

MACROSCOPIC SIMULATION MODEL FOR FREEWAY TRAFFIC  
WITH JAMS AND STOP-START WAVES

Reinhart D. Kühne  
Malte B. Rödiger

Daimler-Benz AG, Research Center  
Traffic Engineering Division  
Stuttgart, Germany

ABSTRACT

A macroscopic freeway model of traffic flow is presented which can be used for simulation of freeway traffic under heavy traffic conditions. The model regards essentially two different reactions of car following behaviour (1) relaxation to a static speed-density relation (empirical fundamental diagram) (2) anticipation of traffic conditions downstream. The model can describe shock wave formation, spreading of stop-start waves, bistability and irregularities due to nonlinearity stochastics. The computation effort for the macroscopic model is described as well as the necessary storage capacity. As an application of the program a simulation procedure for design purposes of traffic control systems is presented.

1 INTRODUCTION

Macroscopic description of traffic flow implies the definition of adequate flow variables expressing the average behaviour of the vehicles on a regarded freeway. Such variables are

traffic volume  $q$   
(vehicles per hour)

traffic density  $\rho$   
(vehicles per km)

mean speed  $v$   
(sampling intervall e. g. 5 min)

Besides the basic relation

$$q = \rho * v$$

(flowrate = density · velocity)  
for a static description an empirical speed density relation holds

$$v = V(\rho)$$

As a functional relation a general expression

$$V(\rho) = v_f \left(1 - \left(\frac{\rho}{\rho_{bump}}\right)^{n_1}\right)^{n_2}$$

is used with  $n_1$  and  $n_2$  as exponents,  $v_f$  as mean free speed, and  $\rho_{bump}$  as density bumper to bumper.

The speed-density relation reflects homogeneous traffic flow with an unambiguous relation between speed and density. This assumption only holds for free traffic flow, therefore a fit of the above  $v$ - $\rho$ -relation to experimental data is only possible with restriction to free traffic flow data. A selfconsistent cut-off procedure is described in a paper by Kühne (1987). The method uses the derived below stability condition for homogeneous traffic flow and a cut-off speed for stable and unstable traffic data discrimination.

For a dynamic description two main effects are regarded: First the retarded reaction on deviations from the equilibrium speed-density relation modelled by a relaxation term. Second the drivers awareness of conditions downstream modelled by an anticipation term like in compressible gas flow. Together with the equation of continuity the dynamic description of freeway traffic reads (Witham (1974)):

$$\rho_t + (\rho v)_x = 0 \tag{1}$$

$$v_t + vv_x = \frac{1}{\tau}(V(\rho) - v) - c_0^2 \frac{\rho_x}{\rho} + \nu_0 v_{xx} \tag{2}$$

A small viscosity term  $\nu_0 v_{xx}$  was introduced to smear out sharp shocks and to guarantee a continuous description of freeway traffic even if bottlenecks are regarded or unstable traffic flow with jams and stop-start waves.

Besides the static speed density relation  $v = V(\rho)$  the dynamic traffic flow model contains as parameters the relaxation time  $\tau$  and the anticipation coefficient  $c_0^2$ . The relaxation term summarizes the characteristic times for acceleration, deceleration and relaxation. It indicates the time after which a speed

variation is compensated. Due to the anticipation, disturbances move along the freeway as waves like in compressible gas flow. Without relaxation these disturbances propagate upstream along the freeway with velocity  $c_0$ . The anticipation coefficient has the meaning of a square of the congestion velocity, when a rapid change in density runs upstream as a shock wave. Without the additional (small) viscosity the model would get difficulties in describing transient effects and high density traffic at bottlenecks (Hauer and Hurdle (1979)). The mathematical reason is that in the case of no second order derivative jumps and piecewise continuous solutions must be included artificially by jump conditions (Dressler (1949)). By discretization of space and time the model can be transformed into a set of difference equations and can reproduce real traffic including transitions from free flow conditions to congestion (Cremer (1979)).

## 2 STABILITY ANALYSIS AND SPEED-DENSITY FIT

For stability analysis of the homogeneous solution  $\rho = \rho_0$   $v = V(\rho_0)$  deviations

$$\rho - \rho_0 = \tilde{\rho} \quad v - V(\rho_0) = \tilde{v} \tag{3}$$

are regarded up to first order. This leads after introduction of dimensionless variables

$$\begin{aligned} \hat{\rho} &= \frac{\tilde{\rho}}{\rho_0} & \hat{v} &= \frac{\tilde{v}}{c_0} \\ \hat{x} &= \frac{x - V(\rho_0)t}{c_0 \tau} & \hat{t} &= \frac{t}{\tau} \end{aligned} \tag{4}$$

to a system of equations which reads ( $\wedge$ suppressed)

$$\begin{pmatrix} \partial_t & \partial_x \\ -\frac{\rho_0}{c_0} V'(\rho_0) + \partial_x & \partial_t + 1 - \nu \partial_x^2 \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{v} \end{pmatrix} = 0 \tag{5}$$

and which can be solved by the decomposition

$$\begin{pmatrix} \bar{\rho} \\ \bar{v} \end{pmatrix} \approx e^{\omega t + ikx} \quad (6)$$

This leads to a wave number dependence of the time constant

$$\omega = -\frac{1+\nu k^2}{2} \pm \sqrt{\left(\frac{1+\nu k^2}{2}\right)^2 - k^2 - ik(a+1)} \quad (7)$$

with the traffic parameter

$$a = -1 - \frac{\rho_0}{c_0} \frac{dV(\rho_0)}{d\rho} \quad (8)$$

For the homogeneous solution remains stable. The cross over occurs for

$$a = \nu k^2 \quad (9)$$

To make sure stability is given for

$$a \leq 0 \quad (10)$$

The corresponding critical density can be derived from

$$a(\rho_c) = -1 - \frac{\rho_0}{c_0} \frac{dV(\rho_0)}{d\rho} = 0 \quad (11)$$

This density does not necessarily coincide with the density of the maximum of the fundamental diagram which is derived from the relation

$$Q'(\rho_{max}) = \left. \frac{d\rho V(\rho)}{d\rho} \right|_{\rho_{max}} = 0 \quad (12)$$

or

$$-\frac{1}{c_0} V(\rho_{max}) - \frac{\rho_{max}}{c_0} \frac{dV(\rho_{max})}{d\rho} = 0 \quad (13)$$

The noncoincidence of critical density for which homogenous traffic flow becomes unstable and maximum capacity density corresponding to the maximum capacity  $Q_{max}$  is typical for phase transition phenomena in physics.

Under particular conditions beyond the stability limit it is possible to overheat a phase transition system. The overheating ration can be derived from capacity measurements coming in one case from stable traffic flow and in the other case from unstable traffic flow. The drop in capacity is about 5 % (compare Agyemang-Duah and Hall (1991)).

$$\text{drop ratio} = \frac{Q(\rho_{max}) - Q(\rho_c)}{Q(\rho_{max})} \approx 5\% \quad (14)$$

Maximum capacity  $Q(\rho_{max})$  or  $Q(\rho_c)$ , critical density  $\rho_c$ , free traffic flow mean speed  $v_f$  and density bumper to bumper (depending on mean car length including the actual truck ratio) are parameters which can be derived directly from stable traffic flow measurements. With these 4 figures the static speed density relation is completely determined. To fitting  $v_f$ ,  $\rho_c$ ,  $Q_{max}$ , and  $\rho_{bump}$  to traffic data is possible when restricting to stable flow measurement data with the above quoted cut off procedure. This presents an alternative to least square fits of functional relations which in non carefully selected measurements mix stable traffic flow data and transient data.

For German autobahns the above described fit procedure leads to parameters for the speed density relation (Sick (1989)).

$$\begin{aligned} v_f &= 115 \text{ km/h} \\ n_1 &= 2.05 \\ n_2 &= 21.11 \\ \rho_{bump} &= 180 \text{ veh/km} \end{aligned}$$

The dynamic parameters for German autobahns under normal conditions are

$$\begin{aligned} c_0 &= 63,5 \text{ km/h} \\ \tau &= 30 \text{ s} \end{aligned}$$

They are determined by the cut off procedure (Kühne (1987)) and disturbance spreading measurements (Cremer (1979); Kühne, Zackor, Balz (1988); Sick (1989)).

### 3 TRAFFIC PATTERNS OF UNHOMOGENEOUS TRAFFIC FLOW

Numerical analysis of the basic equations as starting point for a simulation of traffic flow with given initial and boundary conditions is possible only if the analysis adapts the expected traffic patterns. These traffic patterns can be derived from the traffic flow model by analytical investigations. First the universality of the equations is inquired. For this aim dimensionless variables are introduced.

$$\begin{aligned} \hat{\rho} &= \frac{\rho}{\rho_{bump}} \\ \hat{v} &= \frac{v}{c_0} \\ \hat{t} &= \frac{t}{\tau} \\ \hat{x} &= \frac{x}{c_0 \tau} \end{aligned} \tag{15}$$

which lead to the transformed basic equations (^suppressed)

$$\rho_t + (\rho v)_x = 0 \tag{16}$$

$$v_t + v v_x = \frac{1}{c_0} (V(\rho_{bump} \rho) - v - \frac{\rho_x}{\rho} + \frac{1}{Re} v_{xx}) \tag{17}$$

with the dimensionless speed-density relation from the beginning of the paper

$$\frac{1}{c_0} V(\rho_{bump} \rho) = v_f^* (1 - \rho^{n_1})^{n_2} \tag{18}$$

and the dimensionless characteristic numbers

normalized free speed:

$$v_f^* = \frac{v_f}{c_0} \tag{19}$$

Reynolds number:

$$Re = \frac{v_0}{c_0^2 \tau} \tag{20}$$

Besides the exponents of the static equilibrium speed density relation  $n_1, n_2$  which depend on the choice of an analytical expression for  $V(\rho)$  as universal parameters there are occurring only two numbers  $v_f^*$  and  $Re$ . These numbers summarize the whole static and dynamic behaviour of the solutions. An explicit dependence on the parameters and describing the dynamic behaviour is not given, only an implicit dependence via  $v_f^*$  and  $Re$ . For all subsequent calculations the variables are normalized in the above way and the universal parameters  $v_f^*$  and  $Re$  are used.

Looking for time independent solutions the basic equations simplify to (dimensionless variables!).

$$(\rho^{st} v^{st})_x = 0 \tag{21}$$

$$v^{st} + v_x^{st} = \frac{1}{c_0} (V(\rho_{bump} \rho^{st}) - v^{st} - \frac{\rho_x^{st}}{\rho^{st}} + \frac{1}{Re} v_{xx}^{st}) \tag{22}$$

The time independent solution describes a stable profile along the freeway. The equation of continuity (21) can be integrated directly.

$$\rho^{st} v^{st} = \frac{Q_0}{c_0 \rho_{bump}} \tag{23}$$

The integration constant  $Q_0$  ist the external given flow. It has the meaning of the traffic capacity at the bottleneck far away from the regarded section (stationary profiles!). As acceleration equation remains the profile equation

$$\frac{1}{Re} v_{xx}^{st} + \Gamma'(v^{st}) v_x^{st} + H(v^{st}) = 0 \tag{24}$$

with

$$\Gamma'(v^{st}) = \frac{1}{v^{st}} - v^{st} \tag{25}$$

$$H(v^{st}) = \frac{1}{c_0} V\left(\frac{Q_0/c_0}{v^{st}}\right) - v^{st} \tag{26}$$

The profile equation has the form of a Newtonian equation of motion for a pendulum with an amplitude-dependent damping term  $\Gamma'(v^{st})$  and an anharmonic force term  $H(v^{st})$ . As independent variable serves the coordinate  $x$  is used. The inertia term  $\frac{1}{Re} v_{xx}$  from the viscosity supplement to be taken in the limit  $Re \rightarrow \infty$ .

The zeros of the force term correspond to fixpoints. To discuss the position of these zeros and the occurring traffic patterns instead of the control parameter  $Q_0$  an operating point  $\rho_0 | V(\rho_0)$  is introduced which corresponds to an operating point on the fundamental diagram  $Q(\rho) = \rho V(\rho)$  with the bottleneck capacity  $Q_0$  as ordinate.

$$Q_0 \stackrel{!}{=} \rho_0 V(\rho_0) \tag{27}$$

Since the flow-density relation  $Q(\rho) = \rho V(\rho)$  is ambiguous one ends up with the same integration constant  $Q_0$  if one choses an operating point  $\rho_0 | V(\rho_0)$  from the stable or from the corresponding unstable regime. With the interpretation (27) of the bottleneck capacity the absolute term  $H(v)$  reads

$$H(v^{st}) = \frac{1}{c_0} V\left(\rho_0 \frac{\frac{1}{c_0} V(\rho_0)}{v^{st}}\right) - v^{st} \tag{28}$$

Besides the ever existing zero

$$v_1^{st} = \frac{1}{c_0} V(\rho_0) \tag{29}$$

there exist additional equilibrium points depending on the density  $\rho_0$  at the operating point. This density now serves as control parameter instead of the bottleneck capacity  $Q_0$ . Figure 1 shows the shape of the force term  $H(v^{st})$  for different values of the single control parameter  $\rho_0/\rho_{bump}$ .

The fixpoints are stable or unstable if the force in the fixpoint is restoring or repulsive respectively. To decide this the derivative  $H'(v)$  of the force with respect to the independent variable has to be analyzed. The change from one type of behaviour to the other coincides with a change of the structural stability. Depending on the control parameter values the sequence of the zeros in the force term  $H(v)$  can also change.

If the operating point is unstable and is surrounded by a saddle point in a  $v$ - $v_x$ -phase portrait both fixpoints must include a limit cycle. This limit cycle corresponds to stationary stop-start wave traffic.

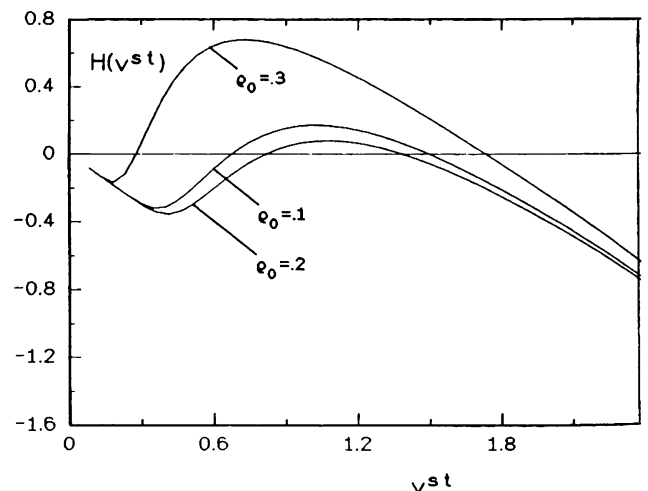


Fig. 1: Shape of the absolute term  $H(v^{st})$  for different operating point densities

For further discussion of the static solutions the substitution

$$v^{st} = 1 - X \tag{30}$$

is introduced. For practical investigations  $X$  can be restricted to small values  $|X| \ll 1$  which implies a restriction to the vicinity of the operating point and which simplifies the damping term to

$$\Gamma' \approx 2X \tag{31}$$

The substitution

$$\frac{1}{c_0} V(\rho_0) = 1 + \alpha(\rho_0) \tag{32}$$

simplifies the absolute term  $H$  together with  $|\alpha| \ll 1$ . The latter assumption restricts the operating point densities to densities slightly above the critical density  $\rho_c$  and to bottleneck capacities  $Q_0$  slightly below the maximum capacity  $Q_{max}$ . For all practical purposes a quadratic approximation for the absolute term  $H$  then is sufficient.

$$H \approx a(X + \alpha)(1 - X) \tag{33}$$

With the desired simplifications the profile equation for time independent unhomogeneous solutions reads

$$\frac{1}{Re} X_{xx} + 2XX_x + a(X + \alpha)(1 - X) = 0 \tag{34}$$

where the limit  $Re \rightarrow \infty$  has to be taken. The profile equation is a nonlinear wave equation of Lienard type (Eckhaus (1983)). It has different types of solutions depending on the control parameter  $\rho_0$  which influences the traffic parameter  $a$ .

$$a(\rho_0) = -1 - \frac{\rho_0}{c_0} \frac{dV(\rho_0)}{d\rho} \tag{35}$$

( $a$  controls the curvature of the absolute term  $H$  within the quadratic approximation) and on the position of the fixpoint  $X_1 = -\alpha$

$$\alpha(\rho_0) = -1 + \frac{1}{c_0} V(\rho_0) \tag{36}$$

Besides a typical saw tooth oscillation (compare Fig. 2), where a regular stop-start wave is shown together with measurements of stop-start waves from congested traffic on a German autobahn, shock fronts and bistability occur, depending on  $\rho_0$ . The whole variety of possible traffic patterns for time independent profiles dividing the

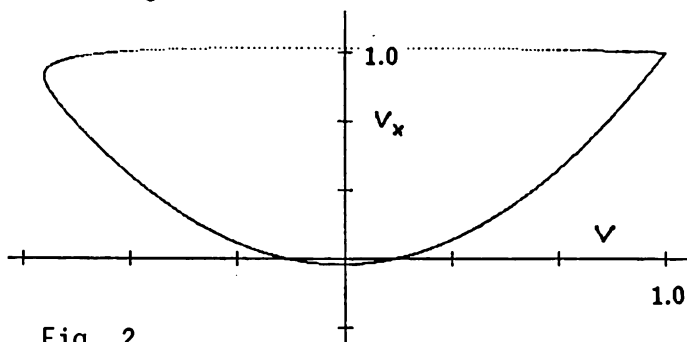


Fig. 2 Saw tooth oscillations in a  $v-v_x$ -phase portrait as example for spatially oscillating time independent profiles (mean density  $\rho_0 = \rho_{bump}$  as density of the operating point and control parameter)

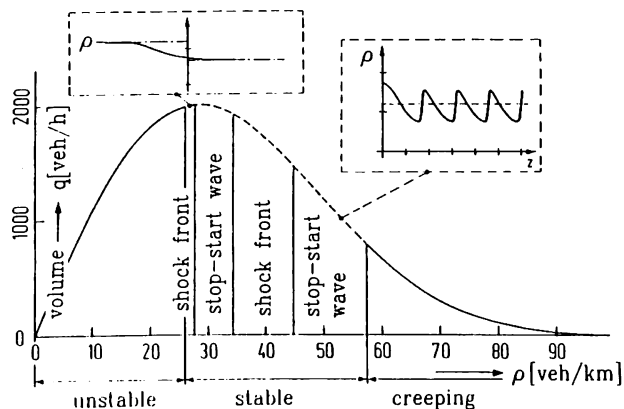


Fig. 3 Fundamental diagram together with different traffic patterns in the unstable regime. The traffic patterns are derived from the Lienard equation for inhomogeneous time independent profiles with mean density  $\rho_0$  as control parameter (compare equ. (34))

Fundamental diagram is shown in Fig. 3 with

$$Q(\rho) = \rho V(\rho) = \rho v_f (1 - (\frac{\rho}{\rho_{bump}})^{n_1})^{n_2} \quad (37)$$

#### 4 NUMERICAL INTEGRATION

For numerical integration the basic equations are transformed with

$$v_x = w \quad (38)$$

into a system of 3 equations for the unknown variables  $r, v, w$  including first derivatives with respect to space and time

$$\begin{aligned} r_t + v r_x + w &= 0 \\ v_t + v w - V(r) + v + r_x - \nu w_x &= 0 \\ v_x - w &= 0 \end{aligned} \quad (39)$$

As a further reparation for numerical integration a logarithmic density.

$$r = \ln(\frac{\rho}{\rho_{bump}}) \quad (40)$$

was introduced which simplifies the speed-density relation to

$$V(r) = \frac{v_f}{c_0} (1 - e^{n_1 r})^{n_2} \quad (41)$$

Due to the nonlinearities  $vw, V(r)$  the system can only be solved iteratively. Starting with an  $n$ th approximation

$$z^{(n)} = \begin{pmatrix} r^{(n)} \\ v^{(n)} \\ w^{(n)} \end{pmatrix} \quad (42)$$

the next  $n + 1$  st approximation is calculated by

$$z^{(n+1)} = z^{(n)} - A^{-1}(n)I(n) \quad (43)$$

with

$$I(n) = \begin{pmatrix} r_t + v^{(n)} r_x + w^{(n)} \\ v_t + v^{(n)} w^{(n)} - V(r^{(n)}) + v^{(n)} \\ v_x - w^{(n)} \\ + r_x - \nu w_x \end{pmatrix} \quad (44)$$

and

$$A(n) = \begin{pmatrix} 0 & r_x & 1 \\ -v'(r^{(n)}) & w^{(n)} + 1 & v^{(n)} \\ 0 & 0 & -1 \end{pmatrix} \quad (45)$$

Inhomogeneity  $I(n)$  and expansion matrix  $A(n)$  are calculated by an Newtonian iteration procedure applied to the system (39).

The derivatives in  $I(n)$  and  $A(n)$  occur as parameters. They are calculated in an implicit discretizing procedure [Cebeci, Smith (1974)]. The continuous arguments

$$z(x, t) = \begin{pmatrix} r(x, t) \\ v(x, t) \\ w(x, t) \end{pmatrix} \quad (46)$$

are replaced by lattice points

$$z(x_0 + i\Delta x, t_0 + j\Delta t) \equiv z_{i,j} \quad (47)$$

All derivatives are replaced by centered-difference quotients

$$\begin{aligned} z_x &\rightarrow \frac{1}{2\Delta x} (z_{i+1,j+1} - z_{i,j+1} + z_{i+1,j} - z_{i,j}) \\ z_t &\rightarrow \frac{1}{2\Delta t} (z_{i+1,j+1} + z_{i,j+1} - z_{i+1,j} - z_{i,j}) \end{aligned} \quad (48)$$

and the function values are replaced by the midpoint values

$$z \rightarrow \frac{1}{4} (z_{i+1,j+1} + z_{i,j+1} + z_{i+1,j} + z_{i,j}) \quad (49)$$

The system (39) then transforms to a difference equations system which can recursively be solved with 3 boundary conditions e. g.

$$\begin{aligned} \rho(x=0, t) \\ v(x=0, t) \\ v_x(x=0, t) \end{aligned} \quad (50)$$

and 2 initial condition e. g.

$$\begin{aligned} \rho(x, t=0) \\ v(x, t=0) \end{aligned} \quad (51)$$

## 5 APPLICATION TO A LINE CONTROL SYSTEM

The macroscopic traffic flow model allows to determine the state of a road for each position at all times. You only have to supply the correct boundary - and starting conditions. This fact can be used for simulation purposes. First, the road which shall be looked at has to be described by the length, the on- and off-ramp facilities and the number of lanes present. These facts have to be considered for the spatial conditions. Also, the traffic has to be described by the macroscopic variables mean speed, density and volume for passenger cars as well as for lorries in respect to each lane. After supplying an initial mesh for the whole freeway, the newly arriving traffic must as well be described. This enables one to study a whole variety of traffic patterns, not only as a static state but also as a dynamical creation and development. All the phenomena which are occurring can be studied and fully derived from starting conditions. So some traffic states may be identified as leading to certain phenomena, for example stop-and-go-traffic.

It has to be considered that macroscopic data are not normally used for traffic control purposes. Most of the common techniques are using data with respect to single cars. So there is the need for a translation from the macroscopic model to the scale of individual cars. This is done by statistical distributions. It is assumed that for each traffic density there exists a unique Poisson distribution which transforms the computed density in to a row of time gaps. In fact, it is a modified Poisson distribution as the time gap cannot be smaller than the average length of a car. For the speed a Gaussian distribution is assumed. This is done separately for trucks and passenger cars.

Such a discretisation is realised at each place of a traffic detector, i. e. an inductive loop. The position of each detector is kept as an absolute kilometer-value in a separate record file describing those parts of the freeway where the data acquisition for the traffic control takes place. A great advantage is the possibility of easily modifying this file. As a result, changes and amendments to the line control can easily be realised, as well as temporarily taking out malfunctioning detector units. Depending on the installation, the data from the sensors are collected for a certain period of time and then sent to the central traffic control centre where the control and the graphical presentation as well as the storage and the possibility of manual interference takes place. As an additional feature, there is the generation of environmental data like wetness, fog and ice. This is done either by random generation, i. e. simulating a patch of drifting fog, or by specifying the values in a separate record file. These data are subsequently added to the output from the traffic algorithm and further processed in the control centre.



With this simulating tool it is possible to test and to properly adjust the whole variety of parameters which are used in the line control strategy. Situations which are only occurring seldom or are posing a risk to drivers can easily be studied. It is also possible to have two sets of parameters, one the simulation tool and the other one for the real freeway, and to study their different behaviours under the same measurement data.

```
;starting density cars in veh/km
20
;percentage of lorries
10
;mean speed cars
130
;standard deviation cars
14
;mean speed lorries
70
;standard deviation lorries
8
```

Fig. 4 Input data for simulation tool

## 5 ACKNOWLEDGEMENT

The authors would thank Dr. Peter Konhäuser from Daimler Benz Research Center Stuttgart (Germany) for stimulating discussions and numerous hints and Roger Feldhoff from University of Münster (Germany) for excellently preparing the numerical integration program

## 6 LITERATURE

Kühne, R. D. 1987.  
Freeway speed distribution and acceleration noise in:  
Gartner, N. Wilson, G. (edtx.) *Transp. on Traffic Theory*, Elsevier, pp 119-137

Hauer, E., Hurdle, V. 1979.  
Discussion of the freeway traffic model. "FREFLO" *Transp. Res. Rec* 722, pp 75 - 77

Dressler, R. F. 1949.  
Mathematical solution of the problem of roll-waves in inclined open channels. *Commun. Pure Appl. Mth* 2, pp 149 - 194

Witham, G. B. 1974.  
*Linear and nonlinear waves*, Wiley & Sons, New York.

Cremer, M. 1979.  
*Der Verkehrsfluß an Fernstraßen*, Springer, Berlin.

Zackor, H.; Kühne, R. D.; Balz, W. 1988.

Investigations on traffic flow in the maximum capacity regime and at instabilities (in German) Report of the Research Program of the Federal Ministry of Transport. Vol. 524

Sick, B. 1989.  
Dynamic effects for nonlinear wave equations with applications to traffic flow theory (in German) Diplom Thesis, University of Ulm

Eckhaus, W. 1983.  
Relaxation oscillations including a standard chase of french ducks in: Verhulst, F. (ed.) *Asymptotic Analysis II*, Springer, Berlin

Cebeci, T., Smith A. M. O. 1974.  
*Analysis of turbulent boundary layers*. Academic press New York

Agyemang-Duah, K.; Hall, F. 1991.  
Some issues regarding the numerical value of freeway capacity in: Branwolk, U. (ed.) *Highway Capacity and level of Service*, Balkema, Rotterdam