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ESTIMATION OF THE SAMPLE SIZE AND COVERAGE FOR GUARANTEED-COVERAGE NONNORMAL TOLERANCE INTERVALS

Huifen Chen
Tsu-Kuang Yang

Department of Industrial Engineering, Da-Yeh University
112 Shan-Jeau Rd., Dah-Tsuen,, Chang-Hwa, 51505, TAIWAN

ABSTRACT

We propose Monte Carlo algorithms to estimate the sample size and coverage of guaranteed-coverage tolerance intervals for nonnormal distributions. The current literature focuses on computation of the tolerance factor, but addresses less on the sample size, coverage, and confidence, which need to be set prior to the tolerance factor. The coverage estimation algorithm, which always converges, is based on our proof that the coverage is a quantile of an observable random variable. The sample-size estimation algorithm, which seems to converge in empirical results, is based on the general stochastic root-finding algorithm, retrospective approximation. Following previous sensitivity analysis for the tolerance factor, we analyze relationships among the sample size, coverage, and confidence.

1 INTRODUCTION

We consider guaranteed-coverage tolerance intervals (GCTIs) for random product characteristics X whose distribution F_X is continuous but has unknown mean μ and unknown variance σ^2. Based on a random sample $\{X_1,\ldots, X_n\}$ from the distribution F_X, a GCTI for X is defined as $I(\overline{X}, S, k)$, where $I(\overline{X}, S, k)$ equals $(\overline{X} - kS,\infty)$ for lower one-sided, $(\infty, \overline{X} + kS)$ for upper one-sided, and $(\overline{X} - kS, \overline{X} + kS)$ for two-sided intervals (Wald and Wolfowitz, 1946), where $\overline{X} = \sum_{i=1}^{n} X_i / n$ and $S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 / (n-1)$. For such intervals, a practitioner can state with confidence γ that the proportion of the population in the random tolerance interval, based on a sample of size n and a tolerance factor k, is at least α. The four tolerance parameters—sample size $n\in \{2,3,\ldots\}$, tolerance factor $k\in R$, coverage $\alpha \in (0,1)$, and confidence $\gamma\in (0,1)$ —are determined so that

$$\mathrm{P}_{\overline{X},S}\ \{ \mathrm{P}_X\ \{X\in I(\overline{X},S,k)\} \ge \alpha\}=\gamma. \qquad (1)$$

GCTIs have wide uses in quality control and system reliability. For specific examples, see Chen and Schmeiser (1995), Patel (1986), and Odeh and Owen (1980).

The existing literature focuses on computation of the tolerance factor k. Most of it assumes normal distributions, e.g., Wald and Wolfowitz (1946), Guttman (1970), Aitchison and Dunsmore (1975), Odeh and Owen (1980), and Eberhardt et al. (1989). The one-sided tolerance factor for normal distributions is a multiple of a noncentral t quantile (see Section 2). The two-sided factor can be computed by solving a nonlinear equation. Nonnormal literature also exists. Chen and Schmeiser (1995) propose quantile estimation methods to compute the tolerance factor for nonnormal distributions. Aitchison and Dunsmore (1975) and Patel (1986) also discuss different forms of tolerance intervals for binomial, Poisson, exponential, and other standard populations. Guenther (1985) provides an extensive discussion of distribution-free tolerance intervals.

Before computation of the tolerance factor, values of n, α, and γ need to be set. The following sample-size determination procedure, used by a rocket manufacturer, motivates our research.

Sample-size determination procedure: Given coverage α and confidence γ:

1. Compute the sample size n satisfying Equation (1) for a given nominal value of k.
2. Collect a real sample $\{x_1,\ldots, x_n\}$ and compute the tolerance factor
$$k = (\overline{X} - L) / s,$$
where L is the constant lower specification of product characteristics.
3. Compute the coverage $\tilde{\alpha}$ so that with confidence γ the interval $I(\overline{X}, S, k)$ contains at least proportion $\tilde{\alpha}$ of the population, i.e., solve the following equation for $\tilde{\alpha}$
$$\mathrm{P}_{\overline{X},S}\ \{ \mathrm{P}_X\ \{X\in I(\overline{X},S,k)\} \ge \tilde{\alpha}\} = \gamma. \qquad (2)$$

4. If $\tilde{\alpha} > \alpha$, stop and return *n*; otherwise, update *n* and go to Step 2.

Our motivation of computing the sample size and coverage for nonnormal distributions arises from the one-dimensional root-finding problems in Steps 1 and 3. Step 1 computes the sample size *n* satisfying Equation 1, given *k* ,α, γ, and the distribution shape; Step 3 computes the coverage $\tilde{\alpha}$ satisfying Equation (2) (or α satisfying Equation 1), given γ, *n*, *k*, and the distribution shape. A traditional approach to these two problems is to build an extensive table of values for the four tolerance parameters in Equation (1) for each distribution of interest. With such a table, practitioners can choose values of the sample size and coverage from the table whenever they are needed. The drawbacks of this approach are that any table with a wide range of tolerance parameter values and distribution types would be huge, and that interpolation—or worse, extrapolation—may be used to approximate values that are not listed in the table.

In this research, we are interested in black-box Monte Carlo algorithms that compute any tolerance parameter of interest. That is, we consider the problem of finding any lower one-sided tolerance parameter (e.g., *n*) that satisfies the tolerance logic (Equation 1) when the other three parameters (e.g., *k*, α, and γ) and distribution shape are known. Specifically, the problem is defined as follows.

Research Problem: Given

(a) the shape of continuous distribution function $F_X(\cdot)$ with unknown mean μ and unknown variance σ^2,

(b) three of the following four tolerance parameters:
- sample size *n*,
- tolerance factor *k*,
- coverage α,
- confidence γ.

Find: the other unknown tolerance parameter satisfying the tolerance logic, i.e.,

$$\mathrm{P}_{\overline{X},S}\{\,\mathrm{P}_X\{X \geq \overline{X} - kS\} \geq \alpha\} = \gamma. \qquad (3)$$

The assumption of known distribution shape means that all standardized moments are known but the mean or variance is unknown. When the real data do not adequately fit any standard distributions (e.g. exponential), practitioners can define the distribution shape through a simulation routine that can generate observations whenever needed. In this paper, we focus on estimation algorithms for lower one-sided GCTIs. The proposed methods in Section 3 can be modified easily for upper one-sided GCTIs and extended for two-sided GCTIs. Furthermore, we propose algorithms only for the sample size and the coverage. The tolerance factor *k* can be estimated by the quantile estimation method proposed in Chen and Schmeiser (1995). Furthermore, estimating the confidence γ is straightforward (Schmeiser, 1990). We review both in Section 2.

The rest of this paper is organized as follows. In Section 2, we review the related literature. In Section 3, we propose estimation algorithms for the coverage and sample size. The coverage estimator always converges; the sample-size estimator seems to converge in our simulation results. In Section 4, we continue Chen and Schmeiser's (1995) sensitivity analysis for the sample size, coverage, and confidence.

2 LITERATURE REVIEW

This section reviews the literature on computation of the four tolerance parameters. The confidence estimator and tolerance-factor estimator discussed here are designed for nonnormal parametric distributions and the coverage estimator is for normal distributions. The literature on sample size is different from, but related to, our problem. We discuss the computation of these four tolerance parameters in turn.

2.1 Confidence

We consider the problem of computing the confidence that the tolerance interval $(\overline{X} - k\,S, \infty)$ contains at least proportion α of the measurements for nonnormal distributions. That is, computing the probability $\gamma = \mathrm{P}_{\overline{X},S}\{\,\mathrm{P}_X\{X \geq \overline{X} - kS\} \geq \alpha\}$, given *n*, *k*, α, and the distribution shape $F_X(\cdot)$. Numerical computation of γ may not be efficient because γ is a (*n*+1)-dimensional integral, except in the normal case where γ is a noncentral *t* percentage point. However, γ can be estimated easily by Monte Carlo simulation. We can generate *m* samples $\{x_{11},\ldots,x_{1n}\},\ldots,\{x_{m1},\ldots,x_{mn}\}$ from the distribution F_X, using any arbitrary values of μ and σ. (Notice that γ does not depend on the unknown mean μ or standard deviation σ.) For each sample, compute the sample mean $\bar{x}_i$ and sample standard deviation s_i for *i* = 1,...,*m*. Then γ can be estimated by

$$\hat{\gamma} = \sum_{i=1}^{m} y_i / m, \qquad (4)$$

where

$$y_i = \begin{cases} 1 & \text{if } \mathrm{P_X}\{X \geq \bar{x}_i - k\,s_i\} \geq \alpha \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to show that the estimate $\hat{\gamma}$ is unbiased with variance $\gamma(1-\gamma)/m$.

2.2 Tolerance Factor

Chen and Schmeiser (1995) propose quantile estimation methods for computing the tolerance factor for nonnormal distributions, as defined earlier in the research problem. They show that the tolerance factor k satisfying Equation (3) is the γth quantile of the random variable $K = [\overline{X} - F_X^{-1}(1-\alpha)]/S$. This result follows from the equivalence of Equation (3) and $P_K\{K \le k\} = \gamma$. Notice that K is observable because K does not depend on the population mean μ or standard deviation σ. Therefore we can generate m observations $k_1,\ldots,k_m$ of the random variable $k_i = [\overline{x}_i - F_X^{-1}(1-\alpha)]/s_i$ for $i = 1,\ldots,m$, where $\overline{x}_i$ and s_i are the sample mean and sample standard deviation of distribution $F_X(\cdot)$, using any arbitrary values of μ and σ. Then the tolerance factor estimate is $\hat{k} = \omega\, k_{(\lfloor (m+1)\gamma \rfloor)} + (1-\omega)\, k_{(\lceil (m+1)\gamma \rceil)}$, the convex combination of the $\lfloor (m+1)\gamma \rfloor$th and $\lceil (m+1)\gamma \rceil$th order statistics, where the weight is $\omega = \lceil (m+)\gamma \rceil - (m+1)\gamma$. (Here, $\lfloor a \rfloor$ is the biggest integer less than or equal to a and $\lceil a \rceil$ is the smallest integer greater than or equal to a.) The asymptotic distribution $\sqrt{m}\,(\hat{k} - k)$ is normal with mean 0 and variance $\gamma(1-\gamma)\,/\,[\,f_K^2(k)]$, where $f_K(\cdot)$ is the density function of the random variable K. (See Lehmann, 1983, page 394.)

If the distribution $F_X(\cdot)$ is normal, then random variable $\sqrt{n}\;K$ is a noncentral t with (n-1) degrees of freedom and noncentrality parameter $\sqrt{n}\,z_\alpha$, where z_α is the αth quantile of the standard normal. Therefore, the tolerance factor is $k = t_{n-1,\gamma}(\sqrt{n}\,z_\alpha)\,/\,\sqrt{n}$, where $t_{\upsilon,\gamma}(\delta)$ is the γth quantile of the noncentral t with υ degrees of freedom and noncentrality δ. For this special case, numerical computation of k would be more efficient.

2.3 Coverage

Owen and Hua (1977) derive methods for computing the coverage for normal distributions, i.e., solving Equation (3) for α, given n, k, γ, and the normal distribution shape. The purpose is to obtain the γ-confidence limit (i.e., α) for the random coverage $P_X\{X \ge \overline{X} - kS\}$. They use the result of $k = t_{n-1,\gamma}(\sqrt{n}\,z_\alpha)/\sqrt{n}$ and suggest searching for the noncentral-t noncentrality $\sqrt{n}z_\alpha$ so that the probability of the noncentral t (i.e., $\sqrt{n}\,K$) being less than $\sqrt{n}\,k$ is γ. However, they did not suggest any search methods. Odeh and Owen (1980) provide tables of α values.

In the case of nonnormal distributions, the random variable $\sqrt{n}\;K$ may not have noncentral t or other standard distributions. Numerical evaluation of α is difficult. In Section 3.1, we propose a quantile estimation approach that requires no root search.

2.4 Sample Size

Most literature on the sample size assumes normal distributions and addresses a problem slightly different from ours. In additional to the criterion of Equation (3), another criterion is added so that the tolerance interval does not cover too large a proportion of the population, relative to the lower limit α. When the sample size is very small, the interval becomes very wide and is of little use. Faulkenberry and Weeks (1968), Faulkenberry and Daly (1970), and Kirkpatrick (1977) suggest a second criterion of $P_{\overline{X},S}\{P_X\,\{X \ge \overline{X} - kS\} \ge \alpha'\,\} = \delta$, where $\alpha' > \alpha$ and δ is a small value. Then, the confidence that the random coverage $P_X\,\{X \ge \overline{X} - k\,S\}$ lies between α and α' is (γ-δ). Other variations of the second criterion include controlling both limits of the random coverage around α (Wilks, 1941, and Odeh et al., 1989) and controlling coverage on both tails of two-sided intervals (Chou and Mee, 1984). Since there are two criteria, this procedure determines not only the sample size but also the tolerance factor, while α and γ are pre-chosen values.

3 METHODS

In Sections 3.1 and 3.2, we propose algorithms to solve Equation (3) for the nonnormal coverage α and for the nonnormal sample size n, respectively. To estimate α, we invert the root-finding equation in the form that α is a quantile of an observable random variable and then estimate α by order statistics. To estimate n, we apply retrospective approximation algorithms developed by Chen and Schmeiser (1995), with small changes for the discrete root-finding function of the sample size n. We show that the estimate of α always converges and that the estimate of n seems to converge in our simulation results.

3.1 Computing the coverage

Here we consider finding the coverage α for nonnormal distributions, such that the tolerance interval contains at least proportion α of the population, with confidence γ. That is, we solve Equation (3) for α, given values of n, k, γ, and the distribution shape. We propose an estimation method similar to the quantile estimation method for the tolerance factor k (Section 2). Analogous to the random variable k for the tolerance factor, we define the random variable $C = \mathrm{P}_X\{X \geq \overline{X} - k\,S\}$. Then Equation (3) is equivalent to

$$\mathrm{P}\{C \geq \alpha\} = \gamma.$$

Hence α is the (1-γ)th quantile of the random variable C. Again, the random variable C is observable because it does not depend on the mean μ or standard deviation σ of distribution $F_X(\cdot)$. Therefore, we can generate m independent Monte Carlo observations $c_1,\ldots,c_m$ of C, using arbitrary values of μ and σ. Then estimate α by

$$\hat{\alpha} = \omega\, c_{(\lfloor (m+1)(1-\gamma)\rfloor)} + (1-\omega)\, c_{(\lceil (m+1)(1-\gamma)\rceil)}, \qquad (5)$$

the convex combination of the $\lfloor (m+1)(1-\gamma)\rfloor$ th and $\lceil (m+1)(1-\gamma)\rceil$th order statistics, where the weight is $\omega = \lceil (m+1)(1-\gamma)\rceil - (m+1)(1-\gamma)$. Specifically, the algorithm performs as follows.

Estimation of coverage α:

Given: sample size n, tolerance factor k, confidence γ, and the distribution shape.

Procedure:

1. Independently generate m random samples $\{x_{11},\ldots,x_{1n}\},\ldots,\{x_{m1},\ldots,x_{mn}\}$ from distribution $F_X(\cdot)$ using any arbitrary values of μ and σ.
2. Compute the sample mean $\bar{x}_i$ and standard deviation s_i for $i = 1,\ldots,m$.
3. Compute $c_i = \mathrm{P}\{X \geq \bar{x}_i - k\,s_i\}$ for $i = 1,\ldots,m$.
4. Compute $\hat{\alpha}$ from $c_1, c_2,\ldots,c_m$ using Equation (5).

By Lehmann (1983, page 394), the asymptotic distribution of $\hat{\alpha}$ is

$$\sqrt{m}\,(\hat{\alpha} - \alpha) \xrightarrow{D} \mathrm{N}\,(0,\ \gamma(1-\gamma)/[\,f_C^2(\alpha)\,]), \qquad (6)$$

where $f_C(\cdot)$ is the density function of C. Hence, the estimator $\hat{\alpha}$ always converges at rate $O(1/\sqrt{m})$.

3.2 Computing the Sample Size

Here we consider finding the sample size n for a nonnormal distribution such that a practitioner can state with confidence γ that the tolerance interval $[\,\overline{X} - k\,S, \infty)$ contains at least the proportion α of the population. That is, we solve Equation (3) for the root n, given values of k, α, γ, and the distribution shape. Unlike in Section 3.1, it is difficult here to invert the root-finding function to express n as a statistical constant, e.g., quantile. Therefore we implement the general stochastic root-finding algorithm, retrospective approximation (RA), with small modifications. Simulation results show that the modified RA seems to converge to the true value despite lack of convergence proof. To emphasize their dependence on the sample size n, we denote the sample mean $\overline{X}$ and sample standard deviation S by $\overline{X}_n$ and S_n. Furthermore, for convenience, we denote the root-finding function of any sample size $\tilde{n}$ by $g(\tilde{n};\,k,\,\alpha) = \mathrm{P}_{\overline{X}_{\tilde{n}},S_{\tilde{n}}}\{\mathrm{P}_X\{X \geq \overline{X}_{\tilde{n}} - k\,S_{\tilde{n}}\} \geq \alpha\}$ Then given $\tilde{n}$, k, α, and the distribution shape, function value $g(\tilde{n};\,k,\,\alpha)$ is the confidence that the random coverage is at least α. Therefore Equation (3) is equivalent to $g(n;\,k,\,\alpha) = \gamma$. We want to solve this equation for n.

Knowing properties of the root-finding function is useful for solving the equation. The function g has four properties: (1) The domain of $g(\tilde{n};\cdot,\cdot)$ is discrete because the sample size $\tilde{n} \in \{2, 3, 4,\ldots\}$; (2) Function g is nonmonotonic with respect to $\tilde{n}$; (3) In the special case of symmetric distributions, then $g(\tilde{n};\,0,\,0.5)$ equals 0.5 for any finite value of $\tilde{n}$; in general,

$$\lim_{\tilde{n}\to\infty} g(\tilde{n};k,\alpha) = \begin{cases} 1 & \text{if } k \geq k^{\infty} \\ 0 & \text{otherwise} \end{cases},$$

where $k^{\infty} = [\mu - F_X^{-1}(1-\alpha)]/\sigma$ (see Section 4); (4) The root n may not be unique or may not exist; for example, in the special case that $F_X(\cdot)$ is symmetric at mean, then $g(\tilde{n};\,0,\,0.5)$ equals 0.5 for any sample size. Therefore, the equation $g(\tilde{n};\,0, 0.5) = \gamma$ has infinite number of roots if γ= 0.5, and has no root, otherwise.

Solving equation $g(\tilde{n};\,k,\alpha) = \gamma$ for the root $\tilde{n} = n$ is a stochastic root-finding problem (SRFP), solving a deterministic equation using only estimates of function values (Chen and Schmeiser, 1994a). Since the function value $g(\tilde{n};\,k,\,\alpha)$ is an $(\tilde{n}+1)$-dimensional integral, numerical computation may not be efficient. However, we can estimate the function value easily via simulation experiment. As shown in Equation (4), an unbiased estimator of $g(\tilde{n};\,k,\alpha)$ is $\hat{g}(\tilde{n};\,k,\,\alpha) = \sum_{i=1}^{m} Y_i/m$, where Y_i equals 1 if $\mathrm{P}\{X \geq \overline{X}_i - k\,S_i\} \geq \alpha$ and equals 0, otherwise, for i=1,...,m.

Chen and Schmeiser (1994b) propose RA algorithms for one-dimensional SRFPs whose root-finding functions are continuous over the real line. Let $\hat{g}(\tilde{n};\,k,\alpha,\underline{\omega})$ denote the estimate $\hat{g}(\tilde{n};\,k,\alpha)$ generated from the simulation experiment using a vector of m pseudo-random number strings $\underline{\omega} = (\omega_1,\ldots,\omega_m)$. Each string ω_i is used to generate the ith sample $\{x_{i1},\ldots,x_{in}\}$ from distribution $F_X(\cdot)$, where i=1,...,m. RA iteratively solves a sequence of sample-path equations $\{\hat{g}(N_i^*;\, k,\alpha,\underline{\omega}_i) = \gamma\colon\ i = 1, 2,...\}$, where $\underline{\omega}_i = (\omega_{i1},...,\omega_{im_i})$ and the sequence $\{m_1, m_2, \ldots\}$ is increasing. In each iteration, the sample-path equation is solved until a bounding interval of the retrospective root N_i^* is found, starting at an initial point and moving by a step-size δ_i, which is doubled each time. The linear interpolate of the bounds, called N_i, is returned. After i iterations, the root estimator $\overline{N}_i$ is then the weighted average of those solutions $N_1, N_2, ..., N_i$; the jth weight is proportional to the number of samples m_j, j = 1,…, i. A specific RA version, called independent bounding RA (IBRA), performs as follows.

IBRA Algorithm:

Given algorithm parameters: the standard error tolerance σ_0, an initial solution N_0, initial step size δ_1, initial number of samples m_1, the number-of-samples multiplier c_1, and the step-size multiplier c_2.

Find: the root n satisfying $g(n;\, k,\alpha) = \gamma$.

0. Initialize the retrospective iteration number $i = 1$.
1. Independently generate $\underline{\omega}_i$.
2. Solve Equation $\hat{g}(N_i^*; k,\alpha,\underline{\omega}_i) = \gamma$ until a bounding interval of the root N_i^* is found, starting at the point $\overline{N}_{i-1}$ (note: $\overline{N}_0 = N_0$) and moving by step size δ_i, which is doubled each time. Return the linear interpolate N_i of the bounds.
3. Compute $\overline{N}_i = \sum_{j=1}^{i} m_j N_j / \sum_{j=1}^{i} m_j$ and its standard error estimate $\hat{se}(\overline{N}_i) = \sigma_N / \sqrt{\sum_{j=1}^{i} m_j}$, where $\sigma_N = \sqrt{(i-1)^{-1}[\sum_{j=1}^{i} m_j N_j^2 - (\sum_{j=1}^{i} m_j)\overline{N}_j^2]}$ (Note that $\sigma_N = 0$ for i=1).
4. If $\hat{se}(\overline{N}_i) < \sigma_0$, stop. Otherwise, compute $\delta_{i+1} = c_2\sqrt{(\sum_{j=1}^{i} m_j)^{-1} + (m_{j+1})^{-1}}\,\sigma_N$ (but $\delta_2 = \delta_1$) and $m_{i+1} = c_1 m_i$, let $i \leftarrow i+1$, and go to Step 1.

RA assumes that the root-finding function is continuous over the whole real line, lies below γ for $\tilde{n}$ below the true root, and lies above γ for $\tilde{n}$ above the root. Additional conditions on g and $\hat{g}$ guarantee that the RA root estimator converges to the true root with probability one (Chen, 1994).

Since the sample size and hence the function g are discrete, the IBRA must be modified. Three rounding steps are added to each IBRA iteration i: (1) Round down the step-size δ_i, since small step size seems to be more efficient, (2) round the retrospective solution N_i to the nearest integer, and (3) round up the root estimator $\overline{N}_i$, to ensure that the confidence is at least γ. Furthermore, before implementing the modified IBRA, if $k < k^{\infty}$, replace g by 1-g so that the root-finding function will not go downward to 0. In this modified IBRA, convergence of the sample-size estimator is not guaranteed because g is discrete and also for $\tilde{n}$ the root may not be unique.

Despite the lack of convergence proof, the modified IBRA algorithm seems to converge to the correct sample size in our simulation results as shown in Table 1. There are twelve design points in Table 1, each consisting of the normal distribution shape and different combination of α, γ, and k:

- $\alpha \in \{.1,.5,.9\}$ and $\gamma \in \{.1,.5,.9\}$ but excluding the combination (α,γ) = (.5, .5) because in this case k must be zero and therefore there are infinite number of solutions (see the fourth property of g). We further delete half the combinations because k only changes sign when (α,γ) becomes (1-α, 1-γ), e.g., (.1, .9) and (.9, .1) have same results. Then only four (α,γ) combinations are included.
- $k \in \{k_5, k_{50}, k_{500}\}$, where $k_{n'}$ is the tolerance factor corresponding to a root of n'; note that $k_{n'}$ is computed by the quantile estimation method in Chen and Schmeiser (1995).

For each design point, the sample size estimate $\hat{n}$ (= $\overline{N}_i$) and its standard error estimate $\hat{se}(\hat{n})$ are computed based on twenty simulation runs. Only significant digits are listed. Table 1 shows that the estimates $\hat{n}$ are very close to the true root n, where the standard error increases with the value of n. The simulation time depends on the value of γ; the simulation time for γ = .5 is about three times larger than for other γ values. This is because the function estimate $\hat{g}(n;\cdot,\cdot)$ has larger variance when $g(n;\cdot,\cdot)$ is around 0.5 (see Equation 4).

Table1: Empirical results for Sample-Size Estimators

α	γ	k	n	$\hat{n}$	$\hat{se}(\hat{n})$
.1	.1	-2.7435	5	4.6	.16
.1	.1	-1.5594	50	49.8	.2
.1	.1	-1.3611	500	508	1.8
.1	.5	-1.3818	5	4.9	.2
.1	.5	-1.2891	50	48.6	.26
.1	.5	-1.2823	500	497	1.2
.1	.9	-.67525	5	4.8	.2
.1	.9	-1.0594	50	50.2	.13
.1	.9	-1.2062	500	493	2
.5	.1	-.68567	5	4.9	.1
.5	.1	-.18372	50	49.9	.28
.5	.1	-.05738	500	499	1.8

4 ANALYSIS

This section is an extension of the sensitivity analysis for the tolerance factor k in Chen and Schmeiser (1995), which shows that k is an increasing function of α and of γ, but is not necessarily a monotonic function of n. Here we continue the analysis for (α, γ), (n, γ), and (n, α). We show that α is a decreasing function of γ, but α or γ is not necessarily monotonic with respect to n. Despite nonmonotonicity, when n goes to infinity, α converges to a constant $1 - F_X(\mu - k\sigma)$. Analogously, when n goes to infinity, α converges to 1 if $k \geq k^{\infty}$ (recall $k^{\infty} = [\mu - F_X^{-1}(1-\alpha)]/\sigma$) and converges to 0, otherwise.

As in Chen and Schmeiser (1995), we use geometric graphs to illustrate the analysis. In the sample plane of (S, $\overline{X}$), define a straight line L as the set of sample points (s, $\bar{x}$) that satisfy $\bar{x} = ks + F_X^{-1}(1-\alpha)$. Then the geometric graph relates to the four tolerance parameters n, k, α, and γ, and the distribution shape as follows: (1) the spread of sample points (s, $\bar{x}$) depends on the sample size n and the distribution shape; (2) the slope of line L is k; (3) the $\bar{x}$-axis intercept $F_X^{-1}(1-\alpha)$ and s-axis intercept $-F_X^{-1}(1-\alpha)/k$ depend on α and the distribution shape; (4) the probability that a random point (s, $\bar{x}$) lies on or below line L is γ. We use these dependencies to analyze the (α,γ), (α,n), and (γ, n) interrelations in turn.

Figure 1 shows that the coverage α is a decreasing function of the confidence γ, given values of n, k, and the distribution shape. Fifty observations of (S, $\overline{X}$) from the Johnson S_B population with μ=0, σ=1, skewness = 4, kurtosis = 35, and sample size n = 10 are plotted. Two parallel lines, with the same slope k = 1, correspond to α= 0.7 and 0.85. As α increases, the $\bar{x}$-axis intercept decreases and the s-axis intercept increases, moving the line L parallel to the right. Therefore, γ, the probability of a point (s, $\bar{x}$) lying on or below L, decreases as α increases.

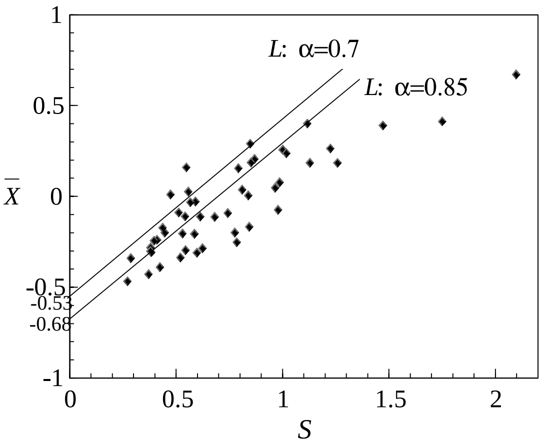


Figure 1: The (α, γ) Relationship: Plot of Line L in the $(S, \overline{X})$ Sample Plane for Johnson S_B Distribution, n = 10, k = 1, and α = 0.7, 0.85

Figure 2 shows that as the sample size n goes to infinity, then the confidence γ nonmonotonically tends to 1 if $k \geq k^{\infty}$, and to 0, otherwise. The constant k^{∞} is the slope of the line joining the $\bar{x}$-axis intersect (0, $F_X^{-1}(1-\alpha)$) and the limiting point (σ, μ) of (s, $\bar{x}$). Fifty observations (s, $\bar{x}$), from the same population as in Figure 1, are plotted for n = 10 and n = 300. The three lines correspond to α= 0.85 and k =0.5, 0.68 (=k^{∞}), and 1. When the sample size n goes to infinity, all sample points (s, $\bar{x}$) degenerate to the limiting point (σ, μ), which is (1, 0) here. For line L with k = 1 (greater than k^{∞} = 0.68), the point (σ, μ) is below the line. Hence as n goes to infinity, the probability of lying on or below the line (i.e., γ) goes to 1. Similarly, for line L with k = 0.5 (less than k^{∞}), all points (s, $\bar{x}$) shrink to the point (σ, μ) above the line, as n goes to infinity, and hence γ goes to 0. The convergence of γ may not be monotonic, however. As mentioned earlier in this section, γ increases with k but k is not necessarily monotonic with n. Therefore γ is not necessarily monotonic with n, even for the normal population.

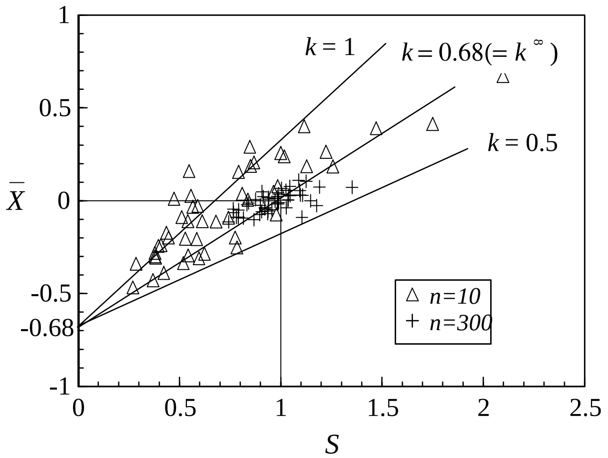
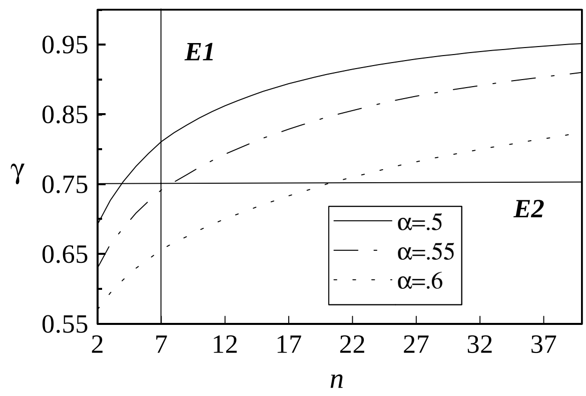


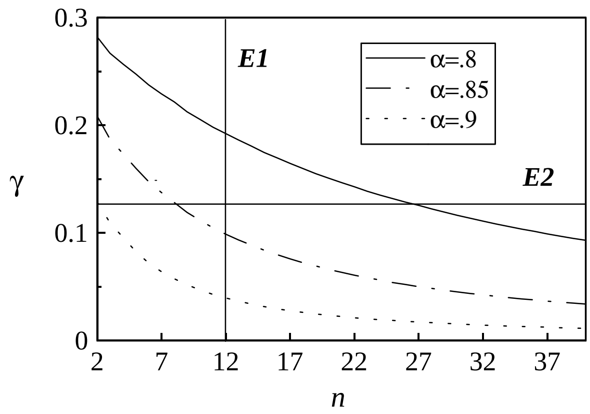
Figure 2: The (n,γ) Relationship: Plot of Line L in $(S, \bar{X})$ Sample Plane for Johnson S_B Distribution, n = 10, 300, k =0.5, 0.68, 1, and α= 0.85

Finally we show that the coverage α converges to the constant $\alpha^{\infty} = 1 - F_X(\mu - k\sigma)$ as n goes to infinity, given values of k and γ, and the distribution shape. As discussed in Section 3.1, α is the (1-γ)th quantile of the random variable $C = \mathrm{P}_X\{X \ge \bar{X} - kS\}$. When n goes to infinity, the sample mean $\bar{X}$ and sample standard deviation S degenerate to μ and σ, respectively. Therefore the random variable C, and every quantile, converge to α^{∞} = $\mathrm{P}_X\{X \ge \mu - k\sigma\}$. Notice that α^{∞} depends on k and the distribution shape but not μ or σ. As for γ, coverage α may not converge monotonically unless k is a monotonic function of n.

For cases that the monotonicity holds, Figures 3(a) and 3(b), respectively, show that α increases with n, converging to α^{∞} for $\alpha \in (0, \alpha^{\infty}]$ and decreases with n but also converges to α^{∞}, otherwise. In Figure 3(a), three curves illustrate that γ is an increasing function of n and converges to 1 for α = 0.5, 0.55, 0.6, k=0.5, and the Johnson S_B distribution with skewness 4 and kurtosis 35. The three α values are less than α^{∞} (0.66 here), therefore k must be greater than their associate k^{∞} values (recall that α^{∞}=1-$F_X(\mu - k\sigma)$ and $k^{\infty} = [\mu - F_X^{-1}(1-\alpha)]/\sigma$), and hence the three curves increase monotonically to γ=1). The two line segments E_1 and E_2 correspond to n = 7 and γ = 0.75, respectively. Since α is decreasing with γ, the intersections of the segment E_1 and the three curves, from top to bottom, correspond to the three increasing α values 0.5, 0.55, 0.6. Furthermore, the intersections of the segment E_2 and the three curves, from left to right, illustrate that α increases as n increases. In the limit, α converges to α^{∞}. Similarly, Figure 3(b) shows that α decreases with n, converging to α^{∞} for $\alpha \in [\alpha^{\infty}, 1)$. The three curves illustrate that γ decreases to 0 as n goes to infinity for α = 0.8, 0.85, 0.9 (larger than α^{∞} and hence $k < k^{\infty}$), where k and the distribution shape are as in Figure 3(a). The intersections of the line segment E_1 (corresponding to n=12) and the three curves illustrate three increasing α values 0.8, 0.85, 0.9, from top to bottom. Therefore the intersections of the line segment E_2 (corresponding to γ = 0.125) and the three curves illustrate that α decreases by n; in the limit, α converges to α^{∞}.



3(a)



3(b)

Figure 3: The (n,α) Relationship: Plot of γ as a Function of n for α = 0.5, 0.55, 0.6 in 3(a) and α = 0.8,0.85, 0.9 in 3(b), where k = 0.5, α^{∞} = 0.66,and the Distribution Shape is Johnson S_B

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AUTHOR BIOGRAPHIES

HUIFEN CHEN is in the faculty of the Department of Industrial Engineering at Da-Yeh University in Taiwan. She completed her Ph.D. in the School of Industrial Engineering at Purdue University in August 1994. She received a B.S. degree in accounting from National Cheng-Kung University in Taiwan in 1986 and an M.S. degree in statistics from Purdue University in 1990. Her research interests include simulation methodology and its application in reliability, transportation, and stochastic operations research.

TSU-KUANG YANG received a B.S. degree in industrial Engineering from Da-Yeh University in Taiwan in June 1998. He then joined the graduate program of the Department of Industrial Engineering at Tung-Hai University in Taiwan. He is an international member of Phi Dau Phi.