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# SENSITIVITY OF OUTPUT PERFORMANCE MEASURES TO INPUT DISTRIBUTIONS IN QUEUEING NETWORK MODELING

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## ABSTRACT

In Gross and Juttijudata (1997) a single node, G/G/1 queue was investigated as to the sensitivity of output performance measures, such as the mean queue wait, to the shape of the interarrival and service distributions selected. Gamma, Weibull, lognormal and Pearson type 5 distributions with identical first and second moments were investigated. Significant differences in output measures were noted for low to moderate traffic intensities (offered load, ρ), in some cases, even as high as 0.8. We continue this type of investigation for two types of queueing networks, namely two versions of a two-node call center, to see if network mixing might reduce the sensitivity effect.

## 1 INTRODUCTION

Certain queueing theoretic results, as mentioned in Gross and Juttijudata (1997), indicate that *only* the first two moments of interarrival and service time distributions may drive output performance measures, such as mean queue wait. This is certainly true for M/G/1 queues (the Pollaczek-Khintchine formula) and for queues in heavy-traffic (the Kingman heavy-traffic approximation and the Kingman-Marshall upper bound). However, in the above referenced study, significant differences in output measures did occur in the G/G/1 queues studied, even for fairly high traffic intensities.

We continue this investigation into sensitivity of mean queue wait, for a small two-node call center network to see if adding nodes changes the sensitivity to higher moments of the input distributions (distribution shapes). Two configurations of the call center network are shown in Figure 1. In both of these configurations, there is an independent stream of arrivals for each single-server node. Arrivals can, however, be rerouted to the other node under certain conditions, in order to decrease waiting time.

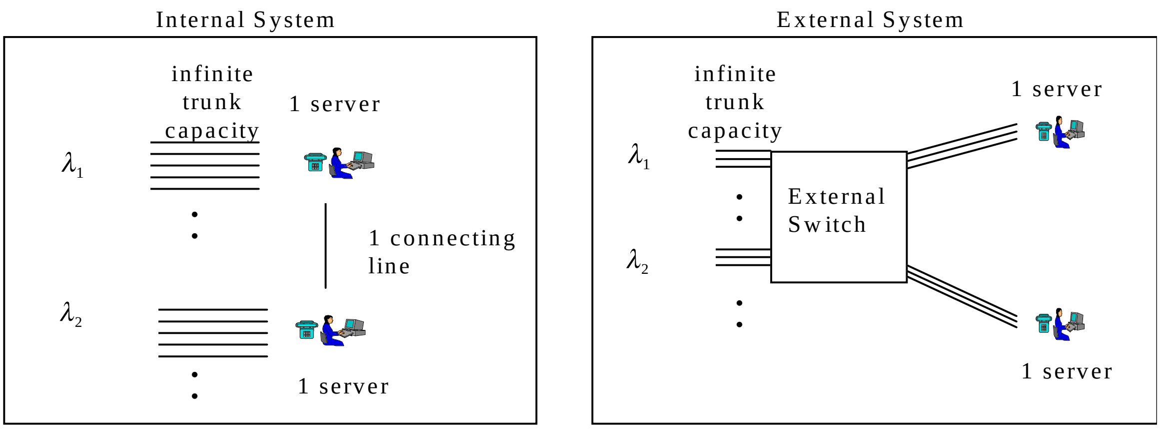



Figure 1: Two-node Call Center Networks

Call centers are being networked together in increasing numbers, as distributed rather than centralized configurations have advantages in emergency backup and call coverage across time zones (Sulkin, 1995; 1993). There are two basic call routing strategies for a networked call center. The internal configuration uses a connecting line between the nodes to re-route calls. In the internal system of interest for this research (Figure 1), a call waiting for an agent at one node can be routed to the other node if the agent at the other node and the connecting line is available. The external configuration has an external switch which routes calls prior to arrival to a node. For this research (Figure 1), a call arriving to the external switch is typically forwarded to its primary node. If, however, the agent at its primary node is busy and the agent at its secondary node is available when the call arrives to the external switch, the call is switched to the secondary node. Hence, the external configuration allows switching at arrival times only, while the internal configuration allows switching after arrival to the primary node, should an agent at the other node and a connecting line become available.

## 2 EXPERIMENTAL DESIGN

The networks were simulated using GPSS/H and 40 replications of 100,000 transactions were performed for each case. Interarrival and service-time distributions had identical means and variances, with coefficients of variation (CV = $\sigma/\mu$) varying among 0.5, 1.0 and 2.0, and differing only in the type of distribution (distributions considered were gamma, lognormal, beta, and Pearson type 5) – see Law and Kelton (1991). These generate a variety of shapes including hyperexponential, unimodal, and bimodal, with a range of skewnesses and kurtoses. Two values of traffic intensity for nodes 1 and 2 ($\rho_1$ and $\rho_2$) of .65 and .85 were also considered. Mean queue waits for each arrival type (Wq1 and Wq2) were compared and percent differences of the lognormal, beta, and Pearson type 5 from the gamma were computed. The 95$^{th}$ percentile of the queue wait distribution for each arrival type (denoted Wq(.95)1 and Wq(.95)2) was also considered for both systems with CV[AT]=1 and CV[ST]=1 at $\rho$=.65.

## 3 VALIDATION

Setting the CV equal to one for the Gamma interarrival and service distributions (which then become exponential) allows comparisons with theoretical queueing results. The stationary state probabilities for these two-node call center networks with exponential interarrival and service distributions can be found numerically using matrix-geometric techniques. The infinitesimal generator matrices of these systems are quasi birth and death processes, which are birth and death processes whose generator matrix entries are themselves matrices. Due to this repetitive structure, the stationary probability vectors for the internal and external systems can be found quite efficiently. After solving a nonlinear matrix equation for a rate matrix $R$ and deriving the invariant probability vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$, the normed stationary probability vector $\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \ldots)$ is computed using $\boldsymbol{x_k} = \boldsymbol{x_1}\boldsymbol{R}^{k-1}$ (for $k \geq 2$), and normalizing. The state probabilities can be used to compute the mean number of each arrival type in the queue, and Wq for each arrival type then found using Little's Formula.

Table 1 shows the simulated Wq for each arrival type, 95% confidence intervals for Wq, and the theoretical Wq computed as described above. We see that the differences in simulated means from theoretical are very small (all quite a bit less than 1%) and within the confidence intervals.

The 95% confidence interval half-width as a percentage of the mean for Wq for each of the four input distributions was also computed. Our estimates for Wq are quite precise, in that the confidence interval half-widths are about 1-2% of the mean for all distributions except the Pearson type 5 distribution, which is about 7% of the mean.

| Center 1 Off. Load | Center 2 Off. Load | System | Simulation: Type 1 Customers Wq1 | CI Lwr Bnd | CI Up Bnd | Theoretical Wq1 | Simulation: Type 2 Customers Wq2 | CI Lwr Bnd | CI Up Bnd | Theoretical Wq2 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.65 | 0.65 | External | 1.042 | 1.032 | 1.052 | 1.050 | 1.050 | 1.041 | 1.059 | 1.050 |
| 0.85 | 0.85 | External | 3.566 | 3.494 | 3.638 | 3.567 | 3.547 | 3.491 | 3.603 | 3.567 |
| 0.65 | 0.85 | External | 1.479 | 1.467 | 1.491 | 1.485 | 2.167 | 2.135 | 2.199 | 2.170 |
| 0.65 | 0.65 | Internal | 0.833 | 0.827 | 0.839 | 0.835 | 0.835 | 0.829 | 0.841 | 0.835 |
| 0.85 | 0.85 | Internal | 2.739 | 2.695 | 2.783 | 2.739 | 2.737 | 2.701 | 2.773 | 2.739 |
| 0.65 | 0.85 | Internal | 1.273 | 1.263 | 1.283 | 1.281 | 1.498 | 1.482 | 1.514 | 1.499 |

Table 1: Validation of Simulation Results

## 4 RESULTS

The main results are presented in Table 2. Figures 2 (external system) and 3 (internal system) show extracts from the table in chart form. Each figure has three charts, the first for offered loads at the nodes of $\rho_1 = \rho_2 = .65$, the second $\rho_1 = .65$, $\rho_2 = .85$, and the third, $\rho_1 = \rho_2 = .85$. Each chart has three sets of 5 pairs of values. The first set are the sensitivities (the percent differences from the gamma simulation) for the lognormal distribution, the second for the Pearson type 5 distribution, and the third for the beta distribution. The five pairs in each set differ in the CV[AT] and CV[ST] used for the runs. In each pair, the solid bar represents the percent difference for Wq1, the wait for arrivals to node one, from the gamma simulation and the striped bar represents the percent difference for Wq2, the wait for arrivals to node two, from the gamma simulation. So, for example, Wq2(.5,2) is the percent difference for the mean wait of arrivals to node two between its particular distribution and the gamma simulation, for a CV[AT] of .5 and a CV[ST] of 2.

We draw several conclusions from these graphs. First, there are significant sensitivities depending on which distribution is chosen, even though the first two moments are identical. For example from Figure 2 for the external system, for the $\rho_1 = \rho_2 = .65$ case, where CV[AT] = 2 and CV[ST] = .5, the lognormal waits are approximately 50% lower than those for the gamma, the Pearson type 5 waits are approximately 80% lower than those for the gamma, while the beta waits are about 60% higher! The sensitivities are about the same for each arrival type (the solid and striped bars in each pair are roughly similar in magnitude). Also, the sensitivities go down as the traffic intensities increase (as we would expect from heavy-traffic theory), but even for the $\rho_1 = \rho_2 = .85$ case, there are significant percentage differences from gamma. Sensitivities seem greatest for large CV[AT] and small CV[ST], the worst cases being the fifth of the set of five (CV[AT] = 2, CV[ST] = .5) and the next worst the third (CV[AT] = 2, CV[ST] = 2). The least sensitive cases were the fourth (CV[AT] = .5, CV[ST] = 2) and the first (CV[AT] = .5, CV[ST] = .5) of the sets of five. In general, CV[AT] seems to have a greater impact than CV[ST]. Note that only five of the nine possible combinations of CV[AT] and CV[ST] were tried, but the evidence certainly indicates that the foregoing conclusions are valid.

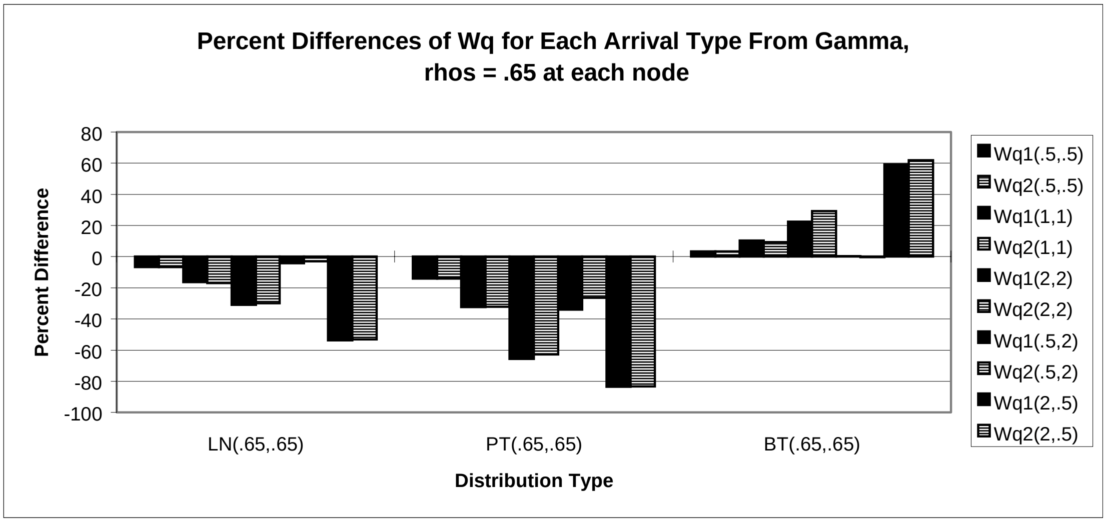
We see further, comparing Figures 2 and 3, that sensitivities are about the same for both systems, with the internal being possibly slightly less sensitive (but not significantly or consistently so). Previous results (Masi, 1998) indicated that the internal system had better performance than the external system with exponential interarrival and service distributions as demonstrated by the expected total number of customers in the system.

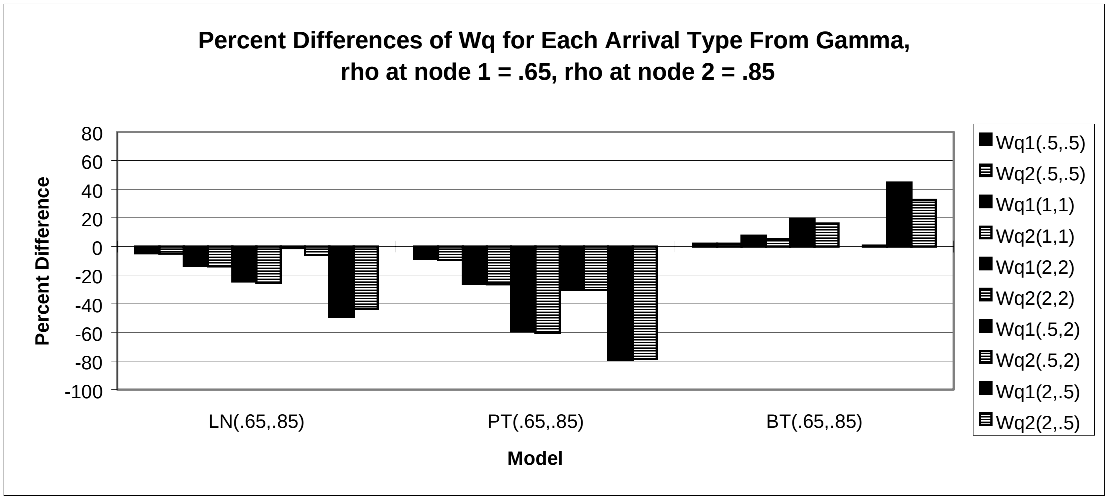
Some of the results for the two-node case under study here can be compared with results obtained for the single-node G/G/1 system studied in Gross and Juttijudata (1997). Table 3 shows percent differences from gamma for lognormal and Pearson type 5 distributions, and compares cases for “similar” offered loads with CV[AT] = 1 and CV[ST] = 1. These sensitivity percentages for the two-node networks for $\rho_1 = \rho_2 = .65$ and $\rho_1 = \rho_2 = .85$ are compared to the G/G/1 cases for $\rho = .6$ and .7 and $\rho = .8$ and .9 respectively. We see that the network is certainly not less sensitive than the single-node, G/G/1.

In the single-node, G/G/1 system (Gross and Juttijudata, 1997), the 95$^{th}$ percentile of the waiting-time distribution was also observed, and found to have slightly less sensitivity to distribution shape than the mean. We computed the 95$^{th}$ percentile value for the two-node, external and internal systems, for the case of offered loads of .65 at both nodes, and CV[AT] = CV[ST] = 1, with the results shown in Table 4. The results seem to also indicate that the 95$^{th}$ percentile value is somewhat less sensitive to distribution shape, but not dramatically.

Table 2: Sensitivity of Wq to Input Distributions

| Center 1 Off. Load | Center 2 Off. Load | CV[AT] | CV[ST] | Arr/Service Distribution | External System Wq1 | Wq2 | %Diff Wq1 | %Diff Wq2 | Internal System Wq1 | Wq2 | %Diff Wq1 | %Diff Wq2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.65 | 0.65 | 0.5 | 0.5 | Gamma | 0.205 | 0.205 | | | 0.200 | 0.199 | | |
| | | | | LN | 0.191 | 0.191 | -6.829 | -6.829 | 0.187 | 0.187 | -6.500 | -6.030 |
| | | | | Beta | 0.212 | 0.212 | 3.415 | 3.415 | 0.205 | 0.205 | 2.500 | 3.015 |
| | | | | PT5 | 0.176 | 0.176 | -14.146 | -14.146 | 0.174 | 0.174 | -13.000 | -12.563 |
| 0.65 | 0.65 | 1 | 1 | Gamma | 1.042 | 1.050 | | | 0.833 | 0.835 | | |
| | | | | LN | 0.872 | 0.872 | -16.315 | -16.952 | 0.710 | 0.705 | -14.766 | -15.569 |
| | | | | Beta | 1.150 | 1.148 | 10.365 | 9.333 | 0.896 | 0.895 | 7.563 | 7.186 |
| | | | | PT5 | 0.704 | 0.713 | -32.438 | -32.095 | 0.573 | 0.576 | -31.212 | -31.018 |
| 0.65 | 0.65 | 2 | 2 | Gamma | 5.440 | 5.420 | | | 3.786 | 3.790 | | |
| | | | | LN | 3.757 | 3.795 | -30.938 | -29.982 | 2.817 | 2.795 | -25.594 | -26.253 |
| | | | | Beta | 6.938 | 7.009 | 27.537 | 29.317 | 4.440 | 4.424 | 17.274 | 16.728 |
| | | | | PT5 | 1.877 | 2.018 | -65.496 | -62.768 | 1.403 | 1.663 | -62.942 | -56.121 |
| 0.65 | 0.65 | 0.5 | 2 | Gamma | 1.994 | 1.990 | | | 1.627 | 1.629 | | |
| | | | | LN | 1.909 | 1.929 | -4.263 | -3.065 | 1.459 | 1.484 | -10.326 | -8.901 |
| | | | | Beta | 2.001 | 1.985 | 0.351 | -0.251 | 1.682 | 1.685 | 3.380 | 3.438 |
| | | | | PT5 | 1.317 | 1.467 | -33.952 | -26.281 | 0.924 | 1.127 | -43.208 | -30.816 |
| 0.65 | 0.65 | 2 | 0.5 | Gamma | 3.152 | 3.121 | | | 2.094 | 2.093 | | |
| | | | | LN | 1.460 | 1.462 | -53.680 | -53.156 | 1.117 | 1.118 | -46.657 | -46.584 |
| | | | | Beta | 5.021 | 5.058 | 59.296 | 62.063 | 3.059 | 3.070 | 46.084 | 46.679 |
| | | | | PT5 | 0.520 | 0.519 | -83.503 | -83.371 | 0.445 | 0.445 | -78.749 | -78.739 |
| 0.85 | 0.85 | 0.5 | 0.5 | Gamma | 0.815 | 0.817 | | | 0.691 | 0.695 | | |
| | | | | LN | 0.793 | 0.791 | -2.699 | -3.182 | 0.678 | 0.678 | -1.881 | -2.446 |
| | | | | Beta | 0.830 | 0.827 | 1.840 | 1.224 | 0.706 | 0.705 | 2.171 | 1.439 |
| | | | | PT5 | 0.765 | 0.769 | -6.135 | -5.875 | 0.655 | 0.658 | -5.210 | -5.324 |
| 0.85 | 0.85 | 1 | 1 | Gamma | 3.566 | 3.547 | | | 2.739 | 2.737 | | |
| | | | | LN | 3.257 | 3.230 | -8.665 | -8.937 | 2.556 | 2.542 | -6.681 | -7.125 |
| | | | | Beta | 3.686 | 3.735 | 3.365 | 5.300 | 2.833 | 2.825 | 3.432 | 3.215 |
| | | | | PT5 | 2.802 | 2.864 | -21.425 | -19.256 | 2.249 | 2.235 | -17.890 | -18.341 |
| 0.85 | 0.85 | 2 | 2 | Gamma | 16.918 | 16.302 | | | 11.548 | 11.289 | | |
| | | | | LN | 13.198 | 13.081 | -21.988 | -19.758 | 9.686 | 9.736 | -16.124 | -13.757 |
| | | | | Beta | 17.992 | 18.198 | 6.348 | 11.630 | 12.199 | 12.001 | 5.637 | 6.307 |
| | | | | PT5 | 7.217 | 7.618 | -57.341 | -53.270 | 5.420 | 5.859 | -53.065 | -48.100 |
| 0.85 | 0.85 | 0.5 | 2 | Gamma | 7.101 | 7.193 | | | 5.609 | 5.597 | | |
| | | | | LN | 6.913 | 6.876 | -2.648 | -4.407 | 5.427 | 5.351 | -3.245 | -4.395 |
| | | | | Beta | 7.212 | 7.138 | 1.563 | -0.765 | 5.723 | 5.728 | 2.032 | 2.341 |
| | | | | PT5 | 4.837 | 5.169 | -31.883 | -28.138 | 3.783 | 4.047 | -32.555 | -27.693 |
| 0.85 | 0.85 | 2 | 0.5 | Gamma | 9.228 | 9.069 | | | 6.178 | 6.073 | | |
| | | | | LN | 5.959 | 5.999 | -35.425 | -33.852 | 4.494 | 4.464 | -27.258 | -26.494 |
| | | | | Beta | 11.084 | 10.950 | 20.113 | 20.741 | 6.966 | 6.931 | 12.755 | 14.128 |
| | | | | PT5 | 2.619 | 2.627 | -71.619 | -71.033 | 2.129 | 2.146 | -65.539 | -64.663 |
| 0.65 | 0.85 | 0.5 | 0.5 | Gamma | 0.345 | 0.438 | | | 0.355 | 0.344 | | |
| | | | | LN | 0.329 | 0.417 | -4.638 | -4.795 | 0.344 | 0.326 | -3.099 | -5.233 |
| | | | | Beta | 0.352 | 0.447 | 2.029 | 2.055 | 0.362 | 0.349 | 1.972 | 1.453 |
| | | | | PT5 | 0.316 | 0.397 | -8.406 | -9.361 | 0.333 | 0.308 | -6.197 | -10.465 |
| 0.65 | 0.85 | 1 | 1 | Gamma | 1.479 | 2.167 | | | 1.273 | 1.498 | | |
| | | | | LN | 1.283 | 1.866 | -13.252 | -13.890 | 1.140 | 1.339 | -10.448 | -10.614 |
| | | | | Beta | 1.591 | 2.279 | 7.573 | 5.168 | 1.360 | 1.558 | 6.834 | 4.005 |
| | | | | PT5 | 1.096 | 1.595 | -25.896 | -26.396 | 1.002 | 1.112 | -21.288 | -25.768 |
| 0.65 | 0.85 | 2 | 2 | Gamma | 6.945 | 11.148 | | | 5.496 | 6.594 | | |
| | | | | LN | 5.253 | 8.311 | -24.363 | -25.449 | 4.342 | 5.246 | -20.997 | -20.443 |
| | | | | Beta | 8.290 | 12.937 | 19.366 | 16.048 | 6.068 | 6.938 | 10.408 | 5.217 |
| | | | | PT5 | 2.830 | 4.427 | -59.251 | -60.289 | 2.489 | 2.881 | -54.713 | -56.309 |
| 0.65 | 0.85 | 0.5 | 2 | Gamma | 2.957 | 4.165 | | | 2.607 | 3.005 | | |
| | | | | LN | 2.926 | 3.918 | -1.048 | -5.930 | 2.530 | 2.816 | -2.954 | -6.290 |
| | | | | Beta | 2.954 | 4.194 | -0.101 | 0.696 | 2.648 | 3.053 | 1.573 | 1.597 |
| | | | | PT5 | 2.063 | 2.899 | -30.233 | -30.396 | 1.786 | 1.978 | -31.492 | -34.176 |
| 0.65 | 0.85 | 2 | 0.5 | Gamma | 3.894 | 6.428 | | | 2.894 | 3.620 | | |
| | | | | LN | 2.030 | 3.634 | -47.869 | -43.466 | 1.685 | 2.363 | -41.776 | -34.724 |
| | | | | Beta | 5.631 | 8.531 | 44.607 | 32.716 | 3.942 | 4.305 | 36.213 | 18.923 |
| | | | | PT5 | 0.814 | 1.385 | -79.096 | -78.454 | 0.754 | 0.994 | -73.946 | -72.541 |



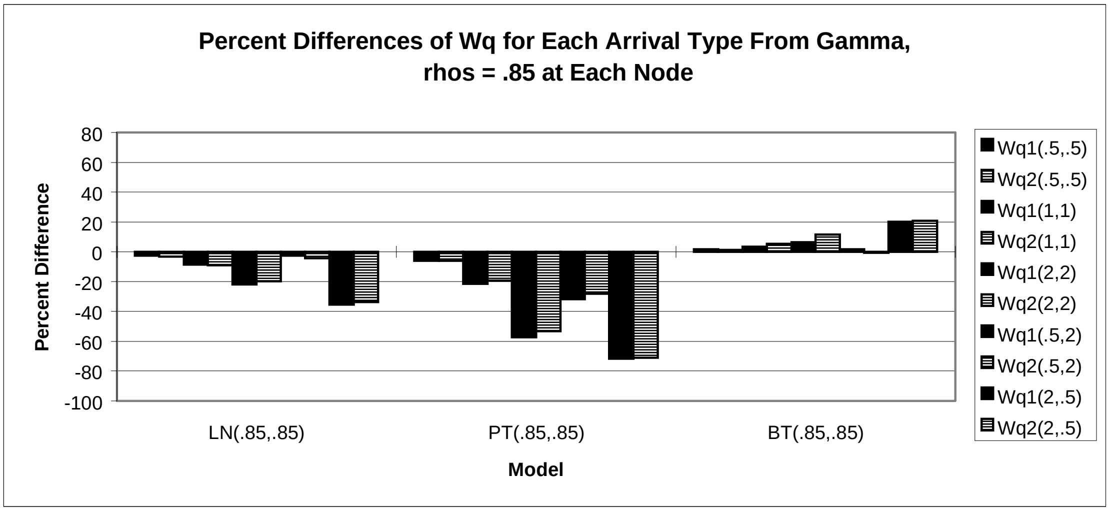




Figure 2: External System

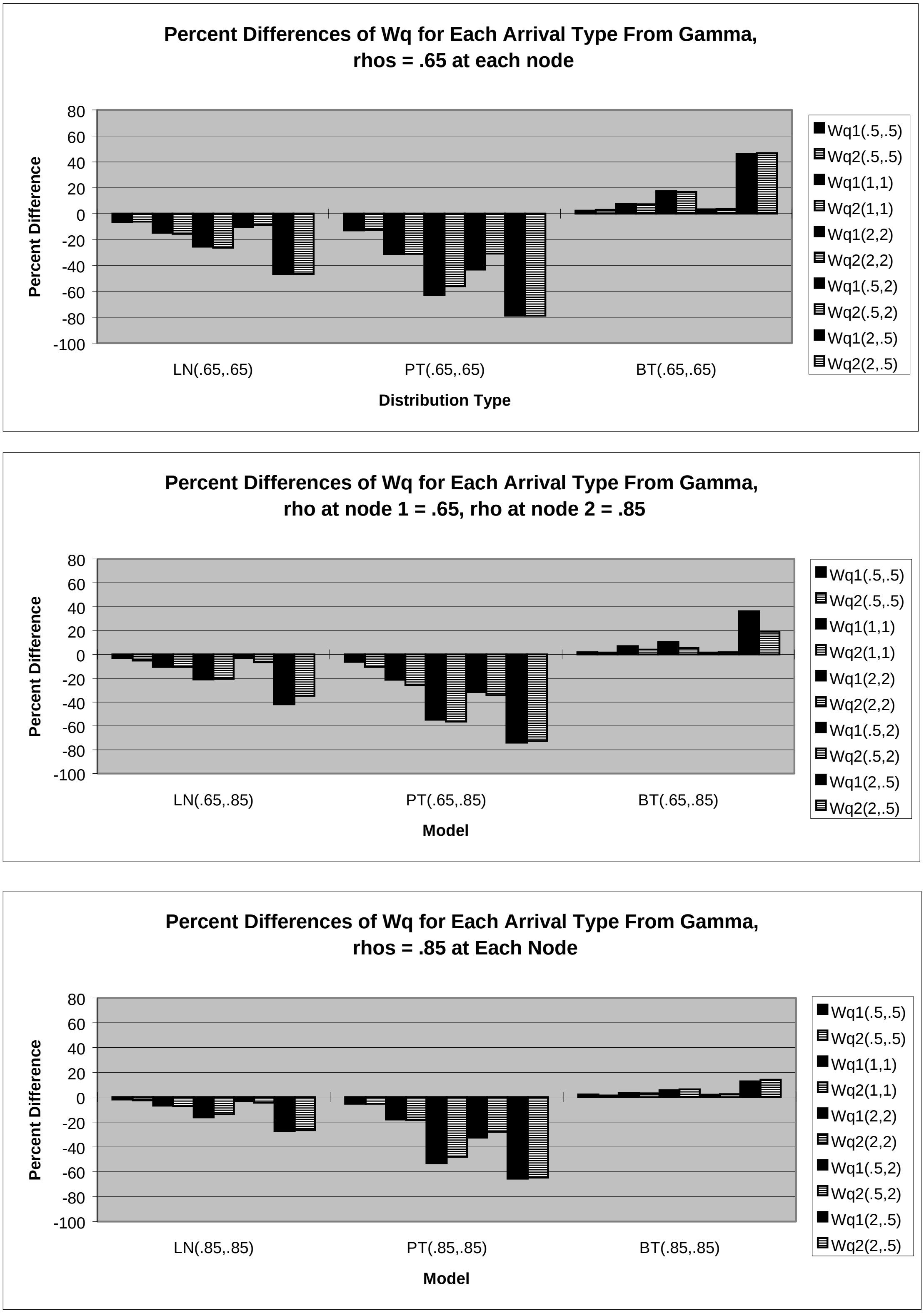



Figure 3: Internal System

Table 3: Comparison of Sensitivity of the Two-Node Networks to the Single Node Case

| % Diff in Wq's from Gamma/Gamma Distributions for CV[AT] = 1 and CV[ST] = 1 | | | | | | |
|---|---|---|---|---|---|---|
| Model | G/G/1 | | Exterior Network | | Interior Network | |
| | | | Type 1 | Type 2 | Type 1 | Type 2 |
| **rho** | **0.6** | **0.7** | **0.65** | **0.65** | **0.65** | **0.65** |
| LN | -14.39 | -10.04 | -16.32 | -16.95 | -14.77 | -15.57 |
| PT5 | -31.17 | -25.26 | -32.44 | -32.1 | -31.21 | -31.02 |
| | | | | | | |
| **rho** | **0.8** | **0.9** | **0.85** | **0.85** | **0.85** | **0.85** |
| LN | -6.7 | -1.65 | -8.67 | -8.94 | -6.68 | -7.13 |
| PT5 | -17.87 | -13.3 | -21.43 | -19.26 | -17.89 | -18.34 |

Table 4: Comparison of Sensitivity of the 95th Percentile of Wq to Wq
Offered Loads at Both Nodes of .65, CV[AT] = CV[ST] = 1

| | External System | | | | | | Internal System | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Arr/Serv Dist. | Wq(.95)1 | Wq(.95)2 | % Diff Wq(.95)1 | % Diff Wq(.95)2 | % Diff Wq1 | % Diff Wq2 | Wq(.95)1 | Wq(.95)2 | % Diff Wq(.95)1 | % Diff Wq(.95)2 | % Diff Wq1 | % Diff Wq2 |
| Gamma | 4.934 | 4.968 | | | | | 3.773 | 3.775 | | | | |
| LN | 4.451 | 4.446 | -9.79 | -10.51 | -16.31 | -16.95 | 3.431 | 3.410 | -9.07 | -9.66 | -14.77 | -15.57 |
| Beta | 5.267 | 5.256 | 6.75 | 5.80 | 10.36 | 9.33 | 3.925 | 3.916 | 4.02 | 3.75 | 7.56 | 7.19 |
| PT5 | 3.509 | 3.536 | -28.87 | -28.82 | -32.44 | -32.10 | 2.732 | 2.755 | -27.58 | -27.00 | -31.21 | -31.02 |

## 5 CONCLUSIONS

Two moments are not enough, in general, to capture the essence of a particular probability distribution with respect to output performance measures. Since most distribution-fitting software consider classic statistical distributions which are, for the most part, families of two-parameter distributions, and each has their own sophisticated formulas for fitting the data, they often give different recommendations as to the particular type of distribution. Even if the resulting means and variances agree, using different distribution families can produce quite large differences in output performance measures. As seen here, it therefore would seem quite necessary for analysts to do sensitivity analysis as to distribution shape.

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