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A SIMULATION-BASED CREDIT DEFAULT SWAP PRICING APPROACH UNDER JUMP-DIFFUSION

Tarja Joro

University of Alberta School of Business
3-40F Business Building
Edmonton, Alberta T6G 2R6, CANADA

Paul Na

Bayerische Landesbank New York Branch
560 Lexington Avenue
New York, NY 10022, U.S.A.

ABSTRACT

Diffusion-based Credit Default Swap (CDS) pricing models produce zero spreads for very short-term contracts, which contradict empirical data. We introduce a simulation-based CDS pricing approach that avoids the zero short-term spreads problem through a jump-diffusion process.

1 INTRODUCTION

In deriving default probabilities, there are two broad modeling approaches: structural approach (see e.g. Merton (1974), Merton (1976), Black and Cox (1976), Longstaff and Schwartz (1995)), and reduced form approach (see e.g. Duffie and Singleton (1995), Jarrow, Lando and Turnbull (1994), Jarrow. and Turnbull (1995), Madan and Unal (1994)). Zhou (1997) characterizes the two approaches as follows:

1. Structural approach proposes that the evolution of the firm's asset value follows a diffusion process, as proposed by Merton (1974). Defaults occur when the value of the asset becomes lower than the debt. According to the structural approach, firms never default by surprise due to the diffusion process, which is continuous.
2. Reduced-form approach assumes that there is no relation between the firm value and default. Default is seen as an unpredictable Poisson event involving a sudden loss in market value. Thus, according to reduced-form approach, firms never default gradually.

As Zhou (1997) argues, in reality, default can occur in both ways: firms can default either gradually or by surprise due to unforeseen external shocks. The philosophies behind the structural and reduced form approaches can be combined by using a jump diffusion model that allows both gradual and sudden defaults (see e.g. Merton (1976), Ahn and Thompson (1988), Kou (2001) and Zhou (1997, 2001)).

Deriving default probabilities is a crucial part in pricing credit derivatives whose market is growing rapidly. CDS is the most popular contract type among credit derivatives. The aim of this paper is to present a simulation-based CDS pricing approach based on jump-diffusion process and demonstrate how the approach overcomes some difficulties encountered in a traditional diffusion-based pricing approach. In particular, a CDS pricing approach based on a diffusion-process produces zero credit spreads for very short maturities. This happens because, if there is a finite distance to the default point (barrier), a continuous process cannot reach it in a very short-time period. This is problematic because in reality the credit spreads would not go to zero even for contracts with very short maturities. This paper will show that the zero short-term spreads problem can be avoided through jump-diffusion process.

The rest of the paper is organized as follows. Section two discusses deriving default probabilities under diffusion and jump-diffusion processes. Section three explains how CDS can be priced in practice. Section four introduces our approach and Section five presents a real-life numerical example. Section six concludes the paper.

2 MODELING DEFAULT PROBABILITIES

Let τ represent the time when a default occurs. When the default occurs only at the maturity T (hereafter, maturity default), τ is defined as

$$\tau = \begin{cases} T : V_T < D \\ \infty : else \end{cases} \tag{1}$$

where V_T is the asset value at T and D the default point (e.g., amount of outstanding debt). When the default can occur anytime $t \in [0,T]$ until the maturity, as the asset value crosses the default point (hereafter, first-passage-time default),

$$\tau = \min\{t > 0 : V_t \le D\}. \tag{2}$$

First, like in Merton (1974), let V follow a Geometric Brownian motion

$$dV = \mu V dt + \sigma V dw \tag{3}$$

where $\mu \in \Re$ is a constant growth rate of the asset value, $\sigma > 0$ is a constant volatility coefficient, and W is a standard Brownian motion (Wiener process) defined on a filtered probability space (Ω, F, P).

By Ito's lemma,

$$V_T = V_0 \exp\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T \tag{4}$$

is solution to Equation (3). Then, the risk-neural default probability under maturity default is

$$\begin{aligned}\tilde{P}(\tau = T) &= \tilde{P}(V_T < D) \\ &= \tilde{P}\left(V_0 \exp\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T < D\right) \\ &= \Phi\left(\frac{\ln\left(\frac{D}{V_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\end{aligned} \tag{5}$$

where r is risk-free rate and $\Phi(\bullet)$ the cumulative standard normal distribution.

The risk-neural default probability under first-passage-time default is

$$\begin{aligned}\tilde{P}(\tau \le T) &= \tilde{P}\left(\min_{t \le T}\left(V_0 \exp\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \le D\right) \\ &= 1 - \Phi\left(\frac{\left(r - \frac{\sigma^2}{2}\right)T - \ln\left(\frac{D}{V_0}\right)}{\sigma\sqrt{T}}\right) \\ &\quad + \exp\left(\frac{2\left(r - \frac{\sigma^2}{2}\right)\ln\left(\frac{D}{V_0}\right)}{\sigma^2}\right) \\ &\quad \times \Phi\left(\frac{\ln\left(\frac{D}{V_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).\end{aligned} \tag{6}$$

For the derivation, refer to Musiela and Rutkowski (1997) and Giesecke (2003).

Now, following Merton (1976), let V follow a jump-diffusion process

$$dV = (\mu - \lambda\kappa)Vdt + \sigma VdW + (J - 1)Vdp \tag{7}$$

where $\kappa = E(J - 1)$, $ln(J) \sim N(\nu, \gamma^2)$, and dp is a Poisson process generating the jumps with the intensity of λ. By Ito's lemma,

$$V_T = V_0 \exp\left(\left(\mu - \frac{\sigma^2}{2} - \lambda\kappa\right)T + \sigma W_T\right)\prod_{i=1}^{n} J_i \tag{8}$$

is the solution to Equation (7). n is Poisson distributed with parameter λT. Then, the risk-neural default probability under maturity-default is

$$\begin{aligned}&\tilde{P}(\tau = T) \\ &= \tilde{P}(V_T < D) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \\ &\quad \times \Phi\left(\frac{\ln\left(\frac{D}{V_0}\right) - \left(r - \frac{\sigma^2}{2} - \lambda\kappa\right)T - n\nu}{\sqrt{\left(\sigma^2 + \frac{n\gamma^2}{T}\right)T}}\right).\end{aligned} \tag{9}$$

A closed-form solution for the first-passage-time default probability under jump-diffusion is not known (see Abrahams (1986)) but can be determined by Monte Carlo simulation.

3 PRICING CREDIT DEFAULT SWAP

A CDS is a contract in which one party buys protection for possible losses of reference asset (for example, a bond or a loan) due to a credit event such as default by the issuer. The protection buyer pays periodic payments to the protection seller until either the maturity of the contract or a credit event occurs. Upon the credit event, the seller pays the loss incurred by the credit event to the buyer and the buyer usually makes a final accrual fee payment to the seller. A CDS can be priced by equating the sum of present values of the fee leg to the sum of present values of the contingent payment leg.

Suppose that the CDS rate S as a fraction of notional in Basis Point (BP, 1% = 100 BP) per year is paid at dates $t_1 < t_2 < \ldots < t_n = T$ with $\Delta(t_{i-1}, t_i)$ representing the interval between payments dates (i.e., 0.5 for semi-annually payments). The sum of present values of fee leg, F, can be written as

$$F = S\sum_{i=1}^{n} \Delta(t_{i-1}, t_i)(1 - \widetilde{p}(t_i))e^{-rt_i} = SL. \qquad (10)$$

The sum of present values of fee accruals, A, can be defined as

$$A = S\sum_{i=1}^{n} \frac{\Delta(t_{i-1}, t_i)}{2}(\widetilde{p}(t_i) - \widetilde{p}(t_{i-1}))e^{-rt_i} = SM \qquad (11)$$

where it is assumed that the default between the regular fee payments always occurs exactly in the middle. The error from this approximation gets smaller as the time step gets smaller.

The contingent leg payoff, C, can be described as

$$C = (1 - R)\sum_{i=1}^{n} (\widetilde{p}(t_i) - \widetilde{p}(t_{i-1}))e^{-rt_i}. \qquad (12)$$

Then,

$$C = F + A = SL + SM = S(L + M). \qquad (13)$$

Therefore,

$$S = \frac{C}{(L + M)}. \qquad (14)$$

4 A SIMULATION-BASED APPROACH TO PRICE CDS

As mentioned in Section 2, a closed-form solution for the first-passage-time default probability under jump-diffusion is not known. However, it is relatively straightforward to compute the default probability by simulation. In addition, simulation brings greater pricing accuracy because we can easily discount contingent payments and the accrual fees from the exact date when the default actually occurs, and there is no need for the relatively crude approximations used in Equations (11) and (12).

It was also mentioned that high short-term spreads cannot be explained by using diffusion-process alone. In the following computational example, we would like to demonstrate that the jump diffusion process could indeed produce high short-term spreads that appear in the markets.

In our approach, daily asset prices are simulated with Formulation (8). The example presented in this paper is implemented with Microsoft Excel and @Risk simulation add-in. The simulation of the diffusion process is straightforward in a spreadsheet environment, however simulating the jumps requires some relatively advanced modeling work. With one-day time step, even a one-year model becomes computationally rather expensive. However, a spreadsheet environment is ideal for prototyping and communicating the model, even if the final implementation ends up being done in some other platform.

5 ILLUSTRATIVE SIMULATIONS

A Fortune 500 company currently has an asset value of \$285 Billion, and a default is assumed to occur when the asset value dips under \$201 Billion. The company's asset sigma is estimated to be 6%. The one-year risk-free rate is known to be 1%. In May 2003, a one-year CDS for the company is traded around 300 BP.

We built a simulation model to price the one-year CDS. We assumed quarterly payment periods, and the recovery rate of 50%. In the diffusion model (without jumps), the company never defaulted (n = 50,000), leading to zero-spread for the CDS, which contradicts the market information.

However, in the jump-diffusion model, the results were very different. For example, with a jump frequency of five jumps per year (Poisson arrival process) and jump amplitude of $ln(J) \sim N(0, 0.0054)$, the spread was around 289 BP, with default probability of 5.64% (n = 50,000). These figures are consistent with the market information.

We want to emphasize that estimating the parameters for a jump diffusion process is a non-trivial endeavor (see, for example, Kou (2001)). In this computational example, the parameters were chosen so that the process yielded a CDS spread that is consistent with the market spread.

6 CONCLUSION

The first-passage-time default probability under jump-diffusion can only be computed by simulation, whereas the other default probabilities can be calculated analytically. However, the first-passage-time default probability has the most important practical implications. The maturity-default assumption is somewhat unrealistic and, as shown in the previous section, jump-diffusion process is needed to explain very high short-term spreads that can be frequently observed in the credit markets. Thus, although the parameterization of the jump diffusion model is non-trivial, these models hold great promise for use in credit derivative pricing.

In literature, models with stochastic default barrier are presented as an alternative to jump-diffusion models (see Finger et al. (2002)). An interesting avenue to pursue might be to seek to combine jump-diffusion first-passage-time default process with stochastic default barrier. This can be done with simulation as well.

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AUTHOR BIOGRAPHIES

TARJA JORO is an Assistant Professor in Management Science at the University of Alberta School of Business in Edmonton, Canada. She received her Ph.D. from Helsinki School of Economics in 2000. Her research interests are in productivity and efficiency studies and financial engineering. Her email and web address are <tarja.joro@ualberta.ca> and <http://www.bus.ualberta.ca/tjoro>.

PAUL NA is a Vice President and Head of Credit Risk Methodology at Bayerische Landesbank New York Branch. He received his Ph.D. from University of Georgia in 2000. His research interests are in credit risk modeling and mutual fund performance. His email is <pna@bayernlbny.com>.