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EXACT SIMULATION OF OPTION GREEKS UNDER STOCHASTIC VOLATILITY AND JUMP DIFFUSION MODELS

Mark Broadie

Graduate School of Business
Columbia University
New York, NY 10027, U.S.A.

Özgür Kaya

Department of IE & OR
Columbia University
New York, NY 10027, U.S.A.

ABSTRACT

This paper derives Monte Carlo simulation estimators to compute option price derivatives, i.e., the 'Greeks,' under Heston's stochastic volatility model and some variants of it which include jumps in the price and variance processes. We use pathwise and likelihood ratio approaches together with the exact simulation method of Broadie and Kaya (2004) to generate unbiased estimates of option price derivatives in these models. By appropriately conditioning on the path generated by the variance and jump processes, the evolution of the stock price can be represented as a series of lognormal random variables. This makes it possible to extend previously known results from the Black-Scholes setting to the computation of Greeks for more complex models. We give simulation estimators and numerical results for some path-dependent and path-independent options.

1 INTRODUCTION

Monte Carlo simulation is a widely used tool in finance for computing the prices of options and their price sensitivities, commonly referred to as Greeks. It is not always possible to derive analytical formulas for option prices and their Greeks, either because the payoff or the underlying model, or both, are too complicated to be tractable. This occurs for many of options under stochastic volatility and jump diffusion models. In these cases, Monte Carlo simulation offers a potential computational approach.

In general, the computation of option Greeks using Monte Carlo simulation is not as straightforward as option prices. Difficulties may be caused by discontinuities in the option payoff function, as in the cases of barrier and digital options, for example. Monte Carlo methods for estimating price sensitivities can be categorized into three groups. Finite-difference approximations resimulate the price process after a small perturbation in the parameter of interest. Simulation estimates generated by finite-difference approximations are biased and computationally expensive because of the resimulation step, and therefore we do not consider them here. The other two methods are the pathwise method (PW) and the likelihood ratio method (LR) first introduced in Broadie and Glasserman (1996). In the PW method, a simulation estimator is derived by differentiating the payoff function inside the expectation operator. Since the PW method requires the interchange of differentiation and expectation, it is not applicable in certain circumstances, e.g., in the computation of the delta of a digital option. In the LR method, the parameter of the price function to be differentiated is viewed as a parameter of the density function rather than the payoff and this density is differentiated inside the expectation. Because the density functions in most financial models are smooth, the LR estimator is more widely applicable than the PW method. However, when applicable, the PW method usually gives better estimates than LR method, since the PW estimator takes advantage of the specific form of the payoff function. See Glasserman (2003) for a more detailed treatment of these methods.

The PW and LR methods can be used to generate unbiased estimators of Greeks in the Black-Scholes setting where the transition density of the underlying price process is known and sampling from the exact distribution is possible. However, for more complex models, e.g., ones involving stochastic volatility, exact simulation of the price and variance processes is not straightforward and discretization (i.e., time-stepping) schemes are often used to generate approximate samples. Discretization approaches have some significant drawbacks. First, discretization introduces bias into simulation estimates and this makes the construction of valid confidence intervals difficult or impossible. Also, methods for estimating Greeks may become harder to implement and computationally inefficient when discretization schemes are used. For example, for the LR method the variance of the simulation estimator typically increases as the time interval is decreased to lower the discretization bias.

An exact simulation algorithm for the stochastic volatility (SV) model of Heston (1993) and other models with jumps in asset prices and jumps in volatility is introduced in Broadie and Kaya (2004). In this paper, we extend that approach to the exact simulation of Greeks under these models. In particular, we use the pathwise and likelihood ratio methods to generate unbiased estimates of option price derivatives, including the delta, gamma and rho. The key idea is that by appropriately conditioning on the path generated by the variance and jump processes, the evolution of the stock price can be represented as a series of lognormal random variables, making it possible to apply previous results for the Black-Scholes setting.

Some authors have proposed a Malliavin calculus to derive simulation estimators for Greeks. This approach can be viewed as an extension of the LR method. Fournié et al. (1999) show that any Greek could be expressed as an expectation of the payoff function times a weight. They show that this weight can be expressed in terms of the Malliavin derivative without knowing the transition density. Fournié et al. (2001) prove that the weight that gives the minimum total estimator variance is the one given by the LR method. This result implies that the estimators derived using Malliavin calculus are not superior to the ones given by LR and PW methods. In particular, Malliavin estimators of the Greeks under the SV model contain various integrals of the variance process making the use of discretization methods necessary (see Benhamou 2002 for the explicit form of the weights for Heston model).

In Section 2 we review the models that we will consider and briefly discuss the exact simulation algorithm of Broadie and Kaya (2004). In Section 3 we present the key idea and show how each of the models can be set-up in a way suitable for application of LR and PW methods. Section 4 gives the simulation estimators for several options and presents numerical results. Section 5 concludes the paper.

2 MODELS AND THE EXACT SIMULATION ALGORITHM

We will consider three different models with increasing complexity. The first model is the stochastic volatility (SV) model of Heston (1993). The next model is an extension of SV to include jumps in the stock price. We will refer to this as the SVJ model. Finally, we will consider a model proposed in Duffie, Singleton and Pan (2000) that includes concurrent jumps in the stock price and the variance processes, which will be referred to as SVCJ.

We give the specification for the SVCJ model since the other two can be expressed as the special cases of this model. The SVCJ model is based on the following stochastic differential equations which are specified under the risk neutral measure:

$$dS_t = (r - \lambda\bar{\mu})S_t dt + \sqrt{V_t}S_t\left[\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right] + S_t(J^s - 1)dN_t, \quad (1)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^{(1)} + J^v dN_t. \quad (2)$$

The first equation gives the dynamics of the stock price: S_t denotes the stock price at time t, r is the interest rate and $\sqrt{V_t}$ is the volatility. The second equation gives the evolution of the variance which follows the square-root process: θ is the long-run mean variance, κ represents the speed of mean reversion, and σ_v is a parameter which determines the volatility of the variance process. $W_t^{(1)}$ and $W_t^{(2)}$ are two independent Brownian motion processes, and ρ represents the instantaneous correlation between the return process and the volatility process. N_t is a Poisson process independent of the Brownian motions and with constant intensity λ, J^s is the relative jump size in the stock price, and J^v is the jump size of the variance. In particular, when a jump occurs at time t, then $S_{t^+} = S_{t^-}J^s$ and $V_{t^+} = V_{t^-} + J^v$. The jumps in stock price and the variance occur concurrently, and their magnitudes have a correlation determined by the parameter ρ_J. The distribution of J^v is exponential with mean μ_v. Given J^v, J^s is lognormally distributed with mean $(\mu_s + \rho_J J^v)$ and variance σ_s^2. The parameters μ_s and $\bar{\mu}$ are related to each other by the equation

$$\mu_s = \log\left[(1+\bar{\mu})(1-\rho_J\mu_v)\right] - \frac{1}{2}\sigma_s^2,$$

and only one of them needs to be specified.

In the SVJ model, there are no jumps in the variance process, so setting $J^v = 0$, $\mu_v = 0$, and $\rho_J = 0$ gives the specification for that model. In the SV model there are no jumps in the stock price nor the variance process, so deleting the dN_t terms in (1) and (2) above gives the specification for SV model.

The stock price at time t, given the values of S_u and V_u for $u < t$, can be written as

$$S_t = S_u \exp\left[(r - \lambda\bar{\mu})(t-u) - \frac{1}{2}\int_u^t V_s ds + \rho\int_u^t \sqrt{V_s}dW_s^{(1)} + \sqrt{1-\rho^2}\int_u^t \sqrt{V_s}dW_s^{(2)}\right] \times \prod_{i=N_u+1}^{N_t} J_i^s \quad (3)$$

and the variance at time t is given by the equation

$$V_t = V_u + \kappa\theta(t-u) - \kappa \int_u^t V_s ds + \sigma_v \int_u^t \sqrt{V_s} dW_s^{(1)} + \sum_{i=N_u+1}^{N_t} J_i^v. \qquad (4)$$

Although there is no simple analytical representation for the distribution of (S_t, V_t), Broadie and Kaya (2004) derive a numerical method to exactly sample from the joint distribution under each of these models. In the case of the SV model, the steps of the algorithm are as follows:

1. Generate a sample from the distribution of V_t given V_u: This is easily done since the distribution of V_t given V_u is, up to a scale factor, a noncentral chi-squared distribution.
2. Generate a sample from the distribution of $\int_u^t V_s ds$ given V_t and V_u: This is done by writing the conditional characteristic function of the integral and then using Fourier transform inversion to obtain the distribution function.
3. Recover $\int_u^t \sqrt{V_s} dW_s^{(1)}$ from (4) given V_t, V_u and $\int_u^t V_s ds$.
4. Generate a sample from the distribution of S_t given $\int_u^t \sqrt{V_s} dW_s^{(1)}$ and $\int_u^t V_s ds$: The conditional distribution of S_t is lognormal, so this is straightforward.

The details of the algorithms and numerical examples are given in Broadie and Kaya (2004). The most important steps for the simulation of Greeks are the first three steps above, where the variance and its integrals are generated. We will use these as conditioning variables, so it is crucial to be able to simulate these exactly.

3 BASIC IDEA, AND LIKELIHOOD RATIO AND PATHWISE METHODS

We first review how the PW and LR estimators are derived in a usual simulation setting. Suppose the option price is given by $\alpha(\theta) = E[f(\theta)]$ where f is the discounted payoff function and we are interested in finding the derivative $\alpha'(\theta)$ with respect to θ. The PW method writes this as

$$\alpha'(\theta) = \frac{d}{d\theta} E\left[f(\theta)\right] = E\left[\frac{d}{d\theta} f(\theta)\right], \qquad (5)$$

assuming the interchange of differentiation and expectation is justified. On the other hand, in the LR method, the payoff is viewed as a function of a random vector X that determines the payoff, and θ is viewed as a parameter of the probability density of X. If this density is denoted by $g_\theta(x)$, then the derivative with respect to θ is written as:

$$\alpha'(\theta) = \frac{d}{d\theta} E\left[f(X)\right] = \int_{R^n} f(x) \frac{d}{d\theta} g_\theta(x) dx. \qquad (6)$$

Now, the integrand can be multiplied and divided by g_θ to give:

$$\alpha'(\theta) = \int_{R^n} f(x) \frac{g'_\theta(x)}{g_\theta(x)} g_\theta(x) dx = E\left[f(X)\frac{g'_\theta(x)}{g_\theta(x)}\right], \qquad (7)$$

where we have written $g'_\theta(x)$ for $dg_\theta/d\theta$. Thus the expression $f(X)g'_\theta(x)/g_\theta(x)$ is an unbiased estimator of $\alpha'(\theta)$. The quantity $g'_\theta(x)/g_\theta(x)$ in (7) is called the score function. It is clear from the above that the score function for a particular sensitivity is independent of the form of the payoff function.

As seen from the above equations, both PW and LR methods use an interchange of integral and differentiation which is only justified if certain regularity conditions are satisfied (see Broadie and Glasserman 1996 for details). In general, it is easier to justify (6) than (5) since density functions are usually smooth functions of their parameters but payoff functions are not.

To be able to use the LR method in the stochastic volatility and jump setting, we will first need to use some conditioning arguments that will allow us to do exact simulation. If the discounted payoff is a function of a vector valued process X, using the law of iterated expectations we write:

$$E\left[f(X)\right] = E\left[E\left[f(X)|Y\right]\right], \qquad (8)$$

where Y is another vector valued process. In the following applications, X will usually be the values of stock price at discrete time intervals along a path, and Y will be a set of state variables recording information about the variance path. To derive the LR and PW estimators, we will differentiate inside the expectation operators similar to the approach given in (5) and (6). If the interchange of integral and differentiation is justified for the left hand side of (8), it will be justified for the right hand side as well. Note that the PW method can be applied directly to the left hand side of (8), however in the numerical results below we use the conditional representation given on the right hand side so that the computation times for the two methods are roughly comparable.

Our aim is to write the values of the stock price as a series of lognormal random variables by conditioning on appropriate state variables. This way, we will be able to use the LR and PW estimators that are derived for the standard Black-Scholes setting. To this end, we show how the conditioning variables are chosen and which form the stock price takes for each of the models we consider.

In the following, let $[0 = t_0 < t_1 < \ldots < t_M = T]$ be a partition of time interval into possibly unequal segments of length Δt_i, for each $i = 0, 1, \ldots, M$. We are assuming that we eventually want to price a path-dependent option whose payoff is a function of the stock price vector $(S_{t_0}, S_{t_1}, \ldots, S_{t_M})$ (note that we can take M=1 for a path-independent option).

SV MODEL: In this case, we will condition on the path generated by the variance process. Willard (1997) introduced a similar conditional Monte Carlo approach for the pricing of path-independent options under multifactor models, but his estimates for price and Greeks are biased since a discretization scheme is used for the simulation of the variance process.

Assume that we have simulated a path of the variance process by using the first three steps of the exact simulation algorithm given in Section 2. Consider two consecutive time steps t_i and t_j on the time grid. The simulation algorithm gives us the values of the quantities $\int_{t_i}^{t_j} V_s ds$ and $\int_{t_i}^{t_j} \sqrt{V_s} dWs^{(1)}$ in equation (3). We define the average variance between t_i and t_j as:

$$\bar{\sigma}_j^2 = \frac{(1-\rho^2)\int_{t_i}^{t_j} V_s ds}{t_j - t_i}.$$

We also define an auxiliary variable as:

$$\xi_j = \exp\left(-\frac{\rho^2}{2}\int_{t_i}^{t_j} V_s ds + \rho \int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}\right).$$

Given the variance path and the stock price S_i, the value of S_j can be written as

$$S_j = S_i \xi_j \exp\left[(r - \frac{\bar{\sigma}_j^2}{2})(t_j - t_i) + \bar{\sigma}_j \sqrt{t_j - t_i}\, Z\right], \quad (9)$$

where Z is a standard normal random variable.

Thus, taking Y in equation (8) to be the variance path, we have reduced the distribution of stock prices in the inner expectation to a series of lognormal random variables. This is crucial for the LR method since we need to know the probability density for derivation of the score function.

SVJ MODEL: In this case, we will condition on the variance path and the number of jumps in each time interval. As before, the exact simulation algorithm can be used to simulate the variance path and the number of jumps is generated independently for each interval.

Consider two consecutive time steps t_i and t_j on the time grid. The simulation algorithm gives us the values of the quantities $\int_{t_i}^{t_j} V_s ds$ and $\int_{t_i}^{t_j} \sqrt{V_s} dWs^{(1)}$ in equation (3). We simulate n_j, the number of stock price jumps between t_i and t_j, by generating a Poisson random variable with mean $\lambda(t_j - t_i)$. We define the average variance between t_i and t_j as:

$$\bar{\sigma}_j^2 = \frac{n_j \sigma_s^2 + (1-\rho^2)\int_{t_i}^{t_j} V_s ds}{t_j - t_i},$$

where σ_s is as defined in Section 2. We also define an auxiliary variable as:

$$\xi_j = \exp\left(n_j(\mu_s + \frac{\sigma_s^2}{2}) - \lambda\bar{\mu}(t_j - t_i) - \frac{\rho^2}{2}\int_{t_i}^{t_j} V_s ds + \rho \int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}\right).$$

With these definitions of $\bar{\sigma}_j^2$ and ξ_j, the value of S_j given the stock price S_i can be written as in (9). We are able to write the distribution of S_j in this way by using the independence of the jump size from the Brownian motion processes which drive the diffusion component of the price process. For simulating S_j, the number of jumps occurring in the time interval is important but the actual jump times are not. We collect the variance from the diffusion and jump components in the value $\bar{\sigma}_j^2$ and simulate S_j as a single lognormal random variable conditional on this information.

SVCJ MODEL: The simulation of SVCJ model is a bit more involved than SV and SVJ models because of the existence of jumps in the variance process. We will again condition on the variance path, but this time we will need the jump times and the jump sizes of the variance. Therefore, we need to stop the simulation at jump times and generate the jumps in the variance and continue simulation of the diffusion part from the updated variance value. Details of this can be found in Broadie and Kaya (2004).

Consider two consecutive time steps t_i and t_j on the time grid. The simulation algorithm will give us the values of the quantities $\int_{t_i}^{t_j} V_s ds$ and $\int_{t_i}^{t_j} \sqrt{V_s} dWs^{(1)}$ in equation (3), where the integrals are split into subintegrals at the simulated jump times between t_i and t_j. If there are n_j jumps in this interval and the kth jump size of the variance is denoted as J_k^v, we define the average variance between t_i and t_j as:

$$\bar{\sigma}_j^2 = \frac{n_j \sigma_s^2 + (1-\rho^2)\int_{t_i}^{t_j} V_s ds}{t_j - t_i},$$

where σ_s is defined as in Section 2 and we also define an auxiliary variable as:

$$\xi_j = \exp\left(\sum_{k=1}^{n_j}(\mu_s + J_k^v \rho_J + \frac{\sigma_s^2}{2}) - \lambda\bar{\mu}(t_j - t_i) - \frac{\rho^2}{2}\int_{t_i}^{t_j} V_s ds + \rho \int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}\right).$$

With these definitions of $\bar{\sigma}_j^2$ and ξ_j, the value of S_j given the stock price S_i can be written as in (9).

4 NUMERICAL EXAMPLES

In this section we illustrate the simulation of Greeks using different options. We first use a European option, since there are analytical formulas for the price and the Greeks in this case, we may compare our simulation estimates with the true values. For the formulas for Greeks, see for example Bakshi, Cao and Chen (1997), and Reiss and Wystup (2001). We then consider several path dependent examples: an Asian option, a discrete knock-out barrier option and a forward start option. The derivations of the estimators given in the following sections are similar to the ones given in Broadie and Glasserman (1996) in the Black-Scholes setting. Therefore, we will just give the estimators and omit the derivations. Since we are using conditional Monte Carlo as given in (8), we can choose different numbers of paths for the two expectations we are evaluating. In the numerical results in this paper, we simulate 10,000 variance paths and we simulate 100 price paths conditional on each variance path. We use the model parameters estimated in Duffie, Singleton and Pan (2000). These were found by minimizing the mean squared errors for market option prices for S&P 500 on November 2, 1993. We also assume a risk-free rate of 3.19%. Table 1 gives these parameters.

Table 1: Model Parameters Used in Simulations

	MODEL		
	SV	SVJ	SVCJ
ρ	-0.70	-0.79	-0.82
θ	0.019	0.014	0.008
κ	6.21	3.99	3.46
σ_v	0.61	0.27	0.14
λ	n/a	0.11	0.47
$\bar{\mu}$	n/a	-0.12	-0.10
μ_s	n/a	0.15	0.0001
$\bar{\mu}_v$	n/a	n/a	0.05
ρ_J	n/a	n/a	-0.38
$\sqrt{V_0}$	10.1%	9.4%	8.7%
r	3.19%	3.19%	3.19%

4.1 European Options

The discounted payoff of a European call option with strike K and maturity T is given by $e^{-rT}(S_T - K)^+$, where S_T is the stock price at time T. In the equations below $\mathbf{1}[A]$ is used to denote the indicator function of the event A. PW derivative estimators for a European call option are given in (10) and (11) below.

PW estimators:

$$\text{Delta: } e^{-rT}\mathbf{1}[S_T \geq K]\frac{S_T}{S_0} \tag{10}$$

$$\text{Rho: } e^{-rT}\mathbf{1}[S_T \geq K]KT \tag{11}$$

To derive the LR estimators using the conditional expectation in (8), we need the conditional density of S_T. Using (9), this density can be written as:

$$g(x) = \frac{1}{x\bar{\sigma}\sqrt{T}}\phi(d(x)),$$

where $\phi(\cdot)$ is the standard normal density function and

$$d(x) = \frac{\ln(x/(S_0\xi)) - (r - \frac{1}{2}\bar{\sigma}^2)T}{\bar{\sigma}\sqrt{T}}.$$

To find the delta estimator, we first take the derivative with respect to S_0. After some simplification, we get:

$$\frac{\partial g(x)}{\partial S_0} = \frac{d(x)\phi(d(x))}{xS_0\bar{\sigma}^2T}.$$

Dividing this by $g(x)$ and evaluating the expression at $x = S_T$ gives the score function for LR delta estimator:

$$\frac{\partial g(S_T)/\partial S_0}{g(S_T)} = \frac{d(S_T)}{S_0\bar{\sigma}\sqrt{T}}.$$

Other estimators can be derived in a similar fashion. For details, see Broadie and Glasserman (1996), and Glasserman (2003).

LR estimators:

$$\text{Delta: } e^{-rT}(S_T - K)^+\left(\frac{d}{S_0\bar{\sigma}\sqrt{T}}\right) \tag{12}$$

$$\text{Gamma: } e^{-rT}(S_T - K)^+\left(\frac{d^2 - d\bar{\sigma}\sqrt{T} - 1}{S_0^2\bar{\sigma}^2T}\right) \tag{13}$$

$$\text{Rho: } e^{-rT}(S_T - K)^+\left(-T + \frac{d\sqrt{T}}{\sigma}\right) \tag{14}$$

where $d = (\ln(S_T/(S_0\xi)) - (r - \frac{1}{2}\bar{\sigma}^2)T)/(\bar{\sigma}\sqrt{T})$ in (12)–(14). If S_T is generated from S_0 using a normal random variable Z via equation (9), then $d = Z$, and these estimators are easily computed in a simulation.

The delta estimator in (10) includes an indicator function, so the PW method cannot be used to take the derivative of this expression to obtain a gamma estimator. For finding estimators for second order derivatives like gamma, we can use a mixed estimator where we use the PW method for one order of differentiation and LR method for the other.

This gives the estimators in (15) and (16) for the gamma of a European call option.

Mixed estimators:

$$\text{LR-PW Gamma: } e^{-rT}\mathbf{1}[S_T \geq K]K \times \left(\frac{d}{S_0^2\bar{\sigma}\sqrt{T}}\right) \quad (15)$$

$$\text{PW-LR Gamma: } e^{-rT}\mathbf{1}[S_T \geq K]\frac{S_T}{S_0^2} \times \left(\frac{d}{\bar{\sigma}\sqrt{T}} - 1\right) \quad (16)$$

where d is as given above.

We give the simulation results along with the true values in Table 2. As expected, the PW estimators have smaller standard deviations than the LR estimators. Also, the mixed estimators for gamma have smaller standard deviations than the LR estimators.

Table 2: Simulation Estimates of Price and Greeks for a European Call Option

	MODEL		
	SV	SVJ	SVCJ
Exact Price	6.8061	6.7578	6.8619
Simulation Price	6.8030	6.7017	6.8787
(sdt. err.)	0.0402	0.0487	0.0593
Exact Delta	0.6958	0.6976	0.6989
PW Delta	0.6952	0.6954	0.7006
(std. err.)	0.0032	0.0036	0.0041
LR Delta	0.6973	0.6925	0.7016
(std. err.)	0.0036	0.0041	0.0048
Exact Gamma	0.0265	0.0264	0.0259
LR Gamma	0.0269	0.0263	0.0260
(std. err.)	0.0005	0.0007	0.0009
LR-PW Gamma	0.0267	0.0266	0.0257
(std. err.)	0.0002	0.0002	0.0003
PW-LR Gamma	0.0267	0.0265	0.0257
(std. err.)	0.0002	0.0002	0.0003
Exact Rho	62.7752	63.0034	63.0329
PW Rho	62.7148	62.8420	63.1775
(std. err.)	0.2774	0.3114	0.3554
LR Rho	62.9256	62.5472	63.2840
(std. err.)	0.3274	0.3663	0.4318

Option Parameters: S_0=100, K=100, T=1 year

4.2 Asian Options

The discounted payoff for an Asian option with strike K and expiration T is given by $e^{-rT}(\bar{S} - K)^+$ where

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i,$$

and S_i is the stock price at time t_i for a time partition $[0 = t_0 < t_1 < \ldots < t_m = T]$. The estimators for an Asian option are given in (17)–(23) below.

PW estimators:

$$\text{Delta: } e^{-rT}\mathbf{1}[\bar{S} \geq K]\frac{\bar{S}}{S_0} \quad (17)$$

$$\text{Rho: } e^{-rT}\mathbf{1}[\bar{S} \geq K]\left(\frac{1}{m}\sum_{i=1}^{m} S_i t_i - T(\bar{S} - K)\right) \quad (18)$$

LR estimators:

$$\text{Delta: } e^{-rT}(\bar{S} - K)^+\left(\frac{d_1}{S_0\bar{\sigma}_1\sqrt{\Delta t_1}}\right) \quad (19)$$

$$\text{Gamma: } e^{-rT}(\bar{S} - K)^+\left(\frac{d_1^2 - d_1\bar{\sigma}_1\sqrt{\Delta t_1} - 1}{S_0^2\bar{\sigma}_1^2\Delta t_1}\right) \quad (20)$$

$$\text{Rho: } e^{-rT}(\bar{S} - K)^+\left(-T + \sum_{i=1}^{m}\frac{d_i\sqrt{\Delta t_i}}{\bar{\sigma}_i}\right) \quad (21)$$

Mixed estimators:

$$\text{LR-PW Gamma: } e^{-rT}\mathbf{1}[\bar{S} \geq K]K \times \left(\frac{d_1}{S_0^2\bar{\sigma}_1\sqrt{\Delta t_1}}\right) \quad (22)$$

$$\text{PW-LR Gamma: } e^{-rT}\mathbf{1}[\bar{S} \geq K]\frac{\bar{S}}{S_0^2} \times \left(\frac{d_1}{\bar{\sigma}_1\sqrt{\Delta t_1}} - 1\right) \quad (23)$$

where in (19)–(23), $\Delta t_i = t_i - t_{i-1}$,

$$d_i = (\ln(S_i/(S_{i-1}\xi_i)) - (r - \frac{1}{2}\bar{\sigma}_i^2)\Delta t_i)/(\bar{\sigma}_i\sqrt{\Delta t_i}),$$

and $\bar{\sigma}_i^2$ is the variance between t_{i-1} and t_i. If S_i is generated from S_{i-1} using a normal random variable Z_i via equation (9), then $d_i = Z_i$, and these estimators are easily computed in a simulation.

We give the simulation results in Table 3. Again, PW estimators dominate the LR estimators in terms of standard error. Also, the mixed estimators for gamma have standard errors that are one-fifth of those for the LR estimator.

4.3 Discrete Barrier Options

The discounted payoff for a discrete knock-out barrier option with strike K, expiration T and barrier $H > S_0$ is given by

$$e^{-rT}(S_T - K)^+\mathbf{1}[\max_{1\leq i\leq m} S_i < H],$$

Table 3: Simulation Estimates of Price and Greeks for an Asian Option

	MODEL		
	SV	SVJ	SVCJ
Simulation Price	4.3736	4.3125	4.4415
(sdt. err.)	0.0254	0.0315	0.0380
PW Delta	0.6733	0.6709	0.6790
(std. err.)	0.0030	0.0034	0.0039
LR Delta	0.6746	0.6736	0.6782
(std. err.)	0.0038	0.0044	0.0050
LR Gamma	0.0384	0.0416	0.0421
(std. err.)	0.0015	0.0018	0.0023
LR-PW Gamma	0.0408	0.0411	0.0400
(std. err.)	0.0003	0.0004	0.0005
PW-LR Gamma	0.0409	0.0412	0.0400
(std. err.)	0.0003	0.0004	0.0005
PW Rho	38.1664	38.0714	38.4530
(std. err.)	0.1658	0.1891	0.2118
LR Rho	38.2272	38.2628	38.4797
(std. err.)	0.2074	0.2377	0.2711

Option Parameters: S_0=100, K=100, T=1 year, m=4 and monitoring times are {0.25, 0.5, 0.75, 1.0}

where S_i is the stock price at time t_i for a time partition $[0 = t_0 < t_1 < \ldots < t_m = T]$. The knock-out feature makes the payoff of a barrier option discontinuous in the path of the underlying, therefore the PW method is not applicable. But the LR estimators are available, and are given by the product of the discounted payoff function and the score function.

LR estimators:

$$\text{Delta: } e^{-rT}(S_T - K)^+ \mathbf{1}[\max_{1\le i\le m} S_i < H] \times \left(\frac{d_1}{S_0\bar{\sigma}_1\sqrt{\Delta t_1}}\right) \quad (24)$$

$$\text{Gamma: } e^{-rT}(S_T - K)^+ \mathbf{1}[\max_{1\le i\le m} S_i < H] \times \left(\frac{d_1^2 - d_1\bar{\sigma}_1\sqrt{\Delta t_1} - 1}{S_0^2\bar{\sigma}_1^2\Delta t_1}\right) \quad (25)$$

$$\text{Rho: } e^{-rT}(S_T - K)^+ \mathbf{1}[\max_{1\le i\le m} S_i < H] \times \left(-T + \sum_{i=1}^{m} \frac{d_i\sqrt{\Delta t_i}}{\bar{\sigma}_i}\right) \quad (26)$$

where in (24)–(26), $\Delta t_i = t_i - t_{i-1}$,

$$d_i = (\ln(S_i/(S_{i-1}\xi_i)) - (r - \frac{1}{2}\bar{\sigma}_i^2)\Delta t_i)/(\bar{\sigma}_i\sqrt{\Delta t_i}),$$

and $\bar{\sigma}_i^2$ is the variance between t_{i-1} and t_i.

We give the simulation results for this option in Table 4.

Table 4: Simulation Estimates of Price and Greeks for a Discrete Knock-Out Barrier Option

	MODEL		
	SV	SVJ	SVCJ
Simulation Price	5.0481	5.1731	5.2487
(sdt. err.)	0.0291	0.0345	0.0394
LR Delta	0.2499	0.2541	0.2207
(std. err.)	0.0029	0.0044	0.0064
LR Gamma	-0.0605	-0.0655	-0.0659
(std. err.)	0.0017	0.0019	0.0025
LR Rho	21.0779	21.2292	18.2402
(std. err.)	0.1955	0.3719	0.5647

Option Parameters: S_0=100, K=100, T=1 year, H=120, m=4 and monitoring times are {0.25, 0.5, 0.75, 1.0}

4.4 Forward Start Options

We next consider forward start options. These are options whose strike is set at a future date. In particular, if T_1 is the time when strike is set, T_2 is the option expiration, S_i is the stock price at time T_i, and k is the constant that determines the strike, then the forward start option payoff at time T_2 is given by $(S_2 - kS_1)^+$. For example, if $k = 1$ then at time T_1, the option becomes an at-the-money option with expiration T_2. Kruse (2003) develops a closed-form solution for this option under the SV model by integrating the pricing formula with the conditional density of the variance value at time T_1. However, the implementation is not very straightforward since it includes another level of integration to already complex integrals in the Heston formula. Therefore simulation may be considered as an alternative for finding prices and sensitivities of this option.

Again, the LR estimators are easily derived by multiplying the discounted payoff with the score function.

LR estimators:

$$\text{Delta: } e^{-rT_2}(S_2 - kS_1)^+ \times \left(\frac{d_1}{S_0\bar{\sigma}_1\sqrt{\Delta t_1}}\right) \quad (27)$$

$$\text{Gamma: } e^{-rT_2}(S_2 - kS_1)^+ \times \left(\frac{d_1^2 - d_1\bar{\sigma}_1\sqrt{\Delta t_1} - 1}{S_0^2\bar{\sigma}_1^2\Delta t_1}\right) \quad (28)$$

$$\text{Rho: } e^{-rT_2}(S_2 - kS_1)^+ \times \left(-T + \sum_{i=1}^{m} \frac{d_i\sqrt{\Delta t_i}}{\bar{\sigma}_i}\right) \quad (29)$$

where in (27)–(29), $\Delta t_i = t_i - t_{i-1}$,

$$d_i = (\ln(S_i/(S_{i-1}\xi_i)) - (r - \frac{1}{2}\bar{\sigma}_i^2)\Delta t_i)/(\bar{\sigma}_i\sqrt{\Delta t_i}),$$

and $\bar{\sigma}_i^2$ is the variance between t_{i-1} and t_i.

We can use an alternative method for simulating a forward start option that will allow us to get more efficient estimators for the price and sensitivities. Note that at time T_1, we know the stock price, strike and the expiration of the option. Therefore the option price at time T_1 can be written using closed form formulas. Let $C_E(S, K, T)$ denote the price of a European call option with initial stock price S, strike K and time to expiration T. Note also that, the option price is linearly homogenous with respect to the stock price and the strike, i.e., we can write $C_E(S, kS, T) = S\, C_E(1, k, T)$. Using this expression, and following the above arguments, the price of a forward start option can be written as:

$$C_{FW} = E[e^{-rT_1} S_1 C_E(1, k, T_2 - T_1)] \tag{30}$$

Note that $C_E(1, k, T_2 - T_1)$ is not a constant since it depends on the realization of the variance at time T_1. Using (30), we can derive PW derivative estimators for a forward start option.

PW estimators:

$$\text{Delta: } e^{-rT_1} C_E(1, k, T_2 - T - 1) \frac{S_1}{S_0} \tag{31}$$

$$\text{Gamma: } 0 \tag{32}$$

$$\text{Rho: } e^{-rT_1} S_1 \left(\frac{\partial C_E}{\partial r} \right) \tag{33}$$

where in (33), $\partial C_E / \partial r$ is the rho of a European call option and can be evaluated using closed form formulas (see Reiss and Wystup 2001). The PW estimator for gamma in (32) is identically zero since the expression S_1/S_0 in the delta estimator in (31) does not actually depend on S_0.

The simulation results are given in Table 5. The price estimator given in (30) is denoted as Formula Sim. Price. When we use the closed form formulas in the simulation, computing time per simulation increases since a numerical integration is done for each formula. We adjust the number of simulation trials for PW method such that it takes roughly the same amount of time as the LR method. As seen from the results, using closed form formulas decreases the standard error significantly.

Table 5: Simulation Estimates of Price and Greeks for a Forward Start Option

	MODEL		
	SV	SVJ	SVCJ
Formula Sim. Price	6.9688	6.8957	7.0593
(sdt. err.)	0.0086	0.0134	0.0127
Plain Sim. Price	7.0502	6.9145	7.1163
(sdt. err.)	0.0422	0.0516	0.0637
PW Delta	0.0697	0.0690	0.0706
(std. err.)	0.0001	0.0001	0.0001
LR Delta	0.0688	0.0671	0.0677
(std. err.)	0.0014	0.0017	0.0022
PW Gamma	0	0	0
(std. err.)	0	0	0
LR Gamma	0.0000	0.0001	0.0006
(std. err.)	0.0003	0.0004	0.0006
PW Rho	62.4620	62.6915	62.1034
(std. err.)	0.0889	0.1469	0.1502
LR Rho	62.8403	62.3380	61.7331
(std. err.)	0.3621	0.4167	0.4909

Option Parameters: S_0=100, k=1, T_1=1 year, T_2=2 years

5 CONCLUSIONS

In this paper we have derived PW and LR methods for the exact simulation of Greeks under stochastic volatility and jump models. After finding an appropriate set of conditioning variables, we have expressed the stock price as a series of lognormal random variables which allowed us to apply standard techniques of PW and LR methods to derive unbiased simulation estimators. The implementation is based on the exact simulation algorithm derived in Broadie and Kaya (2004). Through this algorithm, we are able to simulate the conditioning variables, such as the variance path and the integral of the variance, exactly. This is critical, because using discretization methods such as Euler or Milstein for this step would introduce discretization bias into the simulation estimates.

We gave the simulation estimators for the Greeks of a European call option, and some path-dependent options including a forward start option. As in the case of forward start options, it is sometimes possible to take advantage of the specific form of the payoff to derive more efficient estimators.

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AUTHOR BIOGRAPHIES

MARK BROADIE is a professor at the Graduate School of Business at Columbia University. He received a B.S. from Cornell University and a Ph.D. from Stanford University. His research focuses on problems in the pricing of derivative securities, risk management, and portfolio optimization. Professor Broadie is editor-in-chief of the Journal of Computational Finance, an associate editor of the Mathematical Finance, Operations Research and Computational Management Science. Broadie has served as a consultant for a number of financial firms and previously he was a vice president at Lehman Brothers in their fixed-income research group. His e-mail address is <mnb2@columbia.edu>, and his web page is <www.columbia.edu/~mnb2/broadie>.

ÖZGÜR KAYA is a Ph.D. candidate at the Department of Industrial Engineering and Operations Research at Columbia University. He received a B.S. in Industrial Engineering (2000) from Bilkent University in Turkey, and a M.S in Operations Research (2001) from Columbia University. His main research interests are applications of Monte Carlo simulation and other computational methods to financial engineering problems. His e-mail address is <ok94@columbia.edu> and his webpage is <www.columbia.edu/~ok94>.