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EFFICIENT PRICING OF BARRIER OPTIONS WITH THE VARIANCE-GAMMA MODEL

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ABSTRACT

We develop an efficient Monte Carlo algorithm for pricing barrier options with the variance gamma model (Madan, Carr, and Chang 1998). After generalizing the double-gamma bridge sampling algorithm of Avramidis, L'Ecuyer, and Tremblay (2003), we develop conditional bounds on the process paths and exploit these bounds to price barrier options. The algorithm's efficiency stems from sampling the process paths up to a random resolution that is usually much coarser than the original path resolution. We obtain unbiased estimators, including the case of continuous-time monitoring of the barrier crossing. Our numerical examples show large efficiency gain relative to full-dimensional path sampling.

1 INTRODUCTION

Madan and Seneta (1990), Madan and Milne (1991) and Madan, Carr, and Chang (1998) developed the *variance gamma* (VG) model with application to modeling asset returns and option pricing. The variance gamma process is a Brownian motion with random time change, where the random time change is a gamma process, i.e., a continuous-time process with stationary, independent gamma increments. It was argued that the variance gamma model permits more flexibility in modeling skewness and kurtosis relative to Brownian motion. Closed-form solutions for European option were developed and empirical evidence was provided that the VG option pricing model gives a better fit to market option prices than the classical Black-Scholes model. Except for European options, pricing with variance gamma generally requires numerical techniques; such techniques were developed in Hirsa and Madan (2004) for American options and Ribeiro and Webber (2004), Avramidis, L'Ecuyer, and Tremblay (2003) for path-dependent options.

Bridge sampling of the variance gamma process was independently proposed by Ribeiro and Webber (2004) and Avramidis, L'Ecuyer, and Tremblay (2003) and combined with stratification and Quasi-Monte Carlo, respectively, for pricing path-dependent options efficiently. Large efficiency gains were demonstrated for Asian and lookback options. For barrier options, the results reported in Ribeiro and Webber (2004) do not give a complete picture, but imply the efficiency gain essentially disappears as the barrier approaches the initial asset price.

When the option contract specifies continuous monitoring of the barrier crossing, Monte Carlo-based estimators are generally biased due to the simulation's discrete-time monitoring. For option-pricing models driven by more general Lévy processes, Ribeiro and Webber (2003) develop a correction method for the simulation bias. While empirically found effective, their approach is heuristic and does not yield error bounds, so there is a risk of increasing the error relative to the uncorrected procedure.

Our method is based on *double-gamma bridge sampling* (DGBS) of a variance gamma process (Avramidis, L'Ecuyer, and Tremblay 2003). With DBGS, conditional on sampled values of two gamma processes at any finite set of times containing 0 and T, we can compute bounds on the VG path everywhere on $(0, T]$. For many payoff functions arising in applications, these process-path bounds translate into lower and upper bounds on the conditional payoff; in this paper, we focus on barrier options to convey the main ideas. The algorithm samples a path of the VG process until the gap between the bounding estimators is closed; this ensures unbiasedness, including the case of continuous monitoring of the barrier crossing. In numerical examples, we show that the algorithm's expected work is considerably reduced relative to full-dimensional path sampling.

This paper is an abridged version of Avramidis and L'Ecuyer (2004), who cover more general payoff structures, study the bias of the truncated procedure, and use extrapolation techniques and Quasi Monte-Carlo to improve efficiency. The remainder is organized as follows. Section 2 reviews the essentials of option pricing with the variance gamma model. Section 3.1 generalizes the DGBS algorithm, Section 3.2 develops the process bounds, and

Section 3.3 analyzes the particular case of barrier options. In Section 4 we demonstrate the efficiency gain with two numerical examples.

2 OPTION PRICING WITH VARIANCE GAMMA

Under the variance gamma model, the asset log-return dynamics are characterized by a continuous-time stochastic process obtained as a subordinate to Brownian motion, where the random time change (called *operational time* in Feller (1966)) obeys a gamma process. Let $B = \{B(t;\theta,\sigma) : t \geq 0\}$ be a Brownian motion with drift parameter θ and variance parameter σ. Let $G = \{G(t;\mu,\nu) : t \geq 0\}$ denote a gamma process with mean rate μ and variance rate ν; this is a process with independent gamma increments with $G(t+h;\mu,\nu) - G(t;\mu,\nu)$ having mean μh and variance νh. The variance gamma (VG) process $X(t;\theta,\sigma,\nu)$ is defined as

$$X(t;\theta,\sigma,\nu) := B(G(t;1,\nu),\theta,\sigma).$$

where $G(t;1,\nu)$ is a unit-mean gamma process independent of B.

Option prices under the VG model are expectations of functionals of paths of the asset price process, where expectations are taken with respect to the risk-neutral measure. Under the risk-neutral dynamics, the asset-price process S has paths

$$S(t) = S(0)\exp\{(\omega + r - q)t + X(t)\},$$

where X is a VG process, r is the continuously-compounded, risk-free interest rate, q is the asset's continuously-compounded dividend yield, and $\omega = \log(1-\theta\nu-\sigma^2\nu/2)/\nu$ ensures that $\mathrm{E}[S(t)]=S(0)\exp[(r-q)t]$. In practice, parameter values θ, σ and ν are estimated by calibrating the model against observed option prices. For a more detailed review, see Madan, Carr, and Chang (1998).

3 ALGORITHM AND PROPERTIES

3.1 Generalized Double Gamma Bridge Sampling

The VG process paths have a representation as the difference between two independent gamma processes (Madan, Carr, and Chang 1998):

$$X(t;\theta,\sigma,\nu) = \Gamma^+(t;\mu_p;\nu_p) - \Gamma^-(t;\mu_n;\nu_n), \quad (1)$$

where Γ^+ and Γ^- are independent gamma processes with

$$\begin{aligned}
\mu_p &= (1/2)\sqrt{\theta^2 + 2\sigma^2/\nu} + \theta/2 \\
\mu_n &= (1/2)\sqrt{\theta^2 + 2\sigma^2/\nu} - \theta/2 \\
\nu_p &= \left((1/2)\sqrt{\theta^2 + 2\sigma^2/\nu} + \theta/2\right)^2 \nu \\
\nu_n &= \left((1/2)\sqrt{\theta^2 + 2\sigma^2/\nu} - \theta/2\right)^2 \nu.
\end{aligned}$$

The two gamma processes have common shape parameter per unit-time increment, $\mu_p^2/\nu_p = \mu_n^2/\nu_n = 1/\nu$.

Based on the above representation, Avramidis, L'Ecuyer, and Tremblay (2003) developed double-gamma bridge sampling (DGBS) of a VG process. Their algorithm was stated for *dyadic partitions* of the target time horizon; we make a direct generalization for sampling an arbitrary time partition. We consider a finite time interval $[0,T]$ and in infinite sequence of distinct real numbers $y_o = 0$, $y_1 = T$, and y_2, y_3,..., *dense* in $(0,T)$. This is the sequence of time points at which the two gamma processes are sampled (generated), in order: first at y_1; then at y_2, conditional on their values at y_1; then at y_3, conditional on their values at y_1 and y_2; and so on. For each positive integer m, let $0 = t_{m,0} < t_{m,1} < \ldots < t_{m,m} = T$ denote the values $y_o, y_1, \ldots, y_m$ sorted by increasing order, and let $\iota(m)$ be the index i such that $t_{m,i} = y_m$. That is, $t_{m,\iota(m)}$ is the new observation time added at step m.

We call this more general sampling method the *generalized DGBS* algorithm. Figure 1 outlines the algorithm with an infinite loop. In an actual implementation, the algorithm can be stopped after any number of steps.

3.2 Bounds on the Asset-Price Process

Define

$$\begin{aligned}
\zeta &= \omega + r - q \\
\zeta^+ &= \max(\zeta, 0) \\
\zeta^- &= \max(-\zeta, 0)
\end{aligned}$$

and recall the asset-price process S has representation

$$\begin{aligned}
S(t) &= S(0)\exp[\zeta t + X(t)] \\
&= S(0)\exp[\zeta t + \Gamma^+(t) - \Gamma^-(t)], \quad t \geq 0, \quad (2)
\end{aligned}$$

where Γ^+ and Γ^- are the gamma processes in (1). Define

$$\begin{aligned}
\Delta\Gamma^+_{m,i} &:= \Gamma^+(t_{m,i}) - \Gamma^+(t_{m,i-1}), \\
\Delta\Gamma^-_{m,i} &:= \Gamma^-(t_{m,i}) - \Gamma^-(t_{m,i-1}),
\end{aligned}$$

$t_{1,0} \leftarrow 0;\;\; t_{1,1} \leftarrow T;\;\; \Gamma^+(0) \leftarrow 0;\;\; \Gamma^-(0) \leftarrow 0$
Generate $\Gamma^+(T) \sim \text{Gamma}(T/\nu,\, \nu_\text{p}/\mu_\text{p})$
Generate $\Gamma^-(T) \sim \text{Gamma}(T/\nu,\, \nu_\text{n}/\mu_\text{n})$
For $m = 2$ to ∞ {
  $i \leftarrow \iota(m)$
  $t_{m,i} \leftarrow y_m$
  $t_{m,i-1} \leftarrow t_{m-1,i-1};\quad t_{m,i+1} \leftarrow t_{m-1,i}$
  $\alpha_1 \leftarrow (y_m - t_{m,i-1})/\nu;\quad \alpha_2 \leftarrow (t_{m,i+1} - y_m)/\nu$
  Generate $Y^+ \sim \text{Beta}(\alpha_1, \alpha_2)$
  $\Gamma^+(y_m) \leftarrow \Gamma^+(t_{m,i-1})$
    $+ \left[\Gamma^+(t_{m,i+1}) - \Gamma^+(t_{m,i-1})\right] Y^+$
  Generate $Y^- \sim \text{Beta}(\alpha_1, \alpha_2)$
  $\Gamma^-(y_m) \leftarrow \Gamma^-(t_{m,i-1})$
    $+ \left[\Gamma^-(t_{m,i+1}) - \Gamma^-(t_{m,i-1})\right] Y^-$
  $X(y_m) \leftarrow \Gamma^+(y_m) - \Gamma^-(y_m)$
}

Figure 1: Generalized Double Gamma Bridge Sampling of a VG Process X with Parameters $(1, \nu, \theta, \sigma)$ at an Infinite Sequence of Times $y_o = 0$, $y_1 = T$, and y_2, y_3,... in $(0, T]$

and

$$\begin{aligned} L_{m,i} &= S(t_{m,i-1}) \exp[-\zeta^-(t_{m,i} - t_{m,i-1}) - \Delta\Gamma^-_{m,i}], \\ U_{m,i} &= S(t_{m,i-1}) \exp[\zeta^+(t_{m,i} - t_{m,i-1}) + \Delta\Gamma^+_{m,i}], \\ L_m(t) &= L_{m,i}, \\ U_m(t) &= U_{m,i}, \end{aligned}$$

for $t_{m,i-1} < t < t_{m,i}$, and $L_m(t_{m,i}) = U_m(t_{m,i}) = S(t_{m,i})$, for $i = 1, \ldots, m$.

The following proposition states that the process S is contained between the piecewise constant processes L_m and U_m and that these pathwise bounds are narrowing monotonically with m.

Proposition 1 *For every sample path of S, any integer $m \geq 1$, and all $t \in [0, T]$, we have*

$$L_m(t) \leq L_{m+1}(t) \leq S(t) \leq U_{m+1}(t) \leq U_m(t).$$

Proposition 1 is a consequence of (2) and the fact that the gamma increments are nonnegative. Avramidis and L'Ecuyer (2004) state bounding processes that are tighter bounds than L_m and U_m. The current result is obtained as their Corollary 1.

3.3 Barrier Options

We start with a basic description of the different types of barrier options. A *knock-in* option comes into existence only if the underlying asset price crosses a given barrier. A *knock-out* option ceases to exist whenever the underlying asset price crosses a barrier. Further, we distinguish them as *up* or *down*, depending on the direction of asset-price movement that triggers the barrier crossing. They are further classified as *call* or *put*. For further information, see Hull (2000).

As a prototypical barrier option, we consider the *up-and-in call* with continuous monitoring of the barrier crossing; the payoff, discounted to time zero, is

$$C_\text{B}(\infty) = e^{-rT}\, (S(T) - K)^+\, I \left\{ \sup_{0 \leq t \leq T} S(t) > b \right\}, \quad (3)$$

where $b > S(0)$ is the *barrier*, K is the *strike price*, and I denotes the indicator function. The related option with discrete monitoring has discounted payoff

$$C_\text{B}(d) = e^{-rT}\, (S(T) - K)^+\, I \left\{ \max_{1 \leq i \leq d} S(t_i) > b \right\} \quad (4)$$

for given $t_i \in (0, T]$, $i = 1, \ldots, d$.

Define the sequence of estimators

$$C_{\text{L},m} = e^{-rT}\, (S(T) - K)^+\, I \left\{ \max_{1 \leq i \leq m} S(t_{m,i}) > b \right\}$$

and

$$C_{\text{U},m} = e^{-rT}\, (S(T) - K)^+\, I \left\{ \max_{1 \leq i \leq m} U_{m,i} > b \right\},$$

for $m = 1, 2, \ldots$. An interesting feature of this pair of estimators is that the gap between them vanishes if $S(T) \leq K$ or if the indicator function takes the same value in both cases, i.e., whenever

$$\max_{1 \leq i \leq m} S(t_{m,i}) > b \quad \text{or} \quad \max_{1 \leq i \leq m} U_{m,i} \leq b. \quad (5)$$

Thus, to estimate the continuous-time price, it appears sensible to continue sampling until this gap is closed. Let M denote the random variable defined as the smallest m for which (5) holds. To allow additional deterministic truncation of sampling after k steps, define

$$M(k) = \min(M, k). \quad (6)$$

Proposition 2 below summarizes some properties of the estimators; it is a straightforward consequence of Proposition

1. Let

$$\mathcal{F}_m = (\Gamma^+(t_{m,1}), \Gamma^-(t_{m,1}), \ldots, \Gamma^+(t_{m,m}), \Gamma^-(t_{m,m})).$$

Proposition 2 *(a) For any fixed $m \geq 1$, conditional on $\mathcal{F}_m$,*

$$C_{\mathrm{L},m} \leq C_{\mathrm{B}}(\infty) \leq C_{\mathrm{U},m}.$$

Moreover,

$$C_{\mathrm{L},m} \leq C_{\mathrm{B}}(d) \leq C_{\mathrm{U},m}$$

whenever

$$\{t_{m,1}, \ldots, t_{m,m}\} \subseteq \{t_1, \ldots, t_d\}. \quad (7)$$

The bounding estimators are narrowing monotonically in m.

(b) The estimator $C_{\mathrm{L},M(\infty)} = C_{\mathrm{U},M(\infty)}$ is unbiased for the continuous-time price $\mathrm{E}[C_{\mathrm{B}}(\infty)]$. Moreover, $C_{\mathrm{L},M(d)} = C_{\mathrm{U},M(d)}$ is unbiased for the discrete-time price $\mathrm{E}[C_{\mathrm{B}}(d)]$ whenever (7) holds.

Part (b) states an attractive property of unbiasedness for the case of continuous-time monitoring; this was precisely the goal of the correction procedure of Ribeiro and Webber (2003), which, however, does *not* guarantee unbiasedness. On the other hand, an unresolved issue in our procedure is whether $M(\infty)$ has finite mean. For the case of discrete-time monitoring with finite but large d, our unbiased estimator is likely to require considerably less computation compared to the unbiased estimator that samples full-dimensional paths; empirical evidence supporting this assertion is offered in Section 4. Moreover, part (a) shows a pair of estimators whose expectations bracket the option price; this permits constructing confidence intervals that may be useful in time-critical applications where some pricing accuracy is exchanged for speed of computation.

The above approach and results analogous to Proposition 2 apply with very straightforward modifications to the other types of barrier options. For example, for a *down-and-in call* option, we have $b < S(0)$, we replace "$\sup_{0 \leq t \leq T} S(t) > b$" in the indicator function in (3) by "$\inf_{0 \leq t \leq T} S(t) < b$", and make corresponding replacements in the low and high estimators. The additional variations *up-and-out call*, *down-and-out call*, and the *put* versions can be handled similarly.

4 NUMERICAL RESULTS

We examine the efficiency of the estimator in Proposition 2(b) for two examples of barrier options with discrete monitoring. We consider the up-and-in call (4) and the down-and-out call with discounted payoff

$$e^{-rT}\,(S(T) - K)^+\, I\left\{\min_{1 \leq i \leq d} S(t_i) > b\right\}.$$

Both options have discrete monitoring at $t_i = Ti/d$, $i = 1, \ldots, d$. Option parameters are: $S(0) = 100$ and $K = 100$.

We take VG model parameters from Hirsa and Madan (2004): $T = 0.46575$, $\sigma = 0.19071$, $\nu = 0.49083$, $\theta = -0.28113$, $r = 0.0549$, and $q = 0.011$; these were calibrated against options on the S&P 500 index using data for June 30, 1999 and correspond to intermediate-maturity options.

Given the estimator's unbiasedness, the efficiency gain factor compared to full-dimensional (non-truncated) sampling is the ratio of expected work between the two estimators. Tables 1 and 2 show results for the up-and-in call and the down-and-out call, respectively; we give 95% confidence intervals on the expected work $\mathrm{E}[M(d)]$ and the estimated option prices, varying the barrier b and the problem dimension d. The ratio $d/\mathrm{E}[M(d)]$ may be viewed as a simple, albeit rough, measure of the efficiency gain. In all cases, we see that expected work grows very slowly with the dimension d; equivalently, efficiency increases rapidly.

Table 1: Estimated Expected Work $\mathrm{E}[M(d)]$ and Price (Standard Error in Parentheses) for Up-and-in Call Option, for Selected Barrier Levels b and Dimension d.

b	d	95% C.I. on $\mathrm{E}[M(d)]$	Price
105	4	(2.20, 2.20)	7.3329 (0.008)
	16	(3.64, 3.65)	7.3739 (0.008)
	64	(5.10, 5.13)	7.3843 (0.008)
	256	(6.54, 6.61)	7.3874 (0.008)
110	4	(2.25, 2.26)	6.3883 (0.008)
	16	(3.55, 3.56)	6.5260 (0.008)
	64	(4.83, 4.86)	6.5716 (0.008)
	256	(6.09, 6.15)	6.5833 (0.008)
120	4	(2.11, 2.11)	2.0406 (0.006)
	16	(2.41, 2.42)	2.1238 (0.007)
	64	(2.71, 2.73)	2.1557 (0.007)
	256	(3.01, 3.04)	2.1654 (0.007)

It is interesting to observe the effect of the distance between the barrier and the initial asset price. It is intuitive that expected work decreases as distance increases, because in most sampled paths we determine early in the sampling process (via the bounds) that the barrier is *not* crossed. In the opposite direction, as the distance becomes small, the bounds become less useful and it takes more sampling until we determine the barrier-crossing indicator. On a positive note, our examples suggest that expected work is small throughout and is not very sensitive to this distance.

Table 2: Estimated Expected Work $E[M(d)]$ and Price (Standard Error in Parentheses) for Down-and-out Call Option, for Selected Barrier Levels b and Dimension d.

b	d	95% C.I. on $E[M(d)]$	Price
80	4	(2.16, 2.16)	7.5018 (0.008)
	16	(2.45, 2.46)	7.5011 (0.008)
	64	(2.74, 2.76)	7.5008 (0.008)
	256	(3.03, 3.06)	7.5007 (0.008)
95	4	(2.49, 2.49)	7.3199 (0.008)
	16	(3.45, 3.47)	7.1832 (0.008)
	64	(4.42, 4.44)	7.1368 (0.008)
	256	(5.36, 5.41)	7.1241 (0.008)
99	4	(2.77, 2.77)	6.8283 (0.007)
	16	(4.47, 4.49)	6.3299 (0.007)
	64	(6.20, 6.24)	6.1528 (0.007)
	256	(7.93, 8.00)	6.1021 (0.007)

In the least-favorable (smallest-distance) case across our experiments, the down-and-out option with $b = 99$, we get $E[M(256)] \approx 8$. This should be contrasted to the negative results of Ribeiro and Webber (2004), who found little or no efficiency gain when distance is small.

REFERENCES

Avramidis, A. N., and P. L'Ecuyer. 2004. Efficient Monte Carlo and Quasi-Monte Carlo option pricing with the variance-gamma model. Working paper, Department of Computer Science and Operations Research, Université de Montréal.

Avramidis, A. N., P. L'Ecuyer, and P.-A. Tremblay. 2003. Efficient simulation of gamma and variance-gamma processes. In *Proceedings of the 2003 Winter Simulation Conference*, ed. S. Chick, P. J. Sanchez, D. Ferrin, and D. J. Morrice, 319–326. Piscataway, New Jersey: IEEE Press.

Feller, W. 1966. *An introduction to probability theory and its applications, vol. 2*. first ed. New York: Wiley.

Hirsa, A., and D. B. Madan. 2004. Pricing American options under variance gamma. *The Journal of Computational Finance* 7 (2):63–80.

Hull, J. 2000. *Options, futures, and other derivative securities*. fourth ed. Englewood-Cliff, N.J.: Prentice-Hall.

Madan, D. B., P. P. Carr, and E. C. Chang. 1998. The variance gamma process and option pricing. *European Finance Review* 2:79–105.

Madan, D. B., and F. Milne. 1991. Option pricing with V.G. martingale components. *Mathematical Finance* 1:39–55.

Madan, D. B., and E. Seneta. 1990. The variance gamma (V.G.) model for share market returns. *Journal of Business* 63:511–524.

Ribeiro, C., and N. Webber. 2003. Correcting for simulation bias in Monte Carlo methods to value exotic options in models driven by Lévy processes. Working Paper, Cass Business School, London, UK.

Ribeiro, C., and N. Webber. 2004. Valuing path-dependent options in the variance-gamma model by Monte Carlo with a gamma bridge. *The Journal of Computational Finance* 7 (2):81–100.

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