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MONTE CARLO ESTIMATION FOR GUARANTEED-COVERAGE NONNORMAL TOLERANCE INTERVALS

Huifen Chen
Bruce W. Schmeiser

School of Industrial Engineering
Purdue University
West Lafayette, Indiana 47907-1287, U.S.A.

ABSTRACT

We propose a Monte Carlo sampling algorithm for estimating guaranteed-coverage tolerance factors for nonnormal continuous distributions with known shape but unknown mean and variance. The algorithm is based on reformulating this root-finding problem as a quantile-estimation problem. Our quantile-estimation algorithm always converges, usually faster than stochastic-approximation algorithms, which are designed for general root-finding.

1 INTRODUCTION

Let $\{X_1, X_2, ..., X_n\}$ be a random sample from the distribution of a continuous random variable X with known shape but unknown mean and variance. The guaranteed-coverage tolerance interval $I(k^*)$ for X is $[\bar{X} - k^* S, \infty)$ for lower one-sided, $(-\infty, \bar{X} + k^* S]$ for upper one-sided, or $[\bar{X} - k^* S, \bar{X} + k^* S]$ for two-sided interval. The constant tolerance factor k^* is defined so that with $100\gamma\%$ confidence the random tolerance interval covers the proportion α of the distribution, i.e.,

$$\Pr_{\bar{X},S}\{ \Pr_X\{X \in I(k^*)\} \geq \alpha \} = \gamma. \qquad (1)$$

Here the future X is independent of the sample statistics $\bar{X}$ and S. The value of k^* depends on sample size n, coverage $\alpha \in (0,1)$, confidence $\gamma \in (0,1)$ and the distribution shape of X. For a single application, a single interval is computed from observed values of $\bar{X}$ and S^2; the probability that a future observation X lies in the interval is random, but should be at least α in $100\gamma\%$ of many applications.

Such intervals are used to predict future behavior. In computer simulation of a manufacturing system, X might be the throughput of a single future shift and the observed data $X_1, X_2, ..., X_n$ might be the n simulated shift throughputs. An α proportion of future throughput is predicted to be in the interval with confidence γ. In reliability, based on product test results $X_1, X_2, ..., X_n$ a system is designed at the tolerance bounds to ensure, with confidence γ, that system reliability is at least α; i.e., at least $100\alpha\%$ of the systems built will not fail. In quality control, a contract might specify constants n, k^* and c so that a lot is accepted if a subset $X_1, X_2, ..., X_n$ yields a value of $\bar{X} - k^* S$ less than c (lower tolerance bound); these constants can be chosen using tolerance-interval logic to guarantee that a particular lot containing $100(1-\alpha)\%$ defective items (defined as $X < c$) is accepted with probability γ.

Despite the broad range of applications, most tolerance-interval literature assumes normally distributed X's, e.g., Wald and Wolfowitz (1946), Guttman (1970), Aitchison and Dunsmore (1975), and Eberhardt et al. (1989). The one-sided tolerance factor for the normal distribution is

$$k^* = t_{n-1,\gamma}(\sqrt{n} z_\alpha)/\sqrt{n}, \qquad (2)$$

where $t_{\nu,\gamma}(\lambda)$ is the γ^{th} quantile of the noncentral t distribution with ν degrees of freedom and noncentrality parameter λ, and z_α is the α^{th} quantile of the standard normal. The normal two-sided tolerance factor k^* can be computed by solving the equation

$$\int_{-\infty}^{\infty} \Pr\{ \chi^2_{n-1} \geq \frac{(n-1)v^2}{k^{*2}} \} \frac{e^{-nu^2/2}}{\sqrt{2\pi/n}} \, du = \gamma,$$

where χ^2_ν is chi-square distributed with ν degrees of freedom, and v satisfies $\Phi(u+v) - \Phi(u-v) = \alpha$, where Φ is the standard normal distribution function. Odeh and Owen (1980) provides tables for one-sided and two-sided tolerance factors for normal distributions.

Some nonnormal literature exists. Aitchison and Dunsmore (1975) also propose different forms of tolerance intervals for binomial, Poisson, gamma and two-parameter exponential populations. Guenther (1985) provides an extensive discussion of distribution-free tolerance intervals. Wald (1942) develops maximum-likelihood tolerance limits through asymptotic theory.

We focus on lower one-sided guaranteed-coverage tolerance intervals. Calculating the factor k^* in the upper one-sided guaranteed-coverage tolerance-interval $(-\infty, \bar{X} + k^*\ S]$ is a variation of the lower one-sided problem, as discussed in Appendix A. The factor k^* in the two-sided interval $[\bar{X} - k^*\ S, \bar{X} + k^*\ S]$ can be found by a modified algorithm.

In Section 2 we propose a new Monte Carlo algorithm for the lower one-sided guaranteed-coverage tolerance factor k^* based on reformulating k^* as a distribution quantile. There algorithmic convergence speed is also discussed and compared to that of stochastic approximation. In Section 3 we study the behavior of the lower one-sided factor k^* as a function of n, α, γ, and distribution shape. In Section 4 we show that the two-sided tolerance factor is also a distribution quantile and hence can be solved by modifying the algorithm in Section 2.

2 METHOD

Let $F_X(\cdot)$ denote the distribution function from which the future observation X and the independent sample $\{X_1, X_2, ..., X_n\}$ are drawn. We assume that X is continuous (i.e., there is no point with positive probability mass), the shape of F_X is known, and the mean μ and standard deviation σ are unknown. We want to find the lower one-sided guaranteed-coverage tolerance factor k^*, given sample size n, α, γ and distribution shape, such that Equation 1, i.e.,

$$\Pr_{\bar{X},S}\{\Pr_X\{X \geq \bar{X} - k^*\ S\} \geq \alpha\} = \gamma. \quad (3)$$

is satisfied.

This problem is to find the root k^* of the equation

$$g(k^*) = \gamma, \quad (4)$$

where the function

$$g(k) = \Pr_{\bar{X},S}\{\Pr_X\{X \geq \bar{X} - kS\} \geq \alpha\} \quad (5)$$

is the confidence that the interval $[\bar{X} - kS, \infty)$ contains at least the proportion α of the measurements.

In finding the root, three properties of g are useful:

1. $g : \Re \rightarrow [0, 1]$ is a continuous nondecreasing function and strictly increasing in the set $\{k : g(k) \in (0, 1)\}$,
2. for $0 < \gamma < 1$, equation $g(k) = \gamma$ has a unique solution, k^*, and
3. g does not depend on μ or σ.

These three properties are straightforward. If the value of k increases, the value of $\bar{X} - kS$ decreases. Therefore, $g(k)$, the probability of having coverage at least α, increases. In the limits, $g(-\infty) = 0$ and $g(\infty) = 1$. Continuity follows since F_X has no mass points, so the first property holds. For the second property, existence of the root follows from continuity and the intermediate value theorem; uniqueness follows from g being increasing. To show the third property, let $Y = (X - \mu)/\sigma$ and $Y_i = (X_i - \mu)/\sigma$ for $i = 1, 2, ..., n$. Then

$$\begin{aligned} &\Pr_X\{X \geq \bar{X} - kS\} \\ &= \Pr_Y\{Y \geq \frac{\sum_{i=1}^n Y_i}{n} - k\sqrt{\frac{\sum_{i=1}^n [Y_i - \frac{\sum_{j=1}^n Y_j}{n}]^2}{n-1}}\} \\ &= \Pr_Y\{Y \geq \bar{Y} - kS_Y\}, \end{aligned}$$

where $\bar{Y}$ and $S_Y{}^2$ are the sample mean and sample variance of $Y_1, ..., Y_n$, respectively. Hence,

$$g(k) = \Pr_{\bar{Y},S_Y}\{\Pr_Y\{Y \geq \bar{Y} - kS_Y\} \geq \alpha\}$$

and the third property holds.

Despite not depending on the population mean μ or variance σ^2, g can be easily computed only for special cases, such as the normal distribution, since g depends upon the joint distribution function of $\bar{X}$ and S. However, $g(k)$ can be estimated by Monte Carlo simulation experiment, using any arbitrary values of μ and σ^2.

In Subsection 2.1, we propose interpreting k^* as a quantile, which allows application of our Quantile Estimation (QE) algorithm. In Subsection 2.2, we show that QE is asymptotically more efficient than general-purpose stochastic-approximation algorithms.

2.1 Quantile Estimation Algorithm (QE)

A natural approach to solving for k^* in Equation 3 would be to invert g, defined in Equation 5. By Property 1, the inverse function g^{-1} always exists for domain $(0, 1)$. However, it is easy to compute g^{-1} only for special cases, such as when X is normally distributed.

Nevertheless, we always can simplify g to reformulate k^* as a distribution quantile.

Result 1 *Let* $K = [\bar{X} - F_X^{-1}(1 - \alpha)]\ /\ S$. *Then* $k^* = F_K^{-1}(\gamma)$.

proof:

$$\begin{aligned} g(k) &= \Pr_{\bar{X},S}\{\ \Pr_X\{X \geq \bar{X} - kS\} \geq \alpha\ \} \\ &= \Pr_{\bar{X},S}\{\ \Pr_X\{X < \bar{X} - kS\} \leq 1 - \alpha\ \} \\ &= \Pr_{\bar{X},S}\{\ \bar{X} - kS\ \leq\ F_X^{-1}(1 - \alpha)\ \} \\ &= \Pr_{\bar{X},S}\{\ \frac{\bar{X} - F_X^{-1}(1 - \alpha)}{S}\ \leq\ k\ \} \\ &= F_K(k). \end{aligned}$$

Hence $g^{-1}(\cdot) = F_K^{-1}(\cdot)$. Therefore, $k^* = g^{-1}(\gamma) = F_K^{-1}(\gamma)$, the γ^{th} quantile of K.

Algorithm QE estimates k^* for lower one-sided tolerance intervals $[\bar{X} - k^* \; S, \infty)$ by generating m Monte Carlo independent realizations of K. The estimate is

$$\hat{k}^* = \omega \; k_{(\lfloor (m+1)\gamma \rfloor)} + (1-\omega) \; k_{(\lfloor (m+1)\gamma \rfloor + 1)}, \qquad (6)$$

the convex combination of the $\lfloor (m+1)\gamma \rfloor^{\text{th}}$ and $(\lfloor (m+1)\gamma \rfloor + 1)^{\text{th}}$ order statistics, with the weight $\omega = \lfloor (m+1)\gamma \rfloor + 1 - (m+1)\gamma$ chosen to reduce the first-order bias of the quantile estimate. (See Avramidis (1993) for other possibilities.) Neither $\hat{k}^*$ nor k^* depends on μ or σ.

Algorithm QE(m): Given n, α, γ, and distribution shape, estimate k^*.

Step 0: Set $i = 0$.

Step 1: Independently generate a random sample $\{x_1, x_2, ..., x_n\}$ from the population X with any arbitrary values of μ and σ.

Step 2: Compute the sample mean $\bar{x}$ and standard deviation s from the sample.

Step 3: Compute $k_i = [\bar{x} - F_X^{-1}(1-\alpha)]/s$.

Step 4: If $i < m$, set $i \leftarrow i + 1$ and go to Step 1.

Step 5: Compute $\hat{k}^*$ from $k_1, k_2, ..., k_m$ using Equation 6.

Given $\hat{k}^*$ from QE, a practitioner can form a lower one-sided tolerance interval $[\bar{X} - \hat{k}^* \; S, \infty)$ using observed values of $\bar{X}$ and S from real-world data.

2.2 Asymptotic Efficiency of Algorithms QE and Stochastic Approximation

We show here that the QE algorithm always converges at rate $m^{-1/2}$, the best that stochastic approximation algorithms can achieve. Furthermore, QE has no algorithmic parameter. Hence our QE is easier to apply and asymptotically more efficient.

The asymptotic distribution of the QE estimate $\hat{k}^*$ based on m independent realizations of random variable K is (Lehmann 1983, p. 394)

$$\sqrt{m}(\hat{k}^* - k^*) \xrightarrow{D} N(0, \frac{\gamma(1-\gamma)}{f_K^2(k^*)}), \qquad (7)$$

where $f_K(\cdot)$ is the density function of K. Hence QE always converges at rate $m^{-1/2}$.

Stochastic approximation is a classical Monte Carlo approach first proposed by Robbins and Monro (1951) for root-finding problems, when function value $g(\cdot)$ is difficult to compute. There are several variations (e.g., Kesten 1958, Andradottir 1992, Polyak and Juditsky 1992). All are iterative methods requiring only an ability to estimate $g(k)$. Each has several algorithmic parameters (initial point, step size, etc.) which strongly affect the speed of convergence.

Stochastic approximation has the best asymptotic distribution, that of Equation 7, when the optimal step size which depends on $g'(k^*)$ (Fabian 1973) is chosen each iteration. However, $g'(k^*)$ is unknown since k^* is unknown. Hence QE is asymptotically more efficient than stochastic approximation.

Variations of QE can improve performance. Because the normal distribution yields a fast solution via the noncentral t distribution, the normal-distribution estimator can be used as a control variate. A second variation is to sample K dependently using, for example, Latin hypercube sampling.

3 ANALYSIS

The lower one-sided tolerance factor, k^*, is a function of the parameter values α, γ, and n and the distribution shape. In Subsection 3.1 we show one property of k^* for symmetric distribution shapes, and the limiting value of k^* as sample size n goes to infinity. In Subsection 3.2 we discuss the sensitivity of k^* to the parameter values α, γ, n and distribution shape.

3.1 Symmetric Distribution Shape and Infinite Sample Size

Here we show that for symmetric distributions the value of k^* with coverage α and confidence γ is the negative value of k^* with coverage $1-\alpha$ and confidence $1-\gamma$. We also show that, as sample size n goes to infinity, k^* goes to $[\mu - F_X^{-1}(1-\alpha)]/\sigma$.

Result 2 *Let $k^*_{(n,\alpha,\gamma)}$ denote the tolerance factor for random variable X such that the coverage is α, confidence level is γ and the sample size is n. If the distribution of X is symmetric, then for $0 < \alpha, \gamma < 1$ and $n \in \{2, 3, ...\}$*

$$k^*_{(n,\alpha,\gamma)} = -k^*_{(n,1-\alpha,1-\gamma)}. \qquad (8)$$

The proof is in Appendix B.

The limiting value of k^* as the sample size n goes to infinity, for given values of α and γ, is sometimes useful as a bound, as an initial guess, or as an approximation when n is large.

Result 3 $\lim_{n\to\infty} k^* = [\mu - F_X^{-1}(1-\alpha)] \,/\, \sigma$.

When $n \to \infty$, $\bar{X}$ converges in distribution to μ and S converges in distribution to σ. Therefore, Slutsky's theorem implies from Result 1 that the random variable K converges in distribution to the constant $[\mu - F_X^{-1}(1-\alpha)]/\sigma$. Therefore, all quantiles $g^{-1}(q)$ converge to this same constant, yielding Result 3.

This limiting value is a function of only α and the distribution shape. As always, it is not a function of μ or σ. In addition, the limiting value is not a function of the confidence γ, since the limiting joint distribution of $(S, \bar{X})$ is degenerate at (σ, μ).

3.2 Sensitivity Analysis

We show here that the lower one-sided factor k^* is an increasing function of α and of γ, but that k^* is not necessarily a monotonic function of n. The distribution shape can affect the values of k^* substantially.

To measure distribution shape, we use the skewness α_3 and kurtosis α_4, the third and fourth standardized moments. For any specified point (α_3, α_4), we choose the unique corresponding Johnson distribution. The Johnson family, proposed by Johnson (1949), includes three transformations of the standard normal distribution. Let X and Z denote the Johnson and standard normal random variables, respectively. The three transformations are:

$$S_L: \quad Z = \eta + \delta \ln\left(\frac{X-\xi}{\lambda}\right), \qquad \lambda(X-\xi) \geq 0,$$

$$S_B: \quad Z = \eta + \delta \ln\left(\frac{X-\xi}{\xi+\lambda-X}\right), \quad 0 \leq X-\xi \leq \lambda,$$

$$S_U: \quad Z = \eta + \delta \sinh^{-1}\left(\frac{X-\xi}{\lambda}\right), \quad -\infty < X < \infty.$$

The constants ξ and λ, respectively, are location and scale parameters; η and δ are the shape parameters. The second transformation, S_B, provides a bounded random variable X; the third transformation, S_U, results a unbounded X. For lognormal distributions, S_L, the range is bounded below if $\lambda > 0$ and bounded above if $\lambda < 0$. DeBrota et al. (1989) have developed two public-domain software packages; VISIFIT and FITTR1. VISIFIT allows visual fitting to a desired density shape. FITTR1 fits Johnson distributions to data using any of several criteria. We use the numerical routines of Hill, Hill, and Holder (1976) to find the Johnson distribution having desired moments μ, σ, α_3, and α_4.

Tables 1 and 2 show values of k^* for thirty-six design points: $n \in \{2, 10, 30, \infty\}$, $\alpha \in \{.001, .5, .99\}$, and $\gamma \in \{.001, .5, .99\}$. The normal-distribution results in Table 1 are computed numerically. The $(\alpha_3, \alpha_4) = (4, 30)$ Johnson-distribution results in Table 2 are estimates using the QE algorithm based on 500,000 independent Monte Carlo samples of size n; only significant digits are shown, based on standard errors estimated using Schmeiser et al. (1990), and Hashem and Schmeiser (1993).

Table 1: Tolerance Factors for the Normal Distribution

		n			
α	γ	2	10	30	∞
.001	.001	-2365	-8.93	-5.15	-3.09
.001	.5	-4.53	-3.21	-3.13	-3.09
.001	.99	-0.97	-1.85	-2.28	-3.09
.5	.001	-225	-1.36	-0.62	0.00
.5	.5	0.00	0.00	0.00	0.00
.5	.99	22.49	0.89	0.45	0.00
.99	.001	0.15	1.08	1.49	2.33
.99	.5	3.38	2.41	2.35	2.33
.99	.99	186	5.07	3.45	2.33

These two tables illustrate three points that are true in general: (1) The tolerance factor k^* increases as the coverage α increases. (2) The tolerance factor k^* increases as the confidence γ increases. (3) The sensitivity to n is least when $\alpha \approx .5$ and $\gamma \approx .5$, with $k^* = 0$ in symmetric cases such as the normal.

Table 2: Tolerance Factors for the Johnson S_B Distribution with Skewness 4 and Kurtosis 30

		n			
α	γ	2	10	30	∞
.001	.001	-1.9E+4	-74.9	-32	-8.61
.001	.5	-27.4	-12.6	-10.3	-8.61
.001	.99	-1.5	-2.7	-3.7	-8.61
.5	.001	-400	-1.7	-0.24	0.33
.5	.5	0.30	0.36	0.35	0.33
.5	.99	15	0.89	0.62	0.33
.99	.001	0.43	0.41	0.38	0.74
.99	.5	1.35	0.98	0.86	0.74
.99	.99	67	1.94	1.34	0.74

The behavior of k^* is not always monotonically decreasing as n increases. For the normal distribution in Table 1, it is true that k^* moves monotonically to the limiting value; for the usual case of large confidence values γ, k^* decreases. For the Johnson distribution in Table 2, k^* does not move monotonically to its limiting value in the fifth and seventh rows. If we expanded Table 2, the seventh-row values of k^* would be 0.40, 0.44 and 0.49 when n is 60, 100 and 200, respectively. Hence, the value of k^* decreases with n until some value of n between 30 and 40, where it starts increasing toward the limiting value 0.74. The reason for this non-monotonic behavior is that the shape of the $(S, \bar{X})$ joint distribution changes as n increases. We study this effect geometrically further later in this section. (See Figure 4.)

To interpret the behavior of the tolerance factor k^*, we now view the problem geometrically. Consider the straight line $L : \bar{x} = k^* s + F_X^{-1}(1-\alpha)$ in the sample plane of $(S, \bar{X})$ with given values of n, α, γ and (α_3, α_4), where $\bar{x}$ and s are the realizations of $\bar{X}$ and S, respectively. The $\bar{x}$-axis intercept $F_X^{-1}(1-\alpha)$ is determined only by distribution shape and α. The slope is k^*, determined so that γ is the probability of the random point $(S, \bar{X})$ lying on or below L. The value of k^* is not necessarily positive; negative values of k^* occur when α or γ, or both, are small.

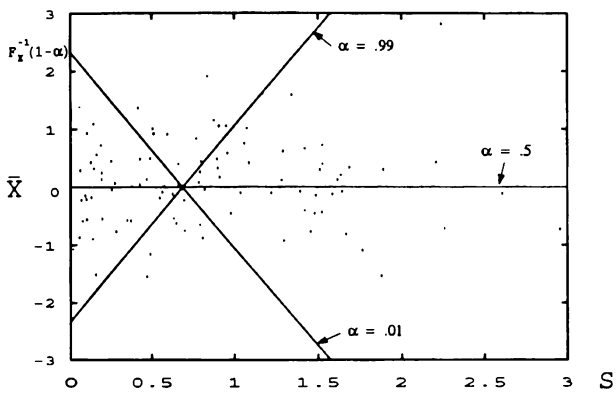

Figure 1: Plot of Lines L in the Sample Plane of $(S, \bar{X})$ for $\alpha = .01, .5$ and $.99$ with Standard Normal Data, $n = 2$, and $\gamma = .5$

Figure 1 is a scatter plot of one hundred independent observations $(s, \bar{x})$ from standard normal samples of size $n = 2$. For $\gamma = 0.5$, the three lines corresponding to $\alpha = 0.01, 0.5, 0.99$ illustrate the change of k^* with change of α. (The slopes are computed numerically.) As α increases, the intercept on the $\bar{x}$-axis moves down so the slope of line L goes up in order to keep half of the observations $(s, \bar{x})$ below line L.

Figure 1 also illustrates that k^* increases with γ, although the change of γ with α fixed is not plotted. As γ increases, the line L pivots counterclockwise at $(0, F_X^{-1}(1-\alpha))$ to increase the proportion of the observations below L; hence, the slope of L increases.

Figures 2 and 3 illustrate k^* increasing and decreasing, respectively, with n. One hundred points $(S, \bar{X})$ are plotted for both $n = 2$ and $n = 30$ for the Johnson S_B population with skewness 4 and kurtosis 30. Lines are shown for $n = 2$, $n = 30$, and $n = \infty$ and $\alpha = 0.5$. (The slopes are estimated using QE with $m = 500{,}000$ samples of size n.) The only difference between Figure 2 and Figure 3 is that the value of γ changes from 0.001 to 0.99. As n increases, the slope of line L passing through point $(0, F_X^{-1}(1-\alpha))$ goes closer to the limiting value of k^* as the joint distribution shrinks toward the point (σ, μ). Since a proportion γ of the points lies below the line, the larger value of γ has the larger slope k^*.

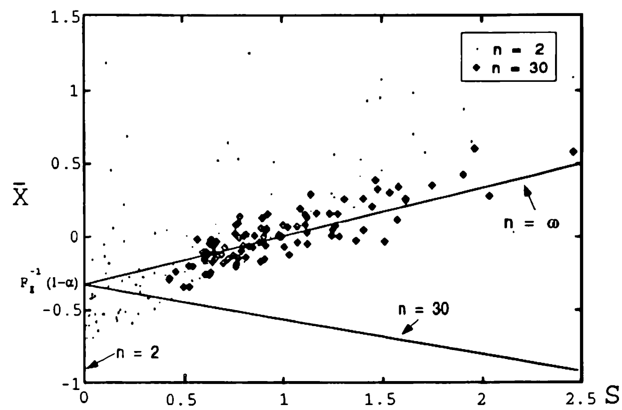

Figure 2: Plot of Line L in Sample Plane of $(S, \bar{X})$ from Johnson S_B Distribution for $n = 2$ and 30, where $\alpha = .5$, and $\gamma = .001$ (When $n = 2$ the Slope Is -400, and the Line L is Hidden in the $\bar{x}$ Axis.)

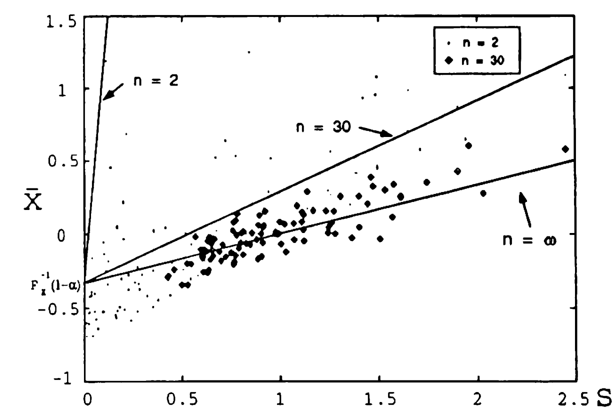

Figure 3: Plot of Line L in Sample Plane of $(S, \bar{X})$ from Johnson S_B Distribution for $n = 2$ and 30, where $\alpha = .5$, and $\gamma = .99$

Figure 4 shows that k^* does not necessarily change monotonically with n. Three sets of thirty points $(S, \bar{X})$ from the same Johnson S_B distribution are shown for each of $n = 2$, $n = 30$ and $n = 200$. The four lines (for $n = 2, 30, 200, \infty$) correspond to $\alpha = 0.99$ and $\gamma = 0.1$. As n increases, the slope first decreases and then increases to the limiting k^*. The graph shows that nonmonotonic behavior occurs because the joint distribution of $(S, \bar{X})$ changes shape as it shrinks to (σ, μ). In this case, the changing shape dominates for small sample sizes and the shrinking dominates for large sample sizes.

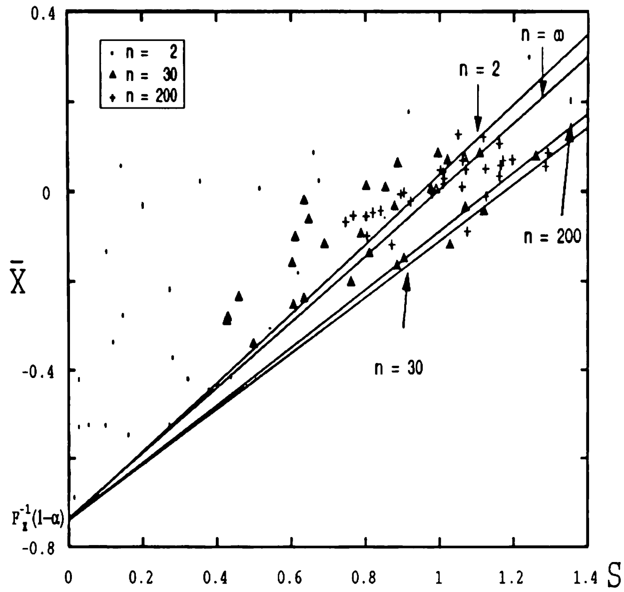

Figure 4: Plot of Line L in Sample Plane of $(S, \bar{X})$ from Johnson S_B Distribution for $n = 2$, 30 and 200, where $\alpha = .99$, and $\gamma = .1$

4 TWO-SIDED TOLERANCE FACTOR

The factor k^* in the two-sided guaranteed-coverage tolerance interval $[\bar{X}-k^* S, \bar{X}+k^* S]$ is also a quantile of an observable random variable. Hence QE can be modified to solve for k^*.

The two-sided factor k^* satisfies Equation 1, i.e.,

$$\Pr_{\bar{X},S}\{\Pr_X\{\bar{X} - k^* S \le X \le \bar{X} + k^* S\} \ge \alpha\} = \gamma. \quad (9)$$

Let $v_\alpha(\bar{X}) = v$ be the random variable satisfying $F_X(\bar{X} + v) - F_X(\bar{X} - v) = \alpha$. Then the event "$\Pr_X\{\bar{X}-k^* S \le X \le \bar{X}+k^* S\} \ge \alpha$" in Equation 9 is equivalent to the event "$k^* S \ge v_\alpha(\bar{X})$". Hence, Equation 9 can be rewritten as

$$\Pr_{\bar{X},S}\{\, v_\alpha(\bar{X})/S \le k^* \,\} = \gamma.$$

Let random variable $K = v_\alpha(\bar{X})/S$, which again does not depend on μ or σ. Then k^* is the γ^{th} quantile of the distribution of K, which can be observed via realizations of $\bar{X}$ and S.

The modification of QE algorithm for two-sided tolerance intervals estimates k^* from order statistics in Equation 6 based on m independent realizations of the observable K. Analogous to the QE Algorithm for lower one-sided stated in Subsection 2.1, Step 3 is changed to

Step 3: Compute $k_i = v_\alpha(\bar{x})/s$, where $v_\alpha(\bar{x}) = v$ satisfies $F_X(\bar{x} + v) - F_X(\bar{x} - v) = \alpha$.

The new Step 3 requires numerical root finding for v.

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APPENDIX A: UPPER ONE-SIDED TOLERANCE FACTORS

Upper one-sided tolerance intervals are closely related to lower one-sided intervals. Let $k^U_{n,\alpha,\gamma}$ and $k^L_{n,\alpha,\gamma}$ denote the factors for upper and lower one-sided intervals, respectively, such that the coverage is α, confidence is γ and the sample size is n. Then

$$\Pr_{\bar{X},S}\{\Pr_X\{X \le \bar{X} + k^U_{n,\alpha,\gamma}\, S\} \ge \alpha\} = \gamma$$

implies that

$$\Pr_{\bar{X},S}\{\Pr_X\{X > \bar{X} + k^U_{n,\alpha,\gamma}\, S\} \le 1 - \alpha\} = \gamma,$$

and therefore

$$\Pr_{\bar{X},S}\{\Pr_X\{X > \bar{X} + k^U_{n,\alpha,\gamma}\, S\} \ge 1 - \alpha\} = 1 - \gamma.$$

Hence,

$$k^U_{n,\alpha,\gamma} = -k^L_{n,1-\alpha,1-\gamma}\,.$$

To estimate the upper one-sided tolerance factor k^* with coverage α and confidence γ, we can estimate the lower one-sided tolerance factor with coverage $1-\alpha$ and confidence $1-\gamma$ and then change the sign. The limiting value of the upper one-sided guaranteed-coverage tolerance factor is then, from Result 3,

$$\lim_{n\to\infty} k^* = [\, F_X^{-1}(\alpha) - \mu \,] \,/\, \sigma, \quad (10)$$

for all positive values of γ.

APPENDIX B: PROOF FOR RESULT 2

Here we prove that if X is symmetrically distributed, then $k^*_{(n,\alpha,\gamma)} = -k^*_{(n,1-\alpha,1-\gamma)}$.

Since changing μ does not affect the tolerance factor, without loss the generality set $\mu = 0$, so that the distribution of X is symmetric at zero. Then, $\bar{X}$ and $-\bar{X}$ have the same distributions. Given a positive integer n and $0 \le \alpha, \gamma \le 1$, then $k^*_{(n,\alpha,\gamma)}$ satisfies

$$\Pr_{\bar{X},S}\{\Pr_X\{X \ge \bar{X} - k^*_{(n,\alpha,\gamma)}\, S\} \ge \alpha\} = \gamma.$$

Also,

$$\begin{aligned}
&\Pr_{\bar{X},S}\{\ \Pr_X\{X \geq \bar{X} - k^*_{(n,\alpha,\gamma)}\ S\} \geq \alpha\ \} \\
=\ &\Pr_{\bar{X},S}\{\ \Pr_X\{X \leq \bar{X} - k^*_{(n,\alpha,\gamma)}\ S\} \leq 1-\alpha\ \} \\
=\ &\Pr_{\bar{X},S}\{\ \Pr_X\{X \geq -\bar{X} + k^*_{(n,\alpha,\gamma)}\ S\} \leq 1-\alpha\ \} \\
&\qquad \text{since } X \text{ is symmetric at } 0 \\
=\ &\Pr_{\bar{X},S}\{\ \Pr_X\{X \geq \bar{X} + k^*_{(n,\alpha,\gamma)}\ S\} \leq 1-\alpha\ \} \\
=\ &1 - \Pr_{\bar{X},S}\{\Pr_X\{X \geq \bar{X} + k^*_{(n,\alpha,\gamma)}\ S\} \geq 1-\alpha\}.
\end{aligned}$$

Hence,

$$\Pr_{\bar{X},S}\{\ \Pr_X\{X \geq \bar{X} + k^*_{(n,\alpha,\gamma)}\ S\} \geq 1-\alpha\ \} = 1-\gamma.$$

Then,

$$k^*_{(n,\alpha,\gamma)} = -k^*_{(n,1-\alpha,1-\gamma)}.$$

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AUTHOR BIOGRAPHIES

HUIFEN CHEN is a Ph.D. student in the School of Industrial Engineering at Purdue University. She received a B.S. degree in accounting from National Cheng-Kung University in Taiwan in 1986 and an M.S. degree in statistics from Purdue University in 1990. Her research interests include simulation and numerical analysis applied to quality control and reliability.

BRUCE SCHMEISER is a Professor in the School of Industrial Engineering at Purdue University. He is the Simulation Area Editor of *Operations Research* and an active participant in the Winter Simulation Conference, including being Program Chairman in 1983 and Chairman of the Board of Directors during 1988-1990.