamics. These levels are presented in Figure 2, and are identical to those in Figure 1 except that there is more than one dependent variable, hence, more than one measure of effectiveness is of interest to the simulation. The multivariate simular model may be visualized as:

$$y_{im} = f_m(x_{i1}, x_{i2}, \ldots, x_{ik}) + \ _{im}, \quad \begin{matrix} i=1,\ldots,n \\ m=1,\ldots,p \end{matrix}$$

where p represents the number of responses, n represents the number of replications, k is the number of input variables, x_{ij} is the value of the jth input variable in the ith replication, y_{im} is the value of the mth response in the ith replication, and ε_{im} is the experimental error of the mth response in the ith replication. This experimental error is approximated by the random number streams upon which the simulation depends.

In a multiple response simulation experiment, a multivariate metamodel must necessarily be proposed. The multivariate extension of the general linear metamodel is:

$$y_{im} = \beta_0 + \sum_j^k \beta_j x_{ij} + \varepsilon_{im}, \quad \begin{matrix} i=1,\ldots,n \\ m=1,\ldots,p \end{matrix}$$

Or, in matrix form,

$$\underline{Y}_{nxp} = \underline{X}_{nxk} \underline{\beta}_{kxp} + \underline{\varepsilon}_{nxp}$$

It can be shown that many multivariate statistical techniques and the univariate techniques of experimental design, are specific cases of this general multivariate linear model.

THE STUDY

A simulation experiment was performed using a SIMSCRIPT II.5 program of an M/M/1 queueing system. The experiment generated 10 independent replications of each of the following five single-server system variations:

ARRIVAL Rate:	9	9	12	15	18	/hr
SERVICE Rate:	10	12	16	20	20	/hr

These systems were selected judgmentally as moderately to heavily congested M/M/1 queueing systems (i.e., utilization factors from .75 to .90). Three measures of effectiveness were output from the simulation runs: The average number of demands in the system (L), the average system waiting time per demand (W), and the average utilization per server (UTIL).

In developing the metamodel, the first impulse was to fit a multivariate linear regression model with three response vari-

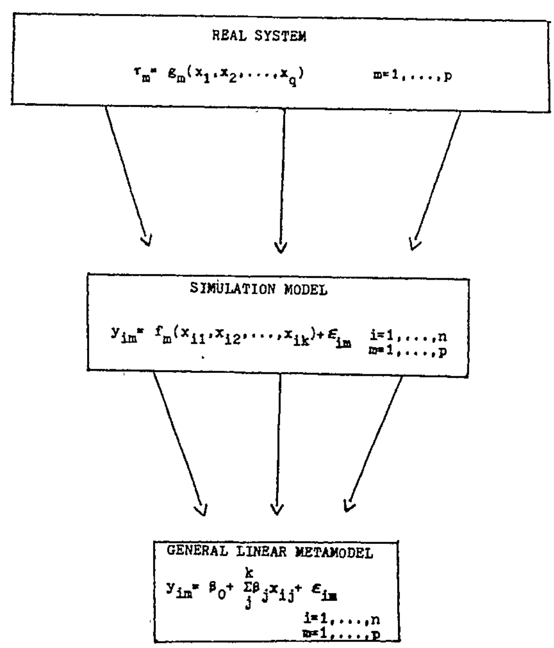

Figure 2: Multiple Response Simulation Analysis

ables (L, W, UTIL), two main effects (ARRIVAL, SERVICE), and the two-way interaction effect. However, a preliminary attempt to fit just such a model showed that it fit the data poorly. In fact, Kleijnen (9) warned of the existence of such a problem in constructing univariate regression metamodels from queueing system simulation data and suggested several solutions.

A different approach was taken based on the knowledge that an important determinant in the behavior of M/M/1 queueing systems is the ratio ARRIVAL/SERVICE (i.e., RHO). This led to the hypothesis of the following functional relationship (MOE = Measure of Effectiveness):

$$(\mathrm{MOE})_{im} = \alpha_m \frac{(\mathrm{ARRIVAL})_i^{\beta_{1m}}}{(\mathrm{SERVICE})_i^{\beta_{2m}}} \nu_{im} \qquad (1)$$

where i, the index for observations, goes from 1 to n, and m, the index for measurements, goes from 1 to p. Taking the natural logarithm of each side gives:

$$\ln(\mathrm{MOE})_{im} = \ln \alpha_m + \beta_{im}\ln(\mathrm{ARRIVAL})_i - \beta_{2m}\ln(\mathrm{SERVICE})_i + \ln \nu_{im} \qquad (2)$$

A simple change of variables, and the familiar linear regression equation is obvious:

$$Y_{im} = \beta_{0m} + \beta_{1m}X_{1i} - \beta_{2m}X_{2i} + \varepsilon_{im} \qquad (3)$$

For the sake of clarity, the change of variables is listed here without i, the index for observations:

$Y_1 = \ln(L)$

$Y_2 = \ln(W)$

$Y_3 = \ln(\mathrm{UTIL})$

$\beta_{0m} = \ln(\alpha)_m$, m=1,2,3

$X_1 = \ln(\mathrm{ARRIVAL})$

$X_2 = \ln(\mathrm{SERVICE})$

$\varepsilon_m = \ln(\nu)_m$

The least squares estimates for β_0, β_1, and β_2 are, respectively, b_0, b_1, and b_2. Table 1 gives these estimates and their standard errors. As Table 2 indicates, the multivariate linear regression model specified was a significant effect in explaining the responses, as were the two main effects. Further model exploration was not necessary as the test for overall lack of fit was not significant, and the model was accepted. For further discussion of the Wilks' lambda (Λ) statistic and the F-test derived from it, the reader is referred to (1). Table 3 displays the results of similar statistical tests for the individual regressions (on each dependent variable), laid out in the familiar analysis of variance table.

Thus, the multivariate linear regression metamodel for the M/M/1 queueing system studied may be expressed either in matrix form

Table 1

Multivariate Regression Model Estimates

Single-Server Queueing System Simulation

	Estimated Regression Coefficients		
	Y_1	Y_2	Y_3
b_0	2.77	2.77	-0.01[a]
b_1	5.75	4.76	0.97
b_2	5.75	5.76	0.97
	Standard Errors of the Coefficients		
	Y_1	Y_2	Y_3
$s(b_0)$	0.108	0.104	0.010
$s(b_1)$	0.125	0.121	0.011
$s(b_2)$	0.124	0.120	0.011

[a]not significantly different from zero

Table 2

Multivariate Regression Model

Tests of Multivariate Hypotheses

Source	Wilks' Λ	F	d.f.	p
Model	0.00000194	10754.93	6, 90	<.01
X_1	0.00275375	5432.13	3, 45	<.01
X_2	0.00518288	2879.15	3, 45	<.01
Lack-of-fit	0.98084468	0.14	6, 86	>.10

Table 3

Regression Analysis Table by Response Variable

Response	Source	d.f.	Sum of Squares	Mean Square	F	p
Y_1	Model	2	13.1870	6.5935	1094.90	<.01
	Residual	47	0.2830	0.0060		
	Lack of fit	2	0.0003	0.0002	0.03	>.10
	Pure error	45	0.2827	0.0063		
	Total	49	13.4701			
Y_2	Model	2	14.9768	7.4484	1332.96	<.01
	Residual	47	0.2640	0.0056		
	Lack of fit	2	0.0004	0.0002	0.03	>.10
	Pure error	45	0.2636	0.0059		
	Total	49	15.2409			
Y_3	Model	2	0.3749	0.1874	3787.02	<.01
	Residual	47	0.0023	0.0001		
	Lack of fit	2	0.92E-5	0.46E-5	0.09	>.10
	Pure Error	45	0.0023	0.0001		
	Total	49	0.3772			

or as p (p=3) individual equations. The coefficients in these equations were taken from Table 1 and then rounded.

$$\ln(L)=2.8+5.8\ln(\text{ARRIVAL})-5.8\ln(\text{SERVICE}) \quad (4)$$
$$\ln(W)=2.8+4.8\ln(\text{ARRIVAL})-5.8\ln(\text{SERVICE}) \quad (5)$$
$$\ln(\text{UTIL})=0+0.97\ln(\text{ARRIVAL})-0.97\ln(\text{SERVICE}) \quad (6)$$

There would be no point in the regression analysis if we could not arrive back at the original (predictive) functional relationships. This we do by raising both sides of each of the three regression equations to a power of e:

$$L = e^{2.8}\left(\frac{\text{ARRIVAL}}{\text{SERVICE}}\right)^{5.8} \quad (7)$$

$$W = e^{2.8}\left(\frac{1}{\text{ARRIVAL}}\right)\left(\frac{\text{ARRIVAL}}{\text{SERVICE}}\right)^{5.8} \quad (8)$$

$$\text{UTIL} = \left(\frac{\text{ARRIVAL}}{\text{SERVICE}}\right)^{0.97} \quad (9)$$

These functions may now be used toward any of the purposes put forth at the beginning of the paper. One of the more interesting tasks, perhaps, is the exploration of the relationships among the response variables. It is clear, for example, that

$$L = (\text{ARRIVAL})(W) \quad (10)$$

which is the well-known queueing theory relationship first demonstrated by Little (6, p.60). Another relationship which comes to the fore is

$$L = e^{2.8}(\text{UTIL})^{6}. \quad (11)$$

Similarly, we see that

$$W = e^{2.8}(1/\text{ARRIVAL})(\text{UTIL})^{6}. \quad (12)$$

For a simulation model which represents an extremely complex and dynamic system, isolating relationships as simple as these would be very valuable and rewarding.

A priori knowledge, in this case, of the true relationships underlying the M/M/1 queue had a beneficial by-product: the determination of whether the metamodel, two steps removed from the original (analytic) queueing model on which the simulation program was based, actually did predict relationships as it should have. For this purpose, 12 new points were selected (having congestion rates similar to those of the original five) and the measures of effectiveness were calculated using both the appropriate queueing theory formulas, and relationships (7), (8), and (9). These two groups of calculations are displayed in Table 4 and are obviously quite close. In fact, using the ratio

$$\frac{|\text{Metamodel} - \text{Analytic}|}{\text{Analytic}}$$

as a measure of error, the average error in L was 7.71%, the average error in W was 7.72%, and the average error in UTIL was 0.50%. Thus, the metamodel was accepted as a valid, simplified, representation of the M/M/1 queueing system in the range studied.

LIMITATIONS

As useful as a multivariate linear regression metamodel is in simulation analysis, it still leaves something to be desired. In fact,

Table 4

A Comparison of the Simulation Metamodel to the Analytic Model Single-Server Queueing System Simulation

Arrival Rate	Service Rate	Analytic L	Analytic W	Analytic UTIL	Metamodel L	Metamodel W	Metamodel UTIL
8	9	8.000	1.000	0.889	8.060	1.007	0.892
8	10	4.000	0.500	0.800	4.374	0.547	0.805
9	11	4.500	0.500	0.818	4.983	0.554	0.823
10	12	5.000	0.500	0.833	5.543	0.554	0.838
11	13	5.500	0.500	0.846	6.056	0.551	0.850
12	14	6.000	0.500	0.857	6.527	0.544	0.861
12	15	4.000	0.333	0.800	4.374	0.364	0.805
13	15	6.500	0.500	0.867	6.959	0.535	0.871
14	16	7.000	0.500	0.875	7.356	0.525	0.878
15	18	5.000	0.333	0.833	5.543	0.369	0.838
16	18	8.000	0.500	0.889	8.060	0.504	0.892
18	21	6.000	0.333	0.857	6.527	0.363	0.861

three separate multiple regression equations were developed. The multivariate aspect came into the analysis only when the models were tested using the statistical tests of multivariate analysis of variance (Table 2). The variance-covariance matrix did not enter into the analysis and, thus, the dynamic interrelationships among the response variables were not used directly in the analysis. Future research should investigate other multivariate techniques which do take the variance-covariance matrix into account in order to aid in the interpretation and exploration of complex system simulations which output data on three or more response variables.

REFERENCES

1. Anderson, T.W. An Introduction to Multivariate Statistical Analysis, Wiley, New York, 1958, 374 pages.

2. Blanning, Robert W., "The Sources and Uses of Sensitivity Information," Interfaces, vol. 4, no. 4, pp.32-38, August 1974.

3. ________, "Response to Michel, Kleijnen, and Permut," Interfaces, vol. 5, no. 3, pp.24-25, may 1975.

4. ________, "The Construction and Implementation of Metamodels," Simulation, vol. 24, no. 6, pp.177-184, June 1975.

5. Geoffrion, Arthur M., "The Purpose of Mathematical Programming is Insight, Not Numbers," Interfaces, vol. 7, no. 1, pp.81-92, November 1976.

6. Gross, Donald and Carl M. Harris. Fundamentals of Queueing Theory, Wiley, New York, 1974, 556 pages.

7. Ignall, Edward J., Peter Kolesar, and Warren E. Walker, "Using Simulation to Develop and Validate Analytic Models: Some Case Studies," Operations Research, vol. 26, no. 2, pp.237-253, Mar-April 1978.

8. Kleijnen, Jack P.C., "A Comment on Blanning's 'Metamodel for Sensitivity Analysis: The Regression Metamodel in Simulation,'" Interfaces, vol. 5, no. 3, pp.21-23, May 1975.

9. ________, "Regression Metamodels for Generalizing Simulation Results," IEEE Transactions on Systems, Man, and Cybernetics, SMC-IX, no. 2, pp.93 pp.93-96, February 1979.

10. ________, "The Role of Statistical Methodology in Simulation," In B.P. Zeigler, M.S. Elias, G.J. Klir, and T.I. Oren (eds.), Methodology in Systems Modeling and Simulation, North-Holland, New York, 1979, pp.425-445.

11. ________, "Regression Analysis for Simulation Practitioners," Journal of the Operational Research Society, vol. 32, no. 1, pp.35-43, January 1981.

12. ________, A.J. van den Burg, and R. Th. van der Ham, "Generalization of Simulation Results," European Journal of Operational Research, vol. 3, no. 1, pp. 50-64, January 1979.

13. Lawless, Robert W., Lewis H. Williams, and Charles G. Richie, "A Sensitivity Analysis Tool for Simulation with Application to Disaster Planning," Simulation, vol. 17, no. 6, pp.217-223, December 1971.

14. Pegels, C. Carl. Systems Analysis for Production Operations, Gordon and Breach, New York, 1976, 400 pages.

15. Rose, Michael R. and R. Harmsen, "Using Sensitivity Analysis to Simplify Ecosystem Models: A Case Study," Simulation, vol. 31, no. 1, pp.15-26, July 1978.

16. Weeks, James K. and John S. Fryer, "A Simulation Study of Operating Policies in a Hypothetical Dual-Constrained Job Shop," Management Science, vol. 23, no. 12, pp.1362-1371, August 1976.

17. ________ and ________, "A Methodology for Assigning Minimum Cost Due-Dates," Management Science, vol. 23, no. 8, pp.872-881, April 1977.