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CRAMÉR-von MISES VARIANCE ESTIMATORS FOR SIMULATIONS

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ABSTRACT

We study estimators for the variance parameter σ^2 of a stationary process. The estimators are based on weighted Cramér-von Mises statistics formed from the standardized time series of the process. Certain weightings yield estimators which are "first-order unbiased" for σ^2 and which have low variance. We also show how the Cramér-von Mises estimators are related to the standardized time series area estimator; we use this relationship to establish additional estimators for σ^2.

1 INTRODUCTION

Suppose $Y_1, Y_2, \ldots, Y_n$ is a stationary process with mean μ. The estimator of choice for μ is usually the sample mean $\overline{Y}_n$, which is unbiased. In order to measure the precision of $\overline{Y}_n$, one often estimates the *variance parameter*, $\sigma^2 \equiv \lim_{n\to\infty} n\mathsf{Var}(\overline{Y}_n)$. There is a wide body of literature devoted to the topic of estimating σ^2 (see Bratley, Fox, and Schrage 1987). This paper studies estimators for σ^2 based on weighted Cramér-von Mises (CvM) statistics formed from the *standardized time series* of the process. The standardized time series is defined as

$$T_n(t) \equiv \frac{\lfloor nt \rfloor(\overline{Y}_n - \overline{Y}_{\lfloor nt \rfloor})}{\sigma\sqrt{n}} \text{ for } 0 \le t \le 1,$$

where $\overline{Y}_j \equiv \sum_{k=1}^{j} Y_k/j$, $j = 1, \ldots, n$, and $\lfloor \cdot \rfloor$ is the greatest integer function. Schruben (1983) and Glynn and Iglehart (1990) show that $T_n \xrightarrow{\mathcal{D}} \mathcal{B}$, where $\mathcal{B}$ is a standard Brownian bridge and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution as $n \to \infty$.

The present article is organized as follows. §2 reviews the standardized time series weighted area estimator for σ^2. §3 presents the CvM estimators, §4 establishes some of their properties, §5 demonstrates empirical work, §6 gives extensions, and §7 concludes.

2 THE WEIGHTED AREA ESTIMATOR

If we define

$$A_0(n) \equiv \frac{\sqrt{12}\sum_{k=1}^{n} \sigma T_n(\frac{k}{n})}{n}$$

and

$$A_0 \equiv \sqrt{12}\int_0^1 \sigma\mathcal{B}(t)\,dt,$$

then it can be shown (see Schruben 1983 and Glynn and Iglehart 1990) that

$$A_0^2(n) \xrightarrow{\mathcal{D}} A_0^2 \sim \sigma^2\chi_1^2.$$

We call $A_0^2(n)$ the *unweighted area* estimator for σ^2.

We can generalize this estimator by setting

$$A(n) \equiv \frac{\sum_{k=1}^{n} f(\frac{k}{n})\sigma T_n(\frac{k}{n})}{n}$$

and

$$A \equiv \int_0^1 f(t)\sigma\mathcal{B}(t)\,dt,$$

where (among other technical conditions) $f(t)$ is continuous and normalized so that $\mathsf{Var}(A) = \sigma^2$. One can show (see Dzhaparidze 1986 and Goldsman, Meketon, and Schruben 1990) that

$$A^2(n) \xrightarrow{\mathcal{D}} A^2 \sim \sigma^2\chi_1^2.$$

We call $A^2(n)$ the *weighted area* estimator for σ^2.

We denote the *covariance function* $R_k \equiv \mathrm{Cov}(Y_1, Y_{1+k})$ and the quantities $\gamma \equiv -2\sum_{k=1}^{\infty} kR_k$, $F \equiv \int_0^1 f(s)\,ds$, and $\overline{F} \equiv \int_0^1 \int_0^t f(s)\,ds\,dt$. Then under mild conditions (see Schmeiser and Song 1989, Foley and Goldsman 1990, and Goldsman, Meketon, and Schruben 1990),

$$\mathsf{E}[A^2(n)] = \sigma^2 + \frac{[(F-\overline{F})^2 + \overline{F}^2]\gamma}{2n} + o(\frac{1}{n}).$$

Example 1 The expected value of the unweighted area estimator is $\mathsf{E}[A_0^2(n)] = \sigma^2 + 3\gamma/n + o(1/n)$. The expected value of the weighted area estimator with weighting function $f(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ is $\mathsf{E}[A^2(n)] = \sigma^2 + o(1/n)$. In this case, we say that $A^2(n)$ is *first-order unbiased* for σ^2.

Further, if $A^4(n)$ is uniformly integrable, then the asymptotic variance of the weighted area estimator is $\mathsf{Var}(A^2) = 2\sigma^4$.

3 THE WEIGHTED CRAMÉR-von MISES ESTIMATOR

In the spirit of §2, we define the *unweighted CvM* estimator for σ^2 by

$$W_0^2(n) \equiv \frac{6\sum_{k=1}^{n}(\sigma T_n(\frac{k}{n}))^2}{n}.$$

One can show that

$$W_0^2(n) \xrightarrow{\mathcal{D}} W_0^2 \equiv 6\int_0^1 (\sigma\mathcal{B}(t))^2\,dt.$$

Cramér (1928) and von Mises (1931) studied statistics nearly of the form of $W_0^2(n)$ for the case of independent and identically distributed $Y_1, Y_2, \ldots$. Anderson and Darling (1952) and Smirnov (1937) derived the distribution of W_0^2.

A generalization of $W_0^2(n)$ is the *weighted CvM* estimator,

$$W^2(n) \equiv \frac{\sum_{k=1}^{n} g(\frac{k}{n})(\sigma T_n(\frac{k}{n}))^2}{n}.$$

Under mild conditions,

$$W^2(n) \xrightarrow{\mathcal{D}} W^2 \equiv \int_0^1 g(t)(\sigma\mathcal{B}(t))^2\,dt,$$

where $g(t)$ is continuous on $[0,1]$ and normalized so that $\mathsf{E}[W^2] = \sigma^2$. Anderson and Darling derived the distribution of W^2 with $g(t) = [t(1-t)]^{-1}$ (which is not continuous on $[0,1]$); the distribution of W^2 with an arbitrary weighting function has not been explicitly determined (see Durbin 1973).

4 PROPERTIES OF CvM ESTIMATORS

We can express the expected value of the CvM estimator in terms of $g(t)$ and R_k. First, we need some standing assumptions.

Assumptions

1. The constants μ and σ^2 satisfy $X_n \xrightarrow{\mathcal{D}} \sigma\mathcal{Z}$, where $\mathcal{Z}$ is a standard Brownian motion and
$$X_n(t) \equiv \frac{\lfloor nt\rfloor(\overline{Y}_{\lfloor nt\rfloor} - \mu)}{\sqrt{n}},$$
2. $\sum_{k=-\infty}^{\infty} R_k = \sigma^2 > 0$,
3. $\sum_{k=1}^{\infty} k^2|R_k| < \infty$,
4. g'' exists and is bounded on $[0,1]$, and
5. $\mathsf{E}[W^2]/\sigma^2 = \int_0^1 g(t)t(1-t)\,dt = 1$.

(Glynn and Iglehart 1990 list various sets of sufficient conditions for Assumption 1 to hold; these usually involve moment and mixing conditions.)

We now state the main theorem. (All proofs are in Goldsman, Kang, and Seila 1991.)

Theorem 1 *Let $G \equiv \int_0^1 g(s)\,ds$. Under the standing assumptions,*

$$\mathsf{E}[W^2(n)] = \sigma^2 + \frac{\gamma}{n}(G-1) + o(\frac{1}{n}).$$

Consider the simplest case in which $g(t)$ is a constant weighting function.

Example 2 If $g(t) = 6$ for all $t \in [0,1]$, then Theorem 1 implies that $\mathsf{E}[W_0^2(n)] = \sigma^2 + 5\gamma/n + o(1/n)$.

If $G = 1$ (and the standing assumptions hold), Theorem 1 says that the bias of $W^2(n)$ as an estimator of σ^2 is $o(1/n)$. In this case, $W^2(n)$ is first-order unbiased for σ^2.

Example 3 Suppose $g(t) = 51 - c/2 + ct - 150t^2$, where $t \in [0,1]$ and c is real. Then Theorem 1 implies that $\mathsf{E}[W^2(n)] = \sigma^2 + o(1/n)$.

We can even give exact small-sample results for certain stochastic processes.

Example 4 Consider an MA(1) process $Y_i = \theta\epsilon_{i-1} + \epsilon_i$, $i = 1, 2, \ldots$, where the ϵ_i's are independent normal $(0,1)$; so $R_0 = 1 + \theta^2$, $R_{\pm 1} = \theta$, and $R_k = 0$, otherwise. For the weights $g(t) = 6$, we have

$$\begin{aligned}\mathsf{E}[W_0^2(n)] \\ &= \sigma^2(1 - \frac{1}{n^2}) + \frac{\gamma(n-1)(5n-1)}{n^3} \\ &= \sigma^2 + \frac{5\gamma}{n} + o(\frac{1}{n}).\end{aligned}$$

For $g(t) = 51 - 150t^2$, some algebra yields

$$\begin{aligned}&\mathsf{E}[W^2(n)]\\ &= \frac{\sigma^2(n^2-1)(n^2+5)}{n^4} + \frac{\gamma(24n^3-29n^2-5)}{n^5}\\ &= \sigma^2 + o(\frac{1}{n}).\end{aligned}$$

These results are in accord with Examples 2 and 3.

The choice of weights clearly affects the variance of $W^2(n)$. In fact, the next theorem gives a useful result on the limiting variance.

Theorem 2 *Suppose $W^4(n)$ is uniformly integrable. Then under the standing assumptions,*

$$\begin{aligned}&\mathsf{Var}(W^2(n)) \to \mathsf{Var}(W^2)\\ &= 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2\,ds\,dt.\end{aligned}$$

Example 5 If $g(t) = 6$ (as in Example 2), then Theorem 2 implies that $\mathsf{Var}(W_0^2) = 4\sigma^4/5$.

Example 6 Suppose $g(t) = 51 - c/2 + ct - 150t^2$, where c is real (as in Example 3). Then $\mathsf{Var}(W^2) = (c^2 - 300c + 26856)\sigma^4/12600$. This quantity is minimized by $c = 150$, in which case $\mathsf{Var}(W^2) = 1.729\sigma^4$.

One would like to choose a weighting function which minimizes the variance of the CvM estimator while satisfying the first-order unbiasedness and normalizing constraints; i.e., find $g(t)$ which minimizes $\mathsf{Var}(W^2)$ subject to $G = 1 = \int_0^1 g(t)t(1-t)\,dt$. It is easy to show via Lagrangian multipliers that the optimal quadratic *and* cubic polynomial weighting function is $g(t) = -24 + 150t - 150t^2$, the choice studied in Example 6. The best quartic is

$$g(t) = \frac{-1310}{21} + \frac{19270t}{21} - \frac{25230t^2}{7} + \frac{16120t^3}{3} - \frac{8060t^4}{3},$$

for which $\mathsf{Var}(W^2) = 1.042\sigma^4$.

5 EMPIRICAL WORK

We present the results of some Monte Carlo simulations to evaluate the performance characteristics of the CvM estimators. Consider the AR(1) process $Y_{i+1} = \phi Y_i + \epsilon_{i+1}$, $i = 1, \ldots, n$, where the ϵ_i's are independent normal $(0, 1-\phi^2)$ random variables, $-1 < \phi < 1$, and Y_1 is initialized as a normal $(0,1)$ random variable independent of the ϵ_i's; so the Y_i's are stationary with normal $(0,1)$ marginals and covariance function $R_k = \phi^{|k|}$. Some algebra shows that the variance parameter is $\sigma^2 = (1+\phi)/(1-\phi)$.

For each value of $n = 2^k$, $k = 3, 4, \ldots, 9$, we ran 10000 independent simulations of the process with $\phi = 0.9$ to estimate the expected values and variances of four estimators for $\sigma^2 = 19$:

- Unweighted area estimator $A_0^2(n)$.
- Weighted area estimator $A^2(n)$ with first-order unbiased weighting function $f(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ (Example 1).
- Unweighted CvM estimator $W_0^2(n)$.
- Weighted CvM estimator $W^2(n)$ with first-order unbiased weighting function $g(t) = -24 + 150t - 150t^2$ (Example 6).

The results are given in Table 1. The table contains the estimated expected values of the four estimators for various n; the numbers in parentheses are the associated standard errors of the entries above them. The standard errors allow us to estimate the variance of the estimators. We first notice that all of the estimators become less biased for σ^2 as n increases. Consider the entries for "large" n, say $n \geq 512$. As predicted by Example 1, the unweighted area estimator $A_0^2(n)$ has comparatively high bias and variance, while the first-order unbiased weighted area estimator $A^2(n)$ has much lower bias and about the same variance. The unweighted CvM estimator $W_0^2(n)$ has high bias (cf. Example 2) but very low variance (cf. Example 5). Finally, the first-order unbiased CvM estimator $W^2(n)$ has very low bias (cf. Example 3) and variance which is slightly lower than those of the area estimators (cf. Example 6).

6 EXTENSIONS

We briefly suggest some extensions to the CvM estimators, all of which are discussed in Goldsman, Kang, and Seila (1991).

6.1 Still More Estimators

Another class of estimators is based on the relationship between the unweighted area and CvM estimators. With the fact that $\mathsf{Cov}(A_0^2, W_0^2) = 6\sigma^4/5$ in mind, we consider the estimator (cf. Durbin 1973 and Watson 1961)

$$\begin{aligned}U_0^2(n) &\equiv \frac{12}{n}\sum_{k=1}^{n}\left(\sigma T_n(\frac{k}{n}) - \frac{A_0(n)}{\sqrt{12}}\right)^2\\ &= 2W_0^2(n) - A_0^2(n)\\ &\xrightarrow{\mathcal{D}} U_0^2 \equiv 2W_0^2 - A_0^2.\end{aligned}$$

Table 1: Sample Expectations of Variance Estimators

n	$A_0^2(n)$	$A^2(n)$	$W_0^2(n)$	$W^2(n)$
8	0.97	0.96	0.69	0.93
	(0.01)	(0.01)	(0.01)	(0.04)
16	2.83	2.69	2.04	2.75
	(0.04)	(0.04)	(0.03)	(0.07)
32	6.50	6.36	4.90	6.52
	(0.09)	(0.09)	(0.06)	(0.11)
64	11.02	11.58	8.99	11.64
	(0.15)	(0.16)	(0.10)	(0.17)
128	14.95	16.45	13.15	16.29
	(0.21)	(0.23)	(0.14)	(0.22)
256	16.80	18.01	15.77	18.07
	(0.24)	(0.25)	(0.15)	(0.24)
512	18.12	18.86	17.41	18.89
	(0.25)	(0.26)	(0.16)	(0.25)
1024	18.44	18.85	18.12	18.89
	(0.26)	(0.27)	(0.17)	(0.25)

As before, we can generalize $U_0^2(n)$ to obtain additional estimators for σ^2. Define

$$U^2(n) \equiv \frac{1}{n}\sum_{k=1}^{n} h(\frac{k}{n})\left(\sigma T_n(\frac{k}{n}) - \frac{A_0(n)}{\sqrt{12}}\right)^2 \xrightarrow{\mathcal{D}} U^2 \equiv \sigma^2 \int_0^1 h(t)(\mathcal{B}(t) - \overline{\mathcal{B}})^2\,dt,$$

where $\overline{\mathcal{B}} \equiv \int_0^1 \mathcal{B}(t)\,dt$ and $h(t)$ is a continuous, bounded weighting function on $[0,1]$, normalized so that $\int_0^1 h(t)\,dt = 12$. Under mild conditions,

$$\mathsf{E}[U^2(n)] \to \mathsf{E}[U^2] = \sigma^2$$

and

$$\mathsf{Var}(U^2(n)) \to \mathsf{Var}(U^2) = 4\sigma^4 \int_0^1 \int_0^t h(s)h(t)c^2(s,t)\,ds\,dt,$$

where $c(s,t) \equiv s(1-t) - \frac{s-s^2}{2} - \frac{t-t^2}{2} + \frac{1}{12}$.

We mention in passing that it is possible to devise estimators for σ^2 based on other functionals of Brownian bridges – for instance, the Anderson-Darling statistic or $\int_0^1 |\mathcal{B}(t)|\,dt$.

6.2 Estimators Using Batching

All of our work so far has assumed that we have one long batch of n observations. Alternatively, we can break the n observations into b contiguous, nonoverlapping batches, each of size m (assume $n = bm$). Then, e.g., let $W_i^2(m)$, $i = 1,\ldots,b$, denote the CvM estimator formed exclusively from the ith batch of observations, $Y_{(i-1)m+1}, Y_{(i-1)m+2},\ldots,Y_{im}$. The CvM batch estimator for σ^2 is $\overline{W}^2(m) \equiv \sum_{i=1}^{b} W_i^2(m)/b$. Of course, $\mathsf{E}[\overline{W}^2(m)] = \mathsf{E}[W^2(m)]$ and, if the $W_i^2(m)$'s are approximately independent, $\mathsf{Var}(\overline{W}^2(m)) \approx \mathsf{Var}(W^2(m))/b$.

6.3 Overlapping Estimators

We can also apply the methodology of Meketon and Schmeiser (1984) in which the n observations are broken into $n-m+1$ *overlapping* batches, each of size m. Then, e.g., let $W^2(i,m)$, $i = 1,\ldots,n-m+1$, denote the CvM estimator formed exclusively from the observations $Y_i, Y_{i+1},\ldots,Y_{i+m-1}$. The CvM overlapping estimator for σ^2 is $\widetilde{W}^2(m) \equiv \sum_{i=1}^{n-m+1} W^2(i,m)/(n-m+1)$. Clearly, $\mathsf{E}[\widetilde{W}^2(m)] = \mathsf{E}[W^2(m)]$; further, in the special case that each $W^2(i,m)$ uses the weighting function $g(t) = 6$, Goldsman and Meketon (1990) show that $\mathsf{Var}(\widetilde{W}^2(m)) \approx \frac{11}{21}\mathsf{Var}(\overline{W}^2(m))$.

7 CONCLUSIONS

In this article, we introduced a class of CvM estimators for σ^2, derived expectation and variance properties, discussed some empirical results, and proposed extensions to the initial work. Although the estimators are all asymptotically unbiased for σ^2, they can be quite biased for finite samples. Luckily, we were able to find first-order unbiased estimators having comparatively low variance.

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