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EFFICIENT SIMULATION FOR DISCRETE PATH-DEPENDENT OPTION PRICING

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ABSTRACT

In this paper we present an algorithm for simulating functions of the minimum and terminal value for a random walk with Gaussian increments. These expectations arise in connection with estimating the value of path-dependent options when prices are monitored at a discrete set of times. The expected running time of the algorithm is bounded above by a constant as the number of steps increases.

1 INTRODUCTION

Let $\{X(t) : 0 \le t \le T\}$ be a Brownian motion, and fix an integer m. We are interested in the expectation of a function f of $\{X(iT/m),\ 0 \le i \le m\}$. The straightforward way to estimate $Ef(X)$ is to simulate the embedded random walk, taking time proportional to m.

In many situations one is interested in functions f that depend only on the minimal (or maximal) and terminal values $\min_{0\le i\le m} X(iT/m)$ and $X(T)$. These expectations arise in connection with pricing certain path-dependent financial options. For example, the value of a discrete *lookback call* is of this form; see Broadie, Glasserman, and Kou (1999). Suppose that the price of an asset at time t, $S(t)$, evolves according to the stochastic differential equation

$$dS(t) = rS(t)\,dt + \sigma\, dZ(t),$$

where Z is a standard Brownian motion and r, $\sigma > 0$ are constants. The solution of the equation is a geometric Brownian motion

$$S(t) = S(0)\exp\Big((r-\sigma^2/2)t + \sigma Z(t)\Big) = S(0)e^{B(t)},$$

where B is a Brownian motion with drift $r - \sigma^2/2$ and variance parameter σ^2. For a discrete lookback call option, the value of the asset is monitored at a finite set of times iT/m, $i = 0, 1, \ldots, m$. The payoff is

$$\begin{aligned} S(T) \quad - \quad & \min_{0\le i\le m} S(iT/m) = S(0)\exp(B(T)) \\ & -S(0)\exp\left(\min_{0\le i\le m} B(iT/m)\right). \end{aligned}$$

Therefore, the value of such an option is the expectation of a function of the terminal value and the minimum of a Brownian motion over a discrete set of points. Such expectations also arise in connection with barrier options, where the payoff depends on whether a certain boundary is crossed.

In situations like these, since not all values of the random walk are needed, one can hope to generate iid copies of $\big(\min_{0\le i\le m} X(iT/m), X(T)\big)$ in time less than linear in m. In this paper we will show that these pairs can be generated in bounded expected time (as $m \to \infty$). The approach is to simulate the continuous process X, "skipping" regions which can not contain the minimum of the discrete process.

Preliminary results are presented in the next section. Section 3 describes the simulation procedure. The results of numerical experiments are presented in Section 4.

2 SETUP

We assume that X is a standard Brownian motion and $T = 1$. The extension to non-unit variance and $T \ne 1$ is simple, and we discuss below how to take into account non-zero drift. Let Θ denote the (first) location where X attains its minimum value. It will be convenient to consider the nonnegative process Y defined by $Y(t) = X(t) - X(\Theta)$ (see Figure 1). With this translation, $f(X) = f(Y - Y(0))$.

The minimizer Θ "splits" the process X into independent pieces, as described below; see Williams (1974) and Fitzsimmons (1985). Let Ω denote the set of continuous nonnegative functions $\omega : [0, 1] \to \mathbb{R}^+$, and define the coordinate mappings $Y(t) : \Omega \to \mathbb{R}^+$ by $Y_t(\omega) = \omega(t)$ for $0 \le t \le 1$. Let $\mathcal{F} = \sigma\{Y(t) : 0 \le t \le 1\}$ be the

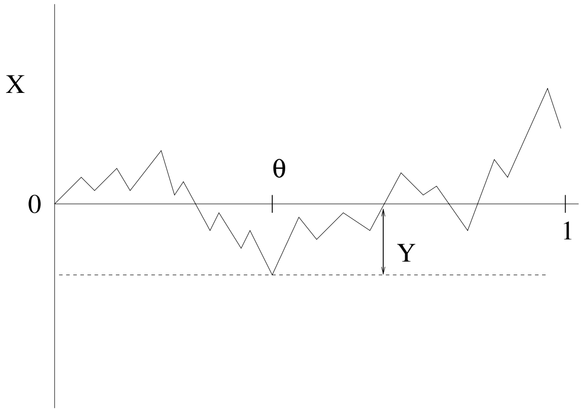

Figure 1: Translated Process

σ-field of subsets of Ω generated by Y. For $0 \le \theta \le 1$, let M_θ (resp. R_θ) denote the probability on $\mathcal{F}$ under which the processes $Y(\theta + t)$, $0 \le t \le 1-\theta$ and $Y(\theta - t)$, $0 \le t \le \theta$, are independent Brownian meanders (resp. 3-dimensional Bessel processes). Conditional on $\Theta = \theta$, the path segments $\{X(\theta+t) - X(\theta) : 0 \le t \le 1-\theta\}$ and $\{X(\theta - t) - X(\theta) : 0 \le t \le \theta\}$, are independent Brownian meanders. The Brownian meander (over $[0, t]$) is absolutely continuous with respect to the 3-dimensional Bessel process, with Radon-Nikodym derivative $\frac{\sqrt{t\pi/2}}{x(t)}$; see Imhof (1984). Therefore,

$$M_\theta(dy) = \frac{\pi}{2}\frac{\sqrt{\theta(1-\theta)}}{y(0)y(1)} R_\theta(dy). \tag{1}$$

The minimizer Θ of X has the arcsine density $\xi(\theta) = \left(\pi\sqrt{\theta(1-\theta)}\right)^{-1}$. Combining these facts yield

Theorem 2.1.

$$Ef(X) = \int_{\theta=0}^{1}\int_{\Omega} \frac{f(y - y(0))}{2\,y(0)y(1)} R_\theta(dy)\,d\theta. \tag{2}$$

The usefulness of (2) is due to the fact that the Bessel process (being a time homogeneous Markov process) is much easier to simulate than the meander.

If X has drift μ, then the law of X is equivalent to the law of standard Brownian motion, with Radon-Nikodym derivative $\exp(\mu X(1) - \mu^2/2)$, and so we would include the factor $\exp(\mu[y(1) - y(0)] - \mu^2/2)$ in (2).

3 SIMULATION PROCEDURE

The overall procedure, of which the key second step will be described in more detail below, is as follows:

(i) Generate θ uniformly distributed over $[0, 1]$.

(ii) Calculate $y(0)$, $y(1)$, and

$$\begin{aligned}\Delta &= \min y\left((\lceil\theta m\rceil + i)/m\right), \\ &\quad i = -\lceil\theta m\rceil, \ldots, m - \lceil\theta m\rceil,\end{aligned}$$

where $y \sim R_\theta$.

(iii) Calculate

$$\frac{f(\Delta - y(0), y(1) - y(0))}{2\,y(0)y(1)}.$$

Repeat the above steps independently and average to obtain an estimate of $Ef(X)$.

We break the calculation of Δ into two steps; calculate the minimum value on the grid to the right of Θ, denoted Δ_R, and then calculate the minimum value to the left of Θ (if it is smaller than Δ_R) in an analogous way. To calculate

$$\Delta_R = \min y\left((\lceil\theta m\rceil + i)/m\right), \quad i = 0, \ldots, m - \lceil\theta m\rceil :$$

(i) Generate $y\left((\lceil\theta m\rceil + i)/m\right)$, starting with $i = 0$, until $c = y\left((\lceil\theta m\rceil + i)/m\right) \ge 2b$, where b is the minimum value seen so far (the expected time until this happens is finite). Let σ denote the location where this occurs. More specifically, let

$$\begin{aligned}\nu &= \min\{i : y\left((\lceil\theta m\rceil + i)/m\right) \ge \\ &\quad 2\min_{0\le j\le i} y\left((\lceil\theta m\rceil + j)/m\right)\},\end{aligned}$$

and for typographical simplicity set

$$\begin{aligned} b &= \min_{0\le j\le \nu} y\left((\lceil\theta m\rceil + j)/m\right)\}, \\ c &= y\left((\lceil\theta m\rceil + \nu)/m\right), \\ \sigma &= (\lceil\theta m\rceil + \nu)/m; \end{aligned}$$

see Figure 2.

(ii) Starting from c, with probability $1 - b/c \ge 1/2$, the Bessel process never returns to level b. In this case, stop with $\Delta_R = b$. Otherwise, generate the time until the next visit to level b, τ_b (see below for the distribution of τ_b). If $\sigma + \tau_b \ge 1$ then stop and return $\Delta_R = b$. Otherwise start generating discrete steps at the next grid point after $\sigma + \tau_b$ and continue as at step 1.

Since starting from $c \ge 2b$ there is probability at least $1/2$ of never returning to the minimal level b, the expected number of steps simulated is bounded.

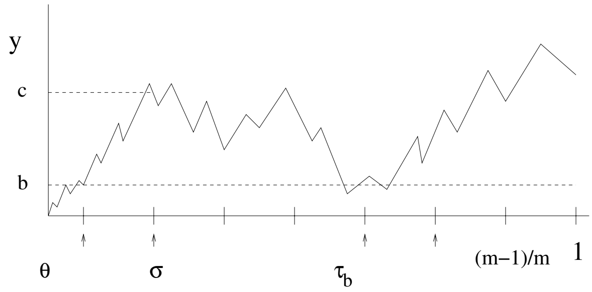

Figure 2: Hitting Times

We now describe how to generate τ_b. For a 3-dimensional Bessel process starting from $c > b > 0$,

$$P(\tau_b \le t \mid \tau_b < \infty) = 2\left(1 - \Phi\left(\frac{c-b}{\sqrt{t}}\right)\right).$$

Therefore, we can generate τ_b, conditional on it being finite, by

$$\tau_b = \left(\frac{c-b}{Z}\right)^2,$$

where $Z \sim \mathcal{N}(0,1)$.

There remains the issue of how to calculate $y(1)$. There are two cases to consider.

If $y(\sigma) = c$ and $\tau_b = +\infty$, then $y(\sigma + t) - b$ is a 3-dimensional Bessel process starting at $c - b$, and so $y(1) \overset{\mathcal{D}}{=} b + \sqrt{(1-\sigma)W}$, where W has the chi-squared distribution with 3 degrees of freedom.

If $1 < \sigma + \tau_b < \infty$, then $\{y(\sigma + t) - b : 0 \le t \le \tau_b\}$ is a 3-dimensional Bessel bridge from $c - b$ at time 0 to 0 at time τ_b (see Figure 3), and $y(1)$ is calculated accordingly (see the next section).

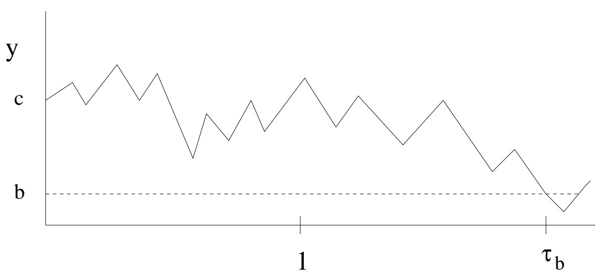

Figure 3: Bessel Bridge

The calculations to the left of θ are the same, except that we start with a “minimal value” of Δ_R instead of 0.

The procedure we have described requires the generation of random variates associated with the 3-dimensional Bessel process and its bridges. One way to do this is to make use of the fact that the 3-dimensional Bessel process has the distribution of the modulus of a 3-dimensional Brownian motion.

Let Y be a 3-dimensional Bessel process and let (B_1, B_2, B_3) be a 3-dimensional Brownian motion. To generate $Y(t+s)$ given $Y(t) = y$, generate

$$\left((y + B_1(s))^2 + B_2(s)^2 + B_3(s)^2\right)^{1/2}.$$

The same approach is used to generate the bridge random variates. Note that generating steps of the Bessel process with this scheme requires more than three times as much work as generating steps of the Brownian motion.

4 NUMERICAL EXPERIMENTS

Recall the lookback call option mentioned in the Introduction: We are interested in estimating the expectation

$$S(0)E\left(\exp(X(T)) - \exp\left(\min_{0\le i\le m} X(iT/m)\right)\right).$$

The straightforward method of simulating the embedded random walk requires the generation of m Gaussian random variates for each replication.

We performed computer experiments to get an idea of how the running time of the algorithm described in this paper increases with m. Table 1 shows the average number of Gaussian random variates generated for various values of m (and the sample standard deviations), based on 100, 000 replications.

Table 1: Average Number of Variates Generated

m	30	100	1000	10,000	100,000
# variates	25	27	29	30	30
std. dev.	9.5	11.5	13.3	14.0	14.0

The number of Gaussian variates generated approximates the actual running time of the computer program. Notice that 30 Gaussian variates corresponds to simulating the random walk at about 10 time points (since the Bessel process is simulated by a 3-dimensional Brownian motion).

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