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A NEW APPROACH TO PRICING AMERICAN-STYLE DERIVATIVES

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ABSTRACT

This paper presents a new approach to pricing American-style derivatives. By approximating the value function with a piecewise linear interpolation function, the option holder's continuation value can be expressed as a summation of European call option values. Thus the pricing of an American option written on a single underlying asset can be converted to the pricing of a series of European call options. We provide two examples of American-style options where this approximation technique yields both upper and lower bounds on the true option price.

1 INTRODUCTION

We consider the problem of pricing American-style derivatives written on a single underlying asset. One general approach in the pricing of such options with "early-exercise" features is to cast the problem in the framework of a stochastic dynamic programming problem and employ a backwards induction algorithm. As is well known, due to the "curse of dimensionality", solving the dynamic programming equations directly can become prohibitively complex and often we need to resort to approximate solutions; see, for example, the methods of Tsitsiklis and Van Roy (2000), Longstaff and Schwartz (2001), and Carriere (1996). In this paper, we present another approach to approximating the dynamic programming equations.

Our approach is to approximate the holding value function by integrating a piecewise linear approximation of the next stage value function. Here we provide the details of using secant lines for the value function. In addition, it is possible in some cases to construct the piecewise linear function with tangent lines; the details of this procedure, as well as the proofs for all of the propositions, can be found in Laprise et al. (2001).

Our contribution is as follows. By approximating the value function using a piecewise linear function, we show that it can be expressed arbitrarily well as a finite sum of European call option payoffs. This enables us to reduce the pricing of an American-style option to that of pricing European call option values. In some settings, European call option values can be determined analytically; otherwise, they can be determined via some numerical method, e.g., simulation. Also, in some cases, it can be shown that the algorithm results in price estimates that bound the correct prices. Further, under certain conditions, as the number of interpolation points goes to infinity, the price estimates converge to the true price.

Related work applying simulation to the pricing of American-style options includes Grant, Vora and Weeks (1996), Tilley (1993), Fu and Hu (1995) Broadie and Glasserman (1997ab), and Fu et al. (2001). Broadie and Detemple (1996) also develop lower and upper bounds on the prices of standard American call and put options written on a single underlying dividend-paying asset.

The rest of the paper is organized as follows. In Section 2, we present the backwards recursion algorithm with the secant interpolation to the value function. Also, we establish criteria for which the approximated value functions result in bounds on the true value functions and present heuristic arguments for the optimal selection of the interpolating points. In Section 3, we apply the algorithm to two pricing problems: an American call option and an American put option. Finally, Section 4 contains some numerical results and Section 5 offers some conclusions.

2 AMERICAN-STYLE OPTION BACKWARDS RECURSION

Consider an American style option written on a single underlying asset with a time homogeneous, Markovian price process (time homogeneity can be relaxed) given by

$$S_{t+\Delta} = h\left(Z; S_t, \theta\right),$$

where θ is a vector of parameters including the riskfree interest rate r and Z is some random vector independent of S_t and θ. Given the asset price at time $t_0 = 0$, S_0, the price of an American-style option can be written as the solution to the following optimal stopping problem:

$$\sup_{\eta} E^{Q}\left[e^{-r\eta} L_{\eta}(S_{\eta})\,|S_0\right], \qquad (1)$$

where Q denotes the appropriate risk-neutral (martingale) measure, $L_t(\cdot)$ represents the payoff at time t (we assume the payoff is only a function of the present asset price), and the supremum is over all stopping times $\eta \in (t_0, T]$ (Henceforth, for ease of notation, we drop the superscript Q on the expectation, but maintain that all subsequent expectations are taken with respect to this measure). Here, we restrict early exercise opportunities to discrete points $\{t_i, i = 1, \ldots, N\}$, where $t_N = T$ represents the option's expiration date; thus, the "sup" operator in (1) can be replaced by a "max" operator. Without loss of generality, we assume a fixed time span τ between exercise dates (τ is written as a fraction of a year), and assume the payoff function is independent of the exercise date - in which case we can drop the subscript on $L_t(\cdot)$.

If we let $V_i(S)$ represent the option value at date t_i as a function of the underlying asset price S, then we can express $V_i(S)$ as the maximum of the option's holding value and exercise value: $V_i(S) = \max(L(S), H_i(S))$, where the holding value, $H_i(S)$, is the present value of the expected one period ahead option value: $H_i(S) = e^{-r\tau} E[V_{i+1}(S_{i+1})\,|S_i = S]$, i.e.,

$$V_i(S) = \max\left(L(S), e^{-r\tau} E\left[V_{i+1}(S_{i+1})\,|S_i = S\right]\right). \quad (2)$$

In particular, at the option's expiration date t_N, as the holding value is zero, we have that $V_N(S) = L(S)$. Further, as the option cannot be exercised at t_0, the option price (1) can be expressed as

$$V_0(S_0) = H_0(S_0) = e^{-r\tau} E[V_1(S_1)\,|S_0\,]$$

Ideally, backwards recursion could be done on (2) and eventually the option price $V_0(S_0)$ could be obtained. However, prior to the expiration date, it is generally impossible to obtain the value function $V_i(S)$ over the entire state space domain; yet this is necessary to calculate the holding value at the previous exercise date. In our approach, we compute the value function at a selected finite number of points in the asset space and then use these points to construct an interpolation function which approximates the value function over the entire state space. We then perform the backwards recursion on this new function, rather than on the value function itself. This interpolation function, which is a piecewise linear function comprised of secant lines, can be conveniently expressed as a summation of European call option payoffs. Therefore, the approximated holding value, as an expectation of this interpolation function, is simply a summation of European call option prices, which are generally straightforward to obtain. We now present the details.

At exercise date t_{N-1},

$$\begin{aligned} H_{N-1}(S) &= e^{-r\tau} E\left[V_N(S_N)\,|S_{N-1} = S\right] \\ &= e^{-r\tau} E\left[L(S_N)\,|S_{N-1} = S\right]. \end{aligned}$$

Thus $H_{N-1}(S)$ is the value of a corresponding European option of length τ, with starting asset price at t_{N-1} equal to S and payoff $L(S_N)$ at t_N. European options such as this can generally be easily evaluated through either a closed form expression or via other methods, such as simulation. Then $V_{N-1}(S) = \max(L(S), H_{N-1}(S))$.

Let $\tilde{V}_{N-1}(\cdot) = V_{N-1}(\cdot)$. Proceeding recursively, at exercise date t_i, $i = N-1, \ldots, 1$, given the value function $\tilde{V}_i(\cdot)$, we construct the interpolation function $\hat{V}_i(\cdot)$: First, we choose $n+1$ points $\{(x_j, y_j)\}_{j=0}^{n}$ on the curve $\tilde{V}_i(\cdot)$ such that $x_0 < x_1 < \cdots < x_n$, and, for $j = 0, \ldots, n$, $y_j = \tilde{V}_i(x_j)$. Generally x_0 is the leftmost endpoint of the domain of $\tilde{V}_i(\cdot)$ (usually, $x_0 = 0$). Similarly, if the domain space is bounded, x_n is generally the rightmost endpoint; otherwise, x_n is a chosen large value of the domain space. Then for $S \in [x_0, x_n]$, the interpolation function is

$$\hat{V}_i(S) = m_j(S - x_{j-1}) + y_{j-1} \text{ if } x_{j-1} \le S < x_j, \quad j = 1, \ldots, n, \qquad (3)$$

where $m_j = \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$ is the slope of the secant line from (x_{j-1}, y_{j-1}) to (x_j, y_j). For $x > x_n$, we let $m_{n+1}(S - x_n) + y_n$ define the limiting line with left endpoint (x_n, y_n), where the slope m_{n+1}, while unconstrained, should be chosen with regard to the right hand limit of $\tilde{V}_i(S)$. If $m_{n+1} < 0$, this limiting line intersects the S-axis at $x_n - \frac{y_n}{m_{n+1}} > x_n$. In this case, we let $x_{n+1} = x_n - \frac{y_n}{m_{n+1}}$ and we consider the S-axis as the "new" limiting line for $x > x_{n+1}$; i.e., $m_{n+2} = 0$. Otherwise, if $m_{n+1} \geq 0$, we let $x_{n+1} = \infty$. Thus, for $S > x_n$, we have

$$\hat{V}_i(S) = \begin{cases} m_{n+1}(S - x_n) + y_n & \text{if } x_n \leq S < x_{n+1}; \\ 0 & \text{if } S \geq x_{n+1}. \end{cases} \tag{4}$$

Therefore, by (3) and (4), for $S \geq x_0$,

$$\hat{V}_i(S) = \sum_{j=1}^{n+1} \left(m_j (S - x_{j-1}) + y_{j-1} \right) \mathbf{1} \left\{ x_{j-1} \leq x < x_j \right\},$$

where $\mathbf{1}\{\cdot\}$ represents the indicator function. As $\mathbf{1}\left\{x_{j-1} \leq S < x_j\right\} = \mathbf{1}\left\{S \geq x_{j-1}\right\} - \mathbf{1}\left\{S \geq x_j\right\}$,

$$\begin{aligned}
&\hat{V}_i(S) \\
&= \sum_{j=1}^{n+1} \left(m_j (S - x_{j-1}) + y_{j-1} \right) \mathbf{1} \left\{ x_{j-1} \leq S < x_j \right\} \\
&= \sum_{j=1}^{n+1} \left(m_j (S - x_{j-1}) + y_{j-1} \right) \mathbf{1} \left\{ S \geq x_{j-1} \right\} \\
&\quad - \sum_{j=1}^{n+1} \left(m_j (S - x_{j-1}) + y_{j-1} \right] \mathbf{1} \left\{ S \geq x_j \right\} \\
&= (m_1(S - x_0) + y_0) \mathbf{1} \{ S \geq x_0 \} \\
&\quad + \sum_{j=2}^{n+1} \left(m_j (S - x_{j-1}) + y_{j-1} \right) \mathbf{1} \left\{ S \geq x_{j-1} \right\} \\
&\quad - [(m_{n+1}(S - x_n) + y_n) \mathbf{1} \{ S \geq x_{n+1} \} \\
&\quad + \sum_{j=1}^{n} \left(m_j (S - x_{j-1}) + y_{j-1} \right) \mathbf{1} \left\{ S \geq x_j \right\}] \\
&= (m_1(S - x_0) + y_0) \\
&\quad - (m_{n+1}(S - x_n) + y_n) \mathbf{1} \{ S \geq x_{n+1} \} \\
&\quad + \sum_{j=1}^{n} \left(m_{j+1} (S - x_j) + y_j \right) \mathbf{1} \left\{ S \geq x_j \right\} \\
&\quad - \sum_{j=1}^{n} \left(m_j (S - x_j) + y_j \right) \mathbf{1} \left\{ S \geq x_j \right\} \\
&= (m_1(S - x_0) + y_0) \\
&\quad - (m_{n+1}(S - x_n) + y_n) \mathbf{1} \{ S \geq x_{n+1} \} \\
&\quad + \sum_{j=1}^{n} \left(m_{j+1} - m_j \right) \left(S - x_j \right)^+,
\end{aligned} \tag{5}$$

where for the fourth equality, $\mathbf{1}\{S \geq x_0\} = 1$, the first summation results from a reindexing, and the second summation from the fact that $m_j(S - x_{j-1}) + y_{j-1}$ and $m_j(S - x_j) + y_j$ define the same line. The final equality results from $(S - x_j)\mathbf{1}\left\{S \geq x_j\right\} = (S - x_j)^+$.

Now for $x_{n+1} < \infty$, i.e., $x_{n+1} = x_n - \frac{y_n}{m_{n+1}}$ and $m_{n+2} = 0$,

$$\begin{aligned}
&-(m_{n+1}(S - x_n) + y_n) \\
&\quad = -(m_{n+1}(S - x_{n+1}) + m_{n+1}(x_{n+1} - x_n) + y_n) \\
&\quad = -(m_{n+1}(S - x_{n+1}) + (-y_n) + y_n) \\
&\quad = (m_{n+2} - m_{n+1})(S - x_{n+1}),
\end{aligned}$$

and for $x_{n+1} = \infty$, $\mathbf{1}\{S \geq x_{n+1}\} = 0$. Thus, by (5),

$$\begin{aligned}
\hat{V}_i(S) &= m_1(S - x_0) + y_0 \\
&\quad + \sum_{j=1}^{n+1} \left(m_{j+1} - m_j \right) \left(S - x_j \right)^+,
\end{aligned} \tag{6}$$

where if $x_{n+1} = \infty$, the last term in the summation is zero since $(S - x_{n+1})^+ = 0$. Thus, the approximated value function $\hat{V}_i(S)$ consists of a linear function of S and the payoff from holding a portfolio of European call options of varying strike prices $\left\{x_j\right\}_{j=1}^n$, all expiring at t_i.

We now define the *approximate* holding value function $\tilde{H}_{i-1}(\cdot)$ as the present value of the expected one period ahead piecewise linear option value $\hat{V}_i(\cdot)$. First, we introduce new notation: we include superscripts on m_j, x_j, and y_j to indicate that these values are taken from the approximate value function $\tilde{V}_i(\cdot)$ at t_i, and, as the number of interpolation points can vary per early exercise date, we include a subscript on n. Thus, from (6),

$$\begin{aligned}
&\tilde{H}_{i-1}(S) \\
&\quad = e^{-r\tau} E\left[\hat{V}_i(S_i) \,|S_{i-1} = S \right] \\
&\quad = e^{-r\tau} \left(m_1^{(i)} \left(E\left[S_i \,|S_{i-1} = S \right] - x_0^{(i)} \right) + y_0^{(i)} \right) \\
&\quad\quad + \sum_{j=1}^{n_i+1} \left(m_{j+1}^{(i)} - m_j^{(i)} \right) V^E(S, x_j^{(i)}, \tau),
\end{aligned} \tag{7}$$

where

$$V^E(S, x, \eta) = e^{-r\eta} E\left[(S_i - x)^+ \,|S_{i-1} = S \right]$$

represents the value of a European call option of maturity η ($\eta = t_i - t_{i-1}$) with starting value S and strike price x. Thus, the approximated holding value $\tilde{H}_{i-1}(\cdot)$ is simply a linear function of the expected asset value at t_i added to a summation of European call option values. $V^E(S, x, \eta)$ can be evaluated either in a closed-form expression, as when the process follows geometric Brownian motion, or via simulation or another numerical method.

Finally, for $i \geq 2$, $\tilde{V}_{i-1}(\cdot)$ is defined as the maximum of the approximated holding value and the exercise value:

$$\tilde{V}_{i-1}(S) = \max(L(S), \tilde{H}_{i-1}(S)), \tag{8}$$

and the recursion continues. We will assume no early exercise at t_0, in which case $\tilde{V}_0(S_0) = \tilde{H}_0(S_0)$ is an estimate for the option's value; if early exercise is allowed at t_0, $\tilde{V}_0(S_0)$ is given by (8) with $i = 1$.

The following steps summarize the backwards recursion algorithm discussed above. Throughout, if $i < N-1$, $\tilde{H}_i(\cdot)$ is calculated via (7).

Algorithm

0: Let $i = N-1$, $\tilde{H}_{N-1}(\cdot) = H_{N-1}(\cdot)$ and $\tilde{V}_{N-1}(\cdot) = V_{N-1}(\cdot)$

1: Choose interpolating points: $\left\{\left(x_j^{(i)}, y_j^{(i)}\right)\right\}_{j=0}^{n_i}$, where, for $j = 1, \ldots, n_i$, $x_{j-1}^{(i)} < x_j^{(i)}$, and, for $j = 0, \ldots, n_i$,

$$y_j^{(i)} = \tilde{V}_i(x_j^{(i)}) = \max(L(x_j^{(i)}), \tilde{H}_i(x_j^{(i)})).$$

2: For $j = 1, \ldots, n_i$, calculate $m_j^{(i)} = \frac{y_j^{(i)} - y_{j-1}^{(i)}}{x_j^{(i)} - x_{j-1}^{(i)}}$.

3: Choose $m_{n_i+1}^{(i)}$. If $m_{n_i+1}^{(i)} < 0$, let $x_{n_i+1} = x_{n_i}^{(i)} - \frac{y_{n_i}^{(i)}}{m_{n_i+1}^{(i)}}$, and $m_{n_i+2}^{(i)} = 0$.

4: Let $i = i-1$. If $i > 0$, return to Step 1. Otherwise, return $\tilde{V}_0(S_0)$.

The interpolation of the value function could begin at the expiration date t_N where $V_N(S) = L(S)$. This may be beneficial if the determination of a sequence of European call option prices is easier than finding the price of a single European option with payoff $L(S)$; for example, if $L(S)$ is a complicated function.

2.1 Criteria for Upper and Lower Bounds

Consider the general backwards recursion algorithm in solving an American style option problem. We use the same notation as above: at exercise date t_i, $H_i(S) = e^{-r\tau} E\left[V_{i+1}(S_{i+1})\,|S_i = S\right]$ and $V_i(S) = \max(L(\cdot), H_i(\cdot))$ represent the true holding value and option value functions, respectively, and $\tilde{H}_i$ and $\tilde{V}_i(\cdot) = \max(L(\cdot), \tilde{H}_i(\cdot))$ represent the approximate holding value and option value functions, respectively. $\hat{V}_i$ is defined as any approximating function (e.g., the secant interpolation function defined above) to $\tilde{V}_i$, so that $\tilde{H}_i(S) = e^{-r\tau} E\left[\hat{V}_{i+1}(S_{i+1})\,|S_i = S\right]$.

At t_{N-1}, let $\tilde{H}_{N-1} \geq H_{N-1}$, so that, by definition, $\tilde{V}_{N-1} \geq V_{N-1}$ (generally, these functions equal, but for now we allow the possibility of the inequality). Next, suppose $\hat{V}_{N-1}$ is constructed such that $\hat{V}_{N-1} \geq \tilde{V}_{N-1}$. Then, as $\hat{V}_{N-1} \geq V_{N-1}$, we have

$$\begin{aligned}\tilde{H}_{N-2}(S) &= e^{-r\tau} E\left[\hat{V}_{N-1}(S_{N-1})\,|S_{N-2} = S\right] \\ &\geq e^{-r\tau} E\left[V_{N-1}(S_{N-1})\,|S_{N-2} = S\right] \\ &= H_{N-2}(S),\end{aligned}$$

which implies, $\tilde{V}_{N-2}(\cdot) \geq V_{N-2}(\cdot)$. Proceeding recursively, if, at exercise date t_{i+1}, $\tilde{V}_{i+1}(\cdot) \geq V_{i+1}(\cdot)$, and $\hat{V}_{i+1}$ is constructed such that $\hat{V}_{i+1} \geq \tilde{V}_{i+1}$, then $\tilde{H}_i(\cdot) \geq H_i(\cdot)$ and $\tilde{V}_i(\cdot) \geq V_i(\cdot)$, similarly. Therefore, constructing $\hat{V}_i$ as an upper bound to $\tilde{V}_i$ at all early exercise dates results in upper bounds on the true holding and value functions. This argument could be repeated with the inequalities reversed to show that constructing $\hat{V}_i$ as a *lower bound* to $\tilde{V}_i$ at all early exercise dates results in *lower bounds* on the true holding and value functions.

In our application, where $\hat{V}_i(\cdot)$ interpolates $\tilde{V}_i(\cdot)$ with secant lines, if $\tilde{V}_i(\cdot)$ is a convex function, then $\hat{V}_i(S) \geq \tilde{V}_i(S)$ for $S \leq x_n^{(i)}$; and if the limiting secant line with slope $m_{n+1}^{(i)}$ is carefully chosen such that $\hat{V}_i(S) \geq \tilde{V}_i(S)$ for $S > x_n^{(i)}$, we have that $\hat{V}_i(\cdot) \geq \tilde{V}_i(\cdot)$. Similarly, if $\tilde{V}_i(\cdot)$ is a concave function, then $\hat{V}_i(S) \leq \tilde{V}_i(S)$ for $S \leq x_n^{(i)}$; and if the limiting secant line with slope $m_{n+1}^{(i)}$ is chosen such that $\hat{V}_i(S) \leq \tilde{V}_i(S)$ for $S > x_n^{(i)}$, we have that $\hat{V}_i(\cdot) \leq \tilde{V}_i(\cdot)$. The following proposition, which we will use in our examples, provides conditions under which the approximating value function, $\tilde{V}_i(\cdot)$, is convex.

Proposition 2.1 *Suppose $L(\cdot)$ is convex. If either $L(\cdot)$ is nondecreasing and $h(Z; \cdot, \theta)$ is convex or $L(\cdot)$ is nonincreasing and $h(Z; \cdot, \theta)$ is concave, then $H_{N-1}(\cdot)$ and $V_{N-1}(\cdot)$ are convex. For $i = 1, \ldots, N-2$, if $\tilde{H}_{i+1}(\cdot)$ and $\tilde{V}_{i+1}(\cdot)$ are convex, $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$, and either $h(Z; \cdot, \theta)$ is linear, or $h(Z; \cdot, \theta)$ is convex and $m_1^{(i+1)} \geq 0$, then $\tilde{H}_i(\cdot)$ and $\tilde{V}_i(\cdot)$ are convex.*

2.2 Efficient Selection of Interpolating Points

The accuracy of the backwards recursion algorithm with the secant interpolation function is inherently dependent on the error in replacing the current value function with the approximating function. In particular, at early exercise date t_i, it is desirable for $\left|\hat{V}_i(\cdot) - \tilde{V}_i(\cdot)\right|$ to be small, and if we are limited to a fixed number of interpolation points, it is important that the interpolation points be chosen so as to minimize this error.

First, the linear interpolation should try to focus on those intervals in the state space where $\tilde{V}_i$ is nonlinear. In other words, if $\tilde{V}_i$ is known to be linear on some interval $[a, b]$, then for some j', we would let $x_{j'} = a$ and $x_{j'+1} = b$ (if $b = \infty$, we would let $x_n = a$ and m_{n+1} equal the slope of $\tilde{V}_i(x)$ for $x > a$; then for $x > a$, $\tilde{V}_i(x) = \hat{V}_i(x)$).

Further, the interpolation points should be concentrated on the areas of the state space where $\tilde{V}(\cdot)$ is most convex or concave. One simple heuristic where the interpolation points are chosen iteratively is as follows. Given a current set of points and the corresponding secant lines, additional interpolation points are inserted into those areas where the absolute difference between the slopes of adjacent secant lines is large, as these areas should correspond to areas of higher convexity. A more rigorously defined heuristic based on this idea is described in Laprise et al. (2001).

Next, as our algorithm essentially reduces the pricing of an American-style option to that of pricing numerous European call options, the computational costs are directly related to the computational costs of pricing the European call options. As each calculation of the approximate holding value (7) requires the determination of a sequence of European call option values of varying strike prices, the number of European call options that require pricing can be large. If we do not have closed-form solutions for these values, and they need to be estimated through a numerical method such as simulation, computation costs can grow very quickly.

In such cases, the total number of European call option prices required can be reduced by reusing a set of state space interpolation points over all early exercise dates; i.e., if $X_i = \left(x_j^{(i)}\right)_{j=0}^{n_i}$, then let $X_{N-1} \subseteq X_{N-2} \subseteq \cdots \subseteq X_1$. For example, suppose we let the set of state space interpolation points be the same for each early exercise date; i.e., we define $X = \left\{x_j\right\}_{j=0}^{n}$ where $X = X_{N-1} = \cdots = X_1$ (as the state space interpolation points are identical across dates, we drop the superscripts and subscripts). Then, referring to the algorithm in Section 2, at t_{N-1}, we compute $y_j^{(N-1)} = \max(L(x_j), H_{N-1}(x_j))$ for $j = 0, \ldots, n$. We then construct the two dimensional array $\bar{X}$ as follows: $\bar{X}_{j,k} = V^E(x_j, x_k, \tau)$ for $j, k = 0, \ldots, n$; i.e., $\bar{X}_{j,k}$ is the value of a European call option of length τ, with starting value x_j and strike price x_k. Then, at early exercise date t_i $(i < N-1)$, as $X_i = X$, $y_k^{(i)} = \max(L(x_k), \tilde{H}_i(x_k))$ for $k = 0, \ldots, n$, where, by (7),

$$
\begin{aligned}
&\tilde{H}_i(x_k) \\
&\quad = e^{-r\tau}\left(m_1^{(i+1)}\left(E\left[S_{i+1}\,|S_i = x_k\right] - x_0\right) + y_0^{(i+1)}\right) \\
&\qquad + \sum_{j=1}^{n+1}\left(m_{j+1}^{(i+1)} - m_j^{(i+1)}\right) V^E(x_k, x_j, \tau).
\end{aligned}
$$

Thus, $\tilde{H}_i(x_k)$ can be determined directly from the kth row of $\bar{X}$ - no further European call values need to be computed. Therefore, once the n^2 European values in $\bar{X}$ are computed initially, the backwards recursion can proceed until t_0 without computing any further European values. In practice, the user may want to add interpolation points at some early exercise date t_i, in which case $X_{i+1} \subset X_i$. Then $\bar{X}$ would need to be updated to include these new interpolating points. However, in total, only n_1^2 European values would need to be computed for the recursion to be completed.

Finally, when the European values need to be estimated via some numerical method, care must be shown in choosing the asset space interpolation points to avoid introducing huge errors. In particular, increasing the number of interpolating points can produce less accurate results if the accuracy of the numerical method is not also improved. As an illustration, consider the backwards recursion at an early exercise date, and let x_{j-1} and x_j be adjacent state space interpolation points (we drop the superscript notation). Let y_{j-1} and y_j be the respective true values of the approximated value function and let $\bar{y}_{j-1}$ and $\bar{y}_j$ be the corresponding values where numerical methods are used to estimate the European values. Define ϵ_{j-1} and ϵ_j as the respective errors resulting from the estimation of the European values, i.e., $\bar{y}_{j-1} = y_{j-1} + \epsilon_{j-1}$ and $\bar{y}_j = y_j + \epsilon_j$. Further, let $\bar{m}_j$ be the slope of the secant line between $\left(x_{j-1}, \bar{y}_{j-1}\right)$ and $\left(x_j, \bar{y}_j\right)$. Then

$$
\begin{aligned}
\bar{m}_j &= \frac{\bar{y}_j - \bar{y}_{j-1}}{x_j - x_{j-1}} \\
&= \frac{(y_j + \epsilon_j) - (y_{j-1} + \epsilon_{j-1})}{x_j - x_{j-1}} \\
&= m_j + \frac{\epsilon_j - \epsilon_{j-1}}{x_j - x_{j-1}},
\end{aligned}
$$

i.e., the error in the slope of the secant line is amplified by $\frac{1}{x_j - x_{j-1}}$. Therefore, if using numerical methods, the state space interpolation points cannot be chosen too close. In particular, if the number of interpolation points are increased, which generally will decrease the distance between adjacent points, the accuracy of the numerical method must also increase. In the context of simulation, increasing the number of interpolation points while maintaining the number of replications used for estimating each European price may result in a less accurate final option price.

3 EXAMPLES

We consider two American style-pricing problems: the American call option and the American put option. Under relatively nonrestrictive conditions, the application of the secant interpolation to these problems results in upper bounds on the true option price. Further, the techniques seen here can generally be applied to more complicated pricing problems.

Example 1: American Call Option

In this case, $L(S) = (S-K)^+$. First, the holding value at the latest early exercise date is simply the value of a European call option:

$$\begin{aligned} H_{N-1}(S) &= e^{-r\tau} E\left[(S-K)^+ \,|S_{N-1} = S\right] \\ &= V^E(S, K, \tau). \end{aligned} \tag{9}$$

Therefore, the European call option is the only option that needs pricing in applying the backwards recursion to this problem. Next, the following property of the American call option helps us achieve stronger results.

Proposition 3.1 *For $i = 1, \ldots, N-1$, if $\frac{\partial H_i(S)}{\partial S} < 1$, then the optimal early exercise policy at t_i is a threshold policy: there exists an $s_i^* > K$ such that $L(s_i^*) = H_i(s_i^*)$, $L(S) < H_i(S)$ for $S < s_i^*$, and $L(S) > H_i(S)$ for $S > s_i^*$, i.e.,*

$$V_i(S) = \begin{cases} H_i(S) & \text{if } S < s_i^*; \\ L(S) = S - K & \text{if } S \geq s_i^*. \end{cases} \tag{10}$$

Further, if $\frac{\partial H_i(S)}{\partial S} \leq \rho < 1$, $s_i^* < \infty$.

Thus, given that the optimal policy at t_i is a threshold policy, if the threshold is finite, the option should only be exercised if $S \geq s_i^*$; otherwise, if $s_i^* = \infty$, i.e., $H_i(\cdot) \geq L(\cdot)$, the option should never be exercised at t_i. The condition, $\frac{\partial H_i(S)}{\partial S} < 1$, is generally satisfied by a smoothness condition on the stock price process $h\,(Z;\cdot,\theta)$. Further, the condition for finite thresholds, $\frac{\partial H_i(S)}{\partial S} \leq \rho < 1$, is generally satisfied for any smooth stock price process with continuous dividends.

The next proposition shows that under some conditions on the price process $h\,(Z;\cdot,\theta)$ that ensure that the optimal policy is a threshold policy, if we apply secant interpolation to the value functions, the estimated optimal policy based on the approximated value function $\tilde{V}_i(\cdot)$ is also a threshold policy. Furthermore, the approximated value functions and thresholds bound the true value functions and thresholds, respectively.

Proposition 3.2 *Assume $h\,(Z;\cdot,\theta)$ is sufficiently smooth such that the optimal policy at each early exercise date is a threshold policy. Also, let $h\,(Z;\cdot,\theta)$ be convex. Then, for $i = 0, \ldots, N-2$, if $m^{(j)}_{n_j+1} = 1$ for $j = i+1, \ldots, N-1$, then*

$$\tilde{V}_i(S) = \begin{cases} \tilde{H}_i(S) & \text{if } S < \tilde{s}_i^*; \\ L(S) = S - K & \text{if } S \geq \tilde{s}_i^* \end{cases}$$

(as early exercise is not allowed at t_0, $\tilde{s}_0^* = s_0^*$ is taken to be infinity). Further, $\tilde{V}_i(\cdot) \geq V_i(\cdot)$, $\tilde{H}_i(\cdot) \geq H_i(\cdot)$, and $\tilde{s}_i^* \geq s_i^*$.

Note: $\tilde{V}_{N-1}(\cdot) = V_{N-1}(\cdot)$, $\tilde{H}_{N-1}(\cdot) = H_{N-1}(\cdot)$, and $\tilde{s}_{N-1}^* = s_{N-1}^*$, where $H_{N-1}(\cdot)$ is given in (9), and $V_{N-1}(\cdot)$ and s_{N-1}^* are given in (10).

Example 2: American Put Option

In this case, $L(S) = (K-S)^+$. For simplicity, we assume the stock price process is free of dividends, in which case $e^{-r\tau} E\left[S_{i+1} \,|S_i = S\right] = S$ by the martingale condition. The holding value at t_{N-1} is the value of a European put option, and as $a^+ = (-a)^+ + a$, we have:

$$\begin{aligned} &H_{N-1}(S) \\ &= e^{-r\tau} E\left[(K - S_N)^+ \,|S_{N-1} = S\right] \\ &= e^{-r\tau} E\left[(S_N - K)^+ + K - S_N \,|S_{N-1} = S\right] \\ &= K e^{-r\tau} - S + V^E(S, K, \tau) \end{aligned} \tag{11}$$

For this example, we show the construction of $V_{N-1}(\cdot)$, $\hat{V}_{N-1}(\cdot)$ and $\tilde{H}_{N-2}(\cdot)$. First, $L(0) = K > Ke^{-r\tau} = H_{N-1}(0)$ and $L(K) = 0 < H_{N-1}(K)$ imply the existence of an $s_{N-1}^* < K$ such $L(s_{N-1}^*) = H_{N-1}(s_{N-1}^*)$. Further,

$$\begin{aligned} \frac{\partial H_{N-1}(S)}{\partial S} &= -1 + \frac{\partial}{\partial S} V^E(S, K, \tau) \\ &> -1, \end{aligned}$$

as $\frac{\partial}{\partial S} V^E(S, K, \tau) > 0$, and $\frac{\partial}{\partial S}(K - S) = -1$ imply the uniqueness of s_{N-1}^*; i.e.,

$$V_{N-1}(S) = \begin{cases} L(S) = K - S & \text{if } S < \tilde{s}_{N-1}^*; \\ H_{N-1}(S) & \text{if } S \geq \tilde{s}_{N-1}^*. \end{cases}$$

Next, in constructing $\hat{V}_{N-1}(\cdot)$, we note that $\left(x_0^{(N-1)}, y_0^{(N-1)}\right) = (0, K)$, and, as $V_{N-1}(S)$ is linear for $S < \tilde{s}_{N-1}^*$, we let $x_1^{(N-1)} = \tilde{s}_{N-1}^*$, so that

$m_1^{(N-1)} = -1$. Thus, by (7),

$$\begin{aligned}\tilde{H}_{N-2}(S) & \\ = \quad & K e^{-r\tau} - S \qquad (12) \\ & + \sum_{j=1}^{n_{N-1}+1} \left(m_{j+1}^{(N-1)} - m_j^{(N-1)} \right) V^E(S, x_j^{(N-1)}, \tau).\end{aligned}$$

Thus the holding value at t_{N-1}, (11), and the approximate holding value at t_{N-2}, (12), are similar. Further, it can be shown that the form of the approximate holding value function at t_{N-2}, (12), is maintained at all early exercise dates.

The following proposition shows that a threshold policy is generally required for optimality. However, unlike the American call, the threshold is always finite.

Proposition 3.3 *For $i = 1, \ldots, N-1$, if $\frac{\partial H_i(S)}{\partial S} > -1$, then the optimal early exercise policy at t_i is a threshold policy: there exists an $s_i^* < K$ such that $L(s_i^*) = H_i(s_i^*)$, $L(S) > H_i(S)$ for $S < s_i^*$, and $L(S) < H_i(S)$ for $S > s_i^*$, i.e.,*

$$V_i(S) = \begin{cases} L(S) = K - S & \text{if } S < s_i^*; \\ H_i(S) & \text{if } S \geq s_i^*. \end{cases}$$

Again, the condition $\frac{\partial H_i(S)}{\partial S} > -1$ is generally satisfied by a smoothly changing stock price. Next, similar to the American call example, if the optimal policy is a threshold policy, then secant interpolation to the value functions will also result in a threshold policy, and the approximated value functions and thresholds bound the true value functions and thresholds, respectively.

Proposition 3.4 *Assume $h(Z; \cdot, \theta)$ is sufficiently smooth such that the optimal policy at each early exercise date is a threshold policy. Also, let $h(Z; \cdot, \theta)$ be linear. Then, for $i = 0, \ldots, N-2$, if $m_{n_j}^{(j)} = 0$ for $j = i+1, \ldots, N-1$, then*

$$\tilde{V}_i(S) = \begin{cases} L(S) = K - S & \text{if } S < \tilde{s}_i^*; \\ \tilde{H}_i(S) & \text{if } S \geq \tilde{s}_i^* \end{cases}$$

(as early exercise is not allowed at t_0, $\tilde{s}_0^* = s_0^*$ is taken to be zero). Further, $\tilde{V}_i(\cdot) \geq V_i(\cdot)$, $\tilde{H}_i(\cdot) \geq H_i(\cdot)$, and $\tilde{s}_i^* \leq s_i^*$.

4 NUMERICAL RESULTS

Numerical results are shown in Table 1. Table 1 shows American call option prices (Example 1) with strike price $K = 100$, where the expiration date is 3.0 yrs and the option is exercisable every 0.5 yrs. The stock price process adopted is geometric Brownian motion with continuous dividends:

$$\begin{aligned} S_{t+\Delta} &= h(Z; S_t, \theta) \\ &= S_t e^{(r-\delta-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} Z}, \end{aligned}$$

where Z is a standard $N(0, 1)$ random variable, r represents the riskfree interest rate, σ the volatility, and δ the continuous dividend rate. For Table 1, $\sigma = .2$, $r = .05$, and $\delta = .04$.

It can be shown via Proposition 3.1 that for $\delta > 0$, the optimal policy at each early exercise date is a finite threshold policy. Thus, by Proposition 3.2, our algorithm will result in upper bounds on the threshold values and option value.

In addition to option price estimates for three starting asset prices ($S_0 = 90, 100, 110$), Table 1 displays the corresponding threshold values (threshold values are independent of the starting prices) where the $t_5 = 2.5$ years threshold is omitted since it is obtained independently of the interpolation algorithms. Also, included are CPU times (in seconds): all computation was implemented in C and carried out on a Sun Ultra 10 running Solaris OS.

The first three rows of Table 1 display results obtained from a tangent interpolation of the value function; details of this are contained in Laprise et al. (2001). It can be shown that tangent interpolation leads to lower bounds on the option values and threshold values, and, as for the secant interpolation, the approximate holding values are summations of European call option prices. The European prices used here are obtained in closed form via the Black-Scholes formula; thus, no numerical method is used. Each row corresponds to a different number of interpolation points.

The second three rows display results from the secant interpolation approach; as previously discussed, the values shown are upper bounds on the true values. As for the first three rows, the European call prices are obtained via the Black-Scholes formula, and each row corresponds to a different number of interpolation points.

The last three rows also display results from the secant interpolation approach, except here, the European call option values are obtained via simulation. The displayed option prices and CPU times are an average of 10 runs of the algorithm. For efficiency, we did not attempt to determine the thresholds, and we followed the approach detailed in Section 2.2 of reusing the state space interpolation points. In particular, we selected n_5 points at $t_5 = 2.5$ years and iteratively add points until $t_1 = .5$ year where we end up with n_1 points; thus, we simulate a total of n_1^2 European call prices. Also listed is m, the number of replications used to estimate each European price.

Our experiments show that the "analytical" upper and lower bounds tighten quickly with respect to the number of interpolating points. For example, with just 50 interpolating points, the upper and lower bounds are able to bracket the true price to within 3 cents. Furthermore, with 200 points,

Table 1: American Call Option on Single Asset under Geometric Brownian Motion: $K = 100$, $r = 0.05$, $\sigma = .2$, $\delta = .04$; $t_i = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ yrs

Algorithm Type	Algorithm Parameters	Option Price $S_0 = 90$	$S_0 = 100$	$S_0 = 110$	Thresholds $t_1 = .5$	$t_2 = 1.0$	$t_3 = 1.5$	$t_4 = 2.0$	CPU
Lower	$n = 20$	8.582	13.507	19.487	158.19	153.87	148.54	141.61	0.07
Bounds	$n = 50$	8.624	13.547	19.526	158.38	154.02	148.65	141.69	0.24
(Anal)	$n = 200$	8.631	13.554	19.533	158.42	154.05	148.67	141.70	3.28
Upper	$n = 200$	8.633	13.556	19.535	158.43	154.06	148.68	141.70	2.21
Bounds	$n = 50$	8.648	13.571	19.551	158.56	154.13	148.74	141.77	0.16
(Anal)	$n = 20$	8.751	13.674	19.656	159.17	154.98	149.37	141.88	0.05
Upper	$n_5 = 15, n_1 \approx 24$	8.774	13.700	19.668					3.25
Bounds	$(m = 10,000)$								
(Sim)	$n_5 = 40, n_1 \approx 52$	8.616	13.498	19.448					12.5
	$(m = 10,000)$								
	$n_5 = 40, n_1 \approx 52$	8.702	13.584	19.571					22.5
	$(m = 20,000)$								

we are able to ascertain the true price to within less than 1 cent. The results are similarly strong for the threshold values.

Difficulties arise when we use simulation for the European values. In particular, as discussed in Section 2.2, we see the errors that can occur when increasing the interpolation points while maintaining the European price accuracy. While keeping the number of simulations fixed at 10,000, increasing the number of interpolating points at t_5 from 15 to 40 actually leads to worse results: for many of the cases, the "upper" bounds fall significantly below the true lower bounds. However, doubling the number of simulations to 20,000, hence improving the European price accuracy, leads to results that are significantly better then when the number of interpolating points at t_5 is 15. Preliminary analysis seems to show that the results can improve with more interpolation points as long as the accuracy of the numerical method also is enhanced.

5 CONCLUSIONS

We have presented a new approach to pricing American-style derivatives through approximating the value function with an interpolation function based on secant lines. With this approximation, we are able to convert the pricing of an American-style derivatives to that of pricing numerous European call options. We show how the algorithm can be applied to American put and call options, and we present numerical results on the application to the American call. For cases where analytical results for the European call are available, the numerical results show rapid convergence of the bounds to the correct price as the number of interpolation points is increased. However, when simulation is needed to estimate the European call prices, preliminary results show that the estimator accuracy must be improved when increasing the number of interpolation points.

Laprise et al. (2001) presents linear interpolation with tangent lines and contains applications to more complicated American-style options. Future work includes the possibility of applying our techniques to multi-dimensional American-style derivatives, such as Asian options.

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