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SENSITIVITY ANALYSIS OF SIMULATION OUTPUT TO PARAMETERS OF NONHOMOGENEOUS POISSON PROCESSES

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ABSTRACT

Nonhomogeneous Poisson processes (NHPPs) are frequently used in stochastic simulations to model nonstationary point processes. These NHPP models are often constructed by estimating the parameters of the rate function from one or more observed realizations of the process. This paper focuses on the degree of accuracy to which the rate function parameters of the NHPP need to be estimated such that the simulation output performance measures are not significantly different from performance measures that would be obtained for the underlying (true) process.

1 INTRODUCTION

Constructing a valid stochastic simulation model involves adequately modeling both the logical flow of entities through the system of interest and the stochastic behavior of the system. The latter includes the selection of appropriate statistical models for interarrival times, delays, processing times, etc. Developing these statistical models may involve estimating parameters of probability distributions based on some observed data. Once the model is complete, one way to verify the simulation model is to compare the output performance measures of the model with those of the real system (Law and Kelton, 1991). If the simulation results are significantly different than the observed measures of performance, then inferences made about the real system based on the model may lead to wrong and costly decisions.

Therefore, accurately modeling the stochastic processes used in simulation models is critical to the success of the model's implementation. However, as the stochastic processes become increasingly complex, so do the methods required to accurately model these processes. Consequently, the time and effort required to construct stochastic models increases along with the time required to generate realizations of these processes during simulation experiments. Our objective in this situation is to, therefore, select the simplest model that accurately models the behavior of the true system.

In this paper we focus on arrival (counting) processes, and more particularly, arrival processes that can be classified as nonstationary point processes. For such processes we are able to observe each arrival time exactly, and in general the arrival intensity (rate) changes over time. Under certain assumptions a nonstationary arrival process can be represented as a nonhomogeneous Poisson process (NHPP) (Çinlar, 1975). Using NHPPs, we can accurately represent a large class of arrival processes encountered in practice.

An NHPP $\{N(t) : t \geq 0\}$ given by

$$N(t) = \text{ \# of arrivals in } [0, t] \quad \text{for all} \quad t \geq 0$$

is a generalization of the Poisson process in which the instantaneous arrival rate $\lambda(t)$ at time t is a nonnegative integrable function of time. The mean-value function of the NHPP is defined by

$$\mu(t) \equiv \mathrm{E}[N(t)] \quad \text{for all} \quad t \geq 0;$$

and the relationship between the rate function and the mean-value function is

$$\mathrm{E}[N(t)] = \int_0^t \lambda(z)\, dz \quad \text{for all} \quad t \geq 0.$$

The probabilistic behavior of the NHPP is completely defined by the rate or mean-value functions. The literature in this area includes both parametric and nonparametric methods for estimating the NHPP rate function. To model arrival processes having several periodic effects or a long-term trend (or both), Kuhl, Wilson, and Johnson (1997) utilized an NHPP having an exponential-polynomial-trigonometric with multiple periodicities (EPTMP). The EPTMP-type rate function has the form

$$\lambda(t) = \exp\{h(t; m, p, \Theta)\}, \qquad t \in [0, S], \qquad (1)$$

with

$$h(t; m, p, \Theta) = \sum_{i=0}^{m} \alpha_i t^i + \sum_{k=1}^{p} \gamma_k \sin(\omega_k t + \phi_k),$$

where

$$\Theta = [\alpha_0, \alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_p, \phi_1, \ldots, \phi_p, \omega_1, \ldots, \omega_p]$$

is the vector of continuous parameters.

Kuhl, Wilson, and Johnson (1997) present a maximum likelihood estimation procedure to fit the rate function $\lambda(t)$ to the observed arrivals over the time interval $[0, S]$. The problem inherent to this methodology is that as the changing arrival rate becomes more complex, so does the NHPP model. Moreover, the computational time required to estimate the parameters of the rate function increases dramatically as the complexity increases. Therefore, in attempt to minimize the amount of work required, we propose to determine the degree of accuracy to which the rate function parameters of the NHPP need to be estimated such that the simulation output performance measures are not significantly different from performance measures that would be obtained for the underlying (true) process.

2 METHODOLOGY

The objective of this research is to perform a sensitivity analysis on the output performance measures of simulation experiments in order to determine the degree of accuracy to which the parameters of nonhomogeneous Poisson processes having an EPTMP-type rate function must be estimated. That is, we want to determine the maximum relative change in the parameters of the rate function that will yield simulation output performance measures that are not significantly different than output performance measures of the underlying NHPP. By doing this, we will be able to establish the amount of relative error allowable in estimating the parameters of the NHPP from observed data.

The sensitivity analysis will be conducted using a simulation model of a single server queueing system. The underlying arrival process will be a nonhomogeneous Poisson process with an EPTMP-type rate function of the form (1). Using this type of rate function, we will be able to model NHPPs having long term trend and one or more cyclic components. The single server will be modeled using an exponential service time with a constant service rate. The queueing discipline employed will be first-in-first-out.

The sensitivity analysis will concentrate on five output performance measures including (a) the number of entities in the system, L; (b) the number of entities in the queue, L_q; (c) the total time an entity spends in the system, W; (d) the time an entity spend waiting in the queue, W_q; and (e) the utilization of the server, ρ.

Several NHPP models will be selected with varying long term trends and cyclic components. Since for these types of queueing systems, there is no general closed form analytical solution for calculating the theoretical values of the performance measures, we first need to simulate the queueing system to estimate each of the output performance measures. Upon doing this, we then vary the parameters of the NHPP rate function to determine the maximum relative variation that will result in the performance measures that are not significantly different than the performance measures obtained using the original rate function parameters.

To determine if the output performance measures obtained under the NHPPs with varied rate function parameters are different from the performance measures obtained under the original parameters, the standard statistical test for testing the difference between two means will be employed. Here we will use a significance level of $\alpha = 0.05$.

To show that this type of sensitivity analysis can provide meaningful results, we have conducted the following preliminary study.

3 PRELIMINARY STUDY

This preliminary study was conducted to show that this method sensitivity analysis produces meaningful result in a case where the theoretical results are known. This analysis involves a simulation model of the above single server queueing system utilizing a Poisson arrival process with a constant average arrival rate. Thus the theoretical output performance measures are those associated with the M/M/1 queueing model (Gross and Harris, 1985).

For this analysis, we used a server utilization of $\rho = 0.5$ by setting the arrival rate $\lambda = 0.5$ arrival per time unit and a mean service time of $\mu = 1$ time unit. Based on these parameters, Table 1 lists the theoretical values of the performance measures.

Table 1: Theoretical Value of Performance Measures for $\lambda = 0.5$ and $\mu = 1$.

L	L_q	W	W_q	ρ
1.0	0.5	2.0	1.0	0.5

After building the simulation model of the above queueing system, we determined an appropriate warm-up period (1000 time units), an appropriate replication length (5000 time units), and an appropriate number of replications (2000 replications) in order to obtain results that approached the theoretical values. We then constructed 95% confidence intervals for each output performance measure. These result are shown in Tables 2–6. Note that in each case the

Table 2: Results for Number In System, L.

λ	Percent Change	Sample Mean	95% Confidence Interval
0.50000	0	1.00	$[0.9961 < L < 1.0022]$
0.50050	0.10	1.00	$[0.9982 < L < 1.0043]$
0.50095	0.19	1.00	$[1.0000 < L < 1.0061]$
0.50100	0.20	1.00	$[1.0002 < L < 1.0063]$
0.50125	0.25	1.00	$[1.0011 < L < 1.0073]$
0.50250	0.50	1.01	$[1.0062 < L < 1.0124]$
0.50500	1.00	1.02	$[1.0165 < L < 1.0227]$
0.52500	5.00	1.11	$[1.1020 < L < 1.1089]$
0.55000	10.0	1.22	$[1.2182 < L < 1.2263]$

Table 3: Results for Number In Queue, L_q.

λ	Percent Change	Sample Mean	95% Confidence Interval
0.50000	0	0.50	$[0.4972 < L_q < 0.5023]$
0.50050	0.10	0.50	$[0.4987 < L_q < 0.5038]$
0.50095	0.19	0.50	$[0.5000 < L_q < 0.5052]$
0.50100	0.20	0.50	$[0.5002 < L_q < 0.5053]$
0.50125	0.25	0.50	$[0.5009 < L_q < 0.5061]$
0.50250	0.50	0.51	$[0.5047 < L_q < 0.5099]$
0.50500	1.00	0.52	$[0.5124 < L_q < 0.5177]$
0.52500	5.00	0.58	$[0.5779 < L_q < 0.5838]$
0.55000	10.0	0.67	$[0.6692 < L_q < 0.6762]$

Table 4: Results for Time In System, W.

λ	Percent Change	Sample Mean	95% Confidence Interval
0.50000	0	2.00	$[1.9923 < W < 2.0028]$
0.50050	0.10	2.00	$[1.9943 < W < 2.0048]$
0.50095	0.19	2.00	$[1.9961 < W < 2.0066]$
0.50100	0.20	2.00	$[1.9963 < W < 2.0069]$
0.50125	0.25	2.00	$[1.9973 < W < 2.0079]$
0.50250	0.50	2.01	$[2.0024 < W < 2.0130]$
0.50500	1.00	2.02	$[2.0127 < W < 2.0234]$
0.52500	5.00	2.10	$[2.0992 < W < 2.1107]$
0.55000	10.0	2.22	$[2.2154 < W < 2.2282]$

Table 5: Results for Time in Queue, W_q.

λ	Percent Change	Sample Mean	95% Confidence Interval
0.50000	0	1.00	$[0.9939 < W_q < 1.0033]$
0.50050	0.10	1.00	$[0.9959 < W_q < 1.0053]$
0.50095	0.19	1.00	$[0.9977 < W_q < 1.0072]$
0.50100	0.20	1.00	$[0.9979 < W_q < 1.0074]$
0.50125	0.25	1.00	$[0.9989 < W_q < 1.0084]$
0.50250	0.50	1.01	$[1.0039 < W_q < 1.0134]$
0.50500	1.00	1.02	$[1.0142 < W_q < 1.0238]$
0.52500	5.00	1.11	$[1.1004 < W_q < 1.1109]$
0.55000	10.0	1.22	$[1.2165 < W_q < 1.2283]$

Table 6: Results for Utilization, ρ.

λ	Percent Change	Sample Mean	95% Confidence Interval
0.50000	0	0.50	$[0.4989 < \rho < 0.5001]$
0.50050	0.10	0.50	$[0.4994 < \rho < 0.5006]$
0.50095	0.19	0.50	$[0.4998 < \rho < 0.5011]$
0.50100	0.20	0.50	$[0.4999 < \rho < 0.5011]$
0.50125	0.25	0.50	$[0.5001 < \rho < 0.5013]$
0.50250	0.50	0.50	$[0.5014 < \rho < 0.5026]$
0.50500	1.00	0.51	$[0.5039 < \rho < 0.5052]$
0.52500	5.00	0.53	$[0.5240 < \rho < 0.5252]$
0.55000	10.0	0.55	$[0.5489 < \rho < 0.5502]$

theoretical value is well within the limits of the confidence interval.

The next step was vary the arrival rate λ (both increasing and decreasing the parameter) to determine first the maximum relative variation resulting in the theoretical value falling outside of the confidence interval indicating a significant difference; and second the maximum relative variation resulting in significant differences between the results and those obtained for the simulation of the system using the original value of the parameter.

The results of this preliminary study are shown in Tables 2–6. In general, these tables show that as the relative change in the arrival rate increases, the simulation output performance measures diverge from their theoretical values. For example, the 95% confidence interval on the average number of entities in the system shown in Table 2 includes the theoretical value of $L = 1$ for the original case where the arrival rate is known and for the cases where the arrival rate has been increased by up to 0.19 percent. However, when the arrival rate is increased beyond 0.20 percent, the confidence intervals do not include the theoretical value which indicates that the average number of customers in the system for the simulation model is significantly different from the theoretical value.

However, for more complex arrival processes such as the NHPP, we cannot, in general, calculate the theoretical values of the output performance measures. Therefore, we will have to compare the simulated results of models having varied arrival process parameters with the simulated results of models having the original (known) arrival process parameters. Table 2 shows that as the arrival rate is increased beyond 0.500 percent, the confidence intervals for L do not overlap with the confidence interval constructed under the original parameter which indicates a significant difference in the resulting output performance measure.

Similar types of results are seen in Tables 3–6 for the other performance measures of interest. In general for the case of the $M/M/1$ queueing system model, these results indicate that a variation in the parameter beyond 0.500

percent will result in output performance measures that are significantly different that those of the original system. Thus, this study has shown that sensitivity analysis of the output performance measures can help to establish the allowable estimation error of the parameters of arrival process models.

The next section, outlines an sensitivity analysis involving arrival processes that can be modeled as nonhomogeneous Poisson processes.

4 CURRENT EXPERIMENTAL STUDY

We are currently conducting the proposed sensitivity analysis of a group to NHPPs having EPTMP-type rate functions of the from (1). The analysis is being conducted on a selected group of rate functions having long term tend and cyclic behavior. The long term trend is represented in the exponent of the rate function by polynomials up to degree four and the cyclic components are represented in the exponent by trigonometric terms. The test cases have up to four periodic components of varying periods.

This experimental procedure will involve using the computer program `mp3sim` from Kuhl, Wilson and Johnson (1997) to generate the arrival times of entities in the simulation model. A constant service rate will be used during each replication. The service rate will be adjusted for different rate functions so that we can determine the effects on the output measures at different traffic intensities. We will adjust the service rate to achieve low ($\rho \approx 0.2$), moderate ($\rho \approx 0.5$), and high ($\rho \approx 0.8$) traffic intensities.

5 SUMMARY

Knowing the accuracy to which parameters of an NHPP rate function need to be estimated has two main benefits. First, the analyst can have confidence in the model (or at least in the model of the arrival process). Second, the analyst has a basis for making judgments about the quality of the fitted arrival process model. That is, one can determine if the current model will be adequate or if more time and effort needs to be spent to achieve more accurate estimates of the arrival process parameters.

Thus, this paper has provided a procedure for conducting a sensitivity analysis of output performance measures with regard to the accuracy of the rate function parameters of NHPP arrival processes. The preliminary study has shown that this type of sensitivity analysis can provide information of the quality of the estimates that need to be achieved in order to obtain results from the simulation model that are not significantly different from the results that would be obtained if the parameters were known. The current experimental study will involve a sensitivity analysis of output performance measure involving simulation models having arrival processes that can be modeled using NHPP having an EPTMP-type rate function. Moreover, this sensitivity analysis can provide some guidance to the analyst about the effects of the relative accuracy of input models has on the output performance measure of simulation experiments.

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