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PIECEWISE-QUADRATIC RATE SMOOTHING: THE CYCLIC CONTEXT

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ABSTRACT

Even when they are known to be continuous, Poisson-process rate functions are sometimes specified as piecewise constant. To better approximate the unknown continuous rate function, we fit a piecewise-quadratic function. In addition to maintaining the rate's integral over each time interval, at each interval's end point we match the rates and their first derivatives. For every interval with negative rates, we force non-negativity by taking the maximum of zero and the quadratic-function value, modifying the quadratic to maintain the integral value. These rate functions can be used alone or applied after one or more iterations of I-SMOOTH, our existing algorithm designed for the same problem. We provide examples. Finally, we discuss random-process generation from piecewise-quadratic rate functions.

1 INTRODUCTION

Because of both easy statistical estimation and easy random-variate generation, Poisson rate functions are often specified as being piecewise constant. Often, however, the unknown rate function is known (or thought to be) continuous. Schmeiser et al. (2003) first considered the problem of better estimating, given only the piecewise-constant function, the unknown rate function. We follow their primary problem assumption, which is to ignore the statistical quality of the piecewise-constant function, which depends upon the method of estimation and the amount of real-world data used. That is, we take each interval's constant rate as given, and require the interval's updated rate function to integrate to the constant. Being a rate function, the updated function must also be non-negative at every point in time.

1.1 Problem statement

We take as given the piecewise-constant rate function: λ_i for the time interval $(t_i, t_{i+1}]$ for $i = 1, 2, \ldots, k$. (Here and throughout, a *piece* and an *interval* are identical.) For simplicity, throughout we assume that each time interval has length one, with $t_i = i$, so the rate function ranges from time 0 to time k. Chen and Schmeiser (2011, 2013) consider two contexts: finite horizon, where time ends at time k, and cyclic, where times extends to infinity and λ_i applies to every time in $(t_i + jk, t_{i+1} + jk]$, where $j = 0, 1, \ldots$. Here, we consider only the cyclic context, so the given piecewise-constant rate function is

$$\lambda(t) = \lambda_i$$

where $j = \lfloor t/k \rfloor$ is the number of previous cycles and $i = \max\{1, \lceil t - jk \rceil\}$ is the interval number.

The problem is to return a better rate function, which requires a definition of *better*. We have no evaluation metric for general rate functions, which might be discontinuous or have times at which the first or second derivatives do not exist. Therefore, our problem definition is ambiguous. Nevertheless, many rate functions are an obvious improvement to a piecewise-constant function.

1.2 Solution Approach

Our solution approach is to fit a piecewise-quadratic function to each interval $i = 1, 2, ..., k$. To begin, consider only the unit interval $[0, 1]$. At time x, the rate is

$$q(x; a, b, c) = ax^2 + bx + c.$$

Define $\underline{a} = (a_1, a_2, ..., a_k)$ and analogously $\underline{b}$ and $\underline{c}$. Then the cyclic piecewise-quadratic rate function is

$$\tau(t; \underline{a}, \underline{b}, \underline{c}) = q(x; a_i x^2 + b_i x + c_i),$$

where again $j = \lfloor t/k \rfloor$, $i = \max\{1, \lceil t - jk \rceil\}$, and $x = t - jk - i + 1$ is the fractional time.

Thus, our specific problem is to determine values for the coefficients $\underline{a}$, $\underline{b}$, and $\underline{c}$ that lead to a better rate function than the given piecewise-constant function. Lacking a definition of *better*, we fit the $3k$ coefficients by matching the k integrals, matching rates at the k interval endpoints, and matching the rates' first derivatives at the k endpoints. We consider this fitting problem in Section 3.

Because the fitted piecewise-quadratic function τ with coefficients $\underline{a}$, $\underline{b}$, and $\underline{c}$ might be negative at some times t, a solution approach must somehow adjust the fitted piecewise-quadratic function. Our approach is to replace each interval's quadratic function with

$$q^+(x; a^+, b^+, c^+) = \max\{0, a^+ x^2 + b^+ x + c^+\},$$

which by definition forces non-negativity. The non-negative rate function is then

$$\tau^+(t; \underline{a}^+, \underline{b}^+, \underline{c}^+) = q(x; a_i^+ x^2 + b_i^+ x + c_i^+),$$

where j, i,and x are defined as before. When τ is entirely nonnegative, $\tau^+ = \tau$. When τ has some negative rates, however, including the maximum function requires updated values $\underline{a}^+$, $\underline{b}^+$, and $\underline{c}^+$ to maintain the required integral for each interval i. Computation of $\underline{a}^+$, $\underline{b}^+$, and $\underline{c}^+$ is discussed in Section 4.

1.3 Organization of paper

Following a literature review in Section 2, Sections 3 and 4 contain details about fitting piecewise-quadratic function τ and the corresponding non-negative rate function τ^+. Numerical examples, with graphs, are provided in Section 5. Finally, logic for Poisson random-process generation is presented in Section 6. In the appendix, for each interval we match the integral and specified end-point rates, in case some future research provides such end-point rates.

2 LITERATURE REVIEW

Chen and Schmeiser (2011, 2013) introduce the algorithm I-SMOOTH, which iteratively smooths any piecewise-constant function by bisecting time intervals to obtain an updated piecewise-constant function with twice as many pieces. At each iteration, each interval's integral is maintained by increasing (decreasing) the left half's rate while decreasing (increasing) the right half's rate. The amount of increase and decrease is chosen to minimize the sum of second differences of the rates. At each iteration, negative rates are avoided by limiting the increase and decrease to being no more than the current rate.

As defined in Chen and Schmeiser (2011, 2013), I-SMOOTH has no stopping rule. Rather, the user can stop iterating whenever the improvement in quality of the resulting piecewise-constant rate function has less benefit than the computational (time and storage) costs of doubling the number of intervals.

The final piecewise-constant rate function can be used directly for process generation or can be modified as desired. As discussed in Schmeiser et al. (2003), each interval's rate could be pivoted about its center to obtain a better (but not continuous) piecewise-linear rate function while maintaining the interval's integral.

Alternatively, the two algorithms in Nicol and Leemis (2014a) could be used to obtain a continuous piecewise-linear rate function.

The piecewise-linear algorithms in Nicol and Leemis (2014a) and our piecewise-quadratic algorithm can be applied with no iterations of I-SMOOTH. Alternatively, since I-SMOOTH always returns a piecewise-constant function, these algorithms can be used as an I-SMOOTH post-processor.

Nicol and Leemis (2014a) consider the same problem as this paper. Their piecewise-linear functions are nonnegative and the rate's integral over each time interval matches the corresponding given rate. They develop two algorithms that maintain five desirable features: equal slopes for each time interval if the specified rates have a constant difference, as few parameters as possible, time reversible, as *smooth* as possible, and as few line segments as possible. Both algorithms begin by computing the rate p_i at time i, where $i = 0,\ 1, \ldots,\ k$; the points (i, p_i) are called knot points. For each time interval, the fitted linear function is the line connecting the knot points at two ends of the interval if the line maintains the integral. Otherwise, more line segments are created by adding knot points within the time interval. Algorithms 1 and 2 differ in the logic of setting the knot points at end points of time intervals and adding knot points when necessary. The authors recommend their Algorithm 2 over their Algorithm 1.

When event-count data are available, Leemis (2004) considers point and interval estimators of the cumulative rate function (that is, the expected number of events) at the interval endpoints. Our problem differs in that we do not consider statistical properties; rather, we match the expected number of events for each interval while providing a piecewise-quadratic (rather than a piecewise-constant) rate function.

Unlike our assumption of beginning with only the k piecewise-constant rates, other literature assumes the richer situation of having the original Poisson event times. Examples include Kuhl and Wilson (2000, 2001) and Leemis (1991) and Arkin and Leemis (2000). Saltzman et al. (2012) generalize to multivariate Poisson processes. Generalizing beyond Poisson processes, Liu (2013) models nonstationary non-Poisson processes that have a given mean-value function but also allows the ratio of mean and variance to take values other than one. Chen and Schmeiser (1992) consider process generation from trigonometric rate functions using the inverse transformation.

3 FITTING PIECEWISE-QUADRATIC FUNCTIONS

As before, let the rate λ_i be the rate over the time interval $(t_{i-1}, t_i]$ for $i = 1, 2, \ldots, k$. We now consider fitting quadratic functions to each interval while maintaining each interval's rate integral, λ_i.

Ideally, we would identify an objective function subject to the k rate-integral constraints. Chen and Schmeiser (2011,2013) use a sum of squared second differences to evaluate piecewise-constant rate functions for piecewise-constant functions. For continuous rate functions, the analogous criterion is

$$z = \int_{t_0}^{t_k} \left[\frac{d^2\tau(t)}{dt^2} \right]^2 dt. \tag{1}$$

For a piecewise-quadratic function over k unit intervals, the second derivative over the entire ith interval is $2a_i$; therefore $z = 4\sum_{i=1}^{k} a_i^2$. So fitting a piecewise-quadratic function seems to be as easy as setting $a_i = 0$, $b_i = 0$ and $c_i = \lambda_i$ for $i = 1, 2, ..., k$. The flaw, of course, is that then the second derivatives do not exist at the interval endpoints. Worse, the function is not continuous at the interval endpoints.

In the hierarchy of criteria, continuity seems highest, then first derivatives, and then higher-order derivatives. We have $3k$ coefficients to determine, so in addition to matching the k intervals' integrals, we fit the piecewise-quadratic function to maintain continuity and first-derivative values at the k intervals' endpoints. We define the fitting equations in Section 3.1 and and then numerically determine the parameter values needed to compute the fit in Section 3.2.

3.1 The fitting equations

The immediate problem is to determine values for the coefficients (a_i, b_i, c_i) for $i = 1, 2, ..., k$ to maintain the k integrals, match the k end-points' function values, and match the k end-points' first derivatives. We write and solve $3k$ simultaneous linear equations that specify the $3k$ coefficient values.

Recall that we assume unit-length intervals, so $t_i = i$ for $i = 1, 2, \ldots, k$. Also we assume the cyclic context, where $\lambda(t) = \lambda(t + jk)$, where j is an integer. If incrementing subscript i makes it larger than k, then interpret it as $i = 1$. For fitting only, we assume that time t lies in $[0, k]$, so $j = 0$. We ignore the nonnegativity constraint, but do consider nonnegativity in Section 4.

For simplicity of computation, we write the piecewise-quadratic rate function in such a way that we can think of every time interval as $[0, 1]$. As before, let $i = \max\{1, \lceil t \rceil\}$ denote the interval into which time t falls, so that $x = t - i + 1$ is the fractional time within that unit interval. Then, the ith interval's quadratic rate is $q(x; a_i, b_i, c_i)$.

There are three types of conditions: maintain the rate's integral, maintain rate continuity, and maintain continuity of the first derivatives. For every interval $i = 1, 2, \ldots, k$ maintaining the rate's integral requires $\int_0^1 a_i x^2 + b_i x + c_i dx = \lambda_i$, which simplifies to

$$a_i/3 + b_i/2 + c_i = \lambda_i; \tag{2}$$

maintaining continuity requires $a_i x^2 + b_i x + c_i|_{x=1} = a_{i+1} x^2 + b_{i+1} x + c_{i+1}|_{x=0}$, which simplifies to

$$a_i + b_i + c_i = c_{i+1}, \tag{3}$$

and maintaining continuity of the first derivatives requires $2a_i x + b_i|_{x=1} = 2a_{i+1} x + b_{i+1}|_{x=0}$, which simplifies to

$$2a_i + b_i = b_{i+1}. \tag{4}$$

We then rewrite Equations (2) to (4) in matrix form as

$$MC = \Lambda, \tag{5}$$

where $C^{3k\times 1} = (a_1, b_1, c_1, \ldots, a_k, b_k, c_k)^T$, $\Lambda^{3k\times 1} = (\lambda_1, 0, 0, \lambda_2, 0, 0, \cdots, \lambda_k, 0, 0)^T$, and

$$M^{3k\times 3k} = \begin{bmatrix} S_1 & S_2 & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & S_1 & S_2 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & S_1 & S_2 \\ S_2 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & S_1 \end{bmatrix}$$

with submatrices

$$S_1^{3\times 3} = \begin{bmatrix} 1/3 & 1/2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad S_2^{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{0}^{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The root of Equation (5) is

$$C = M^{-1}\Lambda.$$

Because M is a block-circulant matrix, its inverse matrix M^{-1} can be computed efficiently using methods such as in De Mazancourt and Gerlic (1983).

3.2 Numerical solutions

The $3k$ equations in Section 3.1 can be solved by matrix inversion for the unique piecewise-quadratic rate function, τ. Here we numerically solve the $3k$ equations for $k = 1,2,\ldots,7$ by computing the matrix inverse. From those solutions we estimate the asymptotic solutions, which can be used for $k > 7$. (The choice of seven was be large enough to establish a pattern and approximate asymptotic results.)

For $k = 1$, trivially $a_1 = 0$, $b_1 = 0$, and $c_1 = \lambda_1$ which yields $\tau(t) = \lambda_1$. Examining the solutions for $k = 2,3,\ldots,7$ quickly yields the form of the solutions for $k > 1$. The solutions for the second-order coefficients are of the form

$$a_i^{(k)} = \alpha_0^{(k)}\lambda_i + \sum_{j=1}^{\lceil k/2 \rceil} \alpha_j^{(k)}(\lambda_{i-j} + \lambda_{i+j}),$$

with α_0 negative, the α values oscillating in sign and decreasing in magnitude, and the sum of all λ coefficients being zero. The solutions for the first-order coefficients are of the form

$$b_i^{(k)} = \sum_{j=0}^{\lceil k/2 \rceil} \beta_j^{(k)}(\lambda_{i+j} - \lambda_{i-j-1}),$$

with $\beta_0 = -\alpha_0$ and the β values oscillating in sign and decreasing in magnitude by roughly a factor of four. The solutions for the constant coefficients are of the form

$$c_i^{(k)} = \sum_{j=0}^{\lceil k/2 \rceil} \gamma_j^{(k)}(\lambda_{i+j} + \lambda_{i-j-1}),$$

with γ_0 positive and the γ values oscillating in sign and decreasing in magnitude by about a factor of four. The sum of λ coefficients is one.

In addition, as k increases $\alpha_j^{(k)}$, $\beta_j^{(k)}$, and $\gamma_j^{(k)}$ converge to their asymptotic counterparts quickly. In particular, informal extrapolation suggests that a good approximation for $k > 8$ is

$$a_i^{(k)} = -2.196 \times \lambda_i + 1.392 \times (\lambda_{i-1} + \lambda_{i+1}) - 0.367 \times (\lambda_{i-2} + \lambda_{i+2}) + 0.073 \times (\lambda_{i-3} + \lambda_{i+3}),$$

$$b_i^{(k)} = 2.196 \times (\lambda_i - \lambda_{i-1}) - 0.5852 \times (\lambda_{i+1} - \lambda_{i-2}) + 0.1463 \times (\lambda_{i+2} - \lambda_{i-3}))$$

and

$$c_i^{(k)} = 0.6339 \times (\lambda_i + \lambda_{i-1}) - 0.1697 \times (\lambda_{i+1} + \lambda_{i-2}) + 0.0458 \times (\lambda_{i+2} + \lambda_{i-3}) - 0.01 \times (\lambda_{i+3} + \lambda_{i-4}).$$

Using these asymptotic approximations yields a piecewise-quadratic rate function in computing time that grows linearly with k. More recently, we have solved analytically for all values of k.

4 MAINTAINING NONNEGATIVITY

From Section 3, we know the piecewise-quadratic function τ, which is specified with the $3k$ coefficients (a_i, b_i, c_i) for $i = 1,2,...,k$. If the function values $\tau(t)$ are nonnegative for all times t, then the piecewise-quadratic function τ can be used as a rate function.

If there are times for which $\tau(t)$ are negative, then our solution is to replace τ with τ^+, where the ith interval uses the rate function $q^+(x) = \max\{0, a_i^+ x^2 + b_i^+ x + c_i^+\}$ and x is the fractional time within the ith interval. When convenient, we refer to (a_i^+, b_i^+, c_i^+) as the updated values of (a_i, b_i, c_i).

4.1 Single-interval updating

Our objective is to choose updated coefficients to maintain each interval's integral with minimal disturbance of continuity and first derivatives. Fortunately, we can think of each interval separately, at least for now. And we only need to consider intervals for which some rates are negative for its quadratic-function coefficients (a,b,c). For the interval of interest, the left- and right-endpoint rates are $q_0 = c$ and $q_1 = a+b+c$.

In addition, define g_0 and g_1 to be the first-derivative values at the left and right endpoints. Before updating, first derivatives are continuous across intervals. But after updating, the first-derivative values might not match those of the adjacent intervals. When these first-derivative values differ between the interval of interest and the adjacent interval, the value should be taken from the adjacent interval.

We need to consider four cases: (1) both q_0 and q_1 negative, (2) only q_0 negative, (3) only q_1 negative, and (4) both q_0 and q_1 nonnegative. For every case, we maintain the integral condition. Therefore, for each case we need to decide which two other conditions to maintain. For Case 4, continuity at both end points is paramount, so the updated coefficients (a^+,b^+,c^+) are chosen to satisfy $c^+ = q_0$ and $a^+ + b^+ + c^+ = q_1$. For Case 3, maintaining the negative right-endpoint rate is unimportant, so the updated coefficients are chosen to maintain the left-endpoint's function value and first derivative; that is, $c^+ = q_0$ and $b^+ = g_0$. Analogously, for Case 2 maintaining the negative left-endpoint rate is unimportant, so the updated coefficients are chosen to maintain the right-endpoint's function value and first derivative; that is, $a^+ + b^+ + c^+ = q_1$ and $2a^+ + b^+ = g_1$. Because both endpoint rates are negative, conditions for Case 1 are less obvious; in light of the lack of important conditions, we choose to maintain continuity, leading to the Case 4 update.

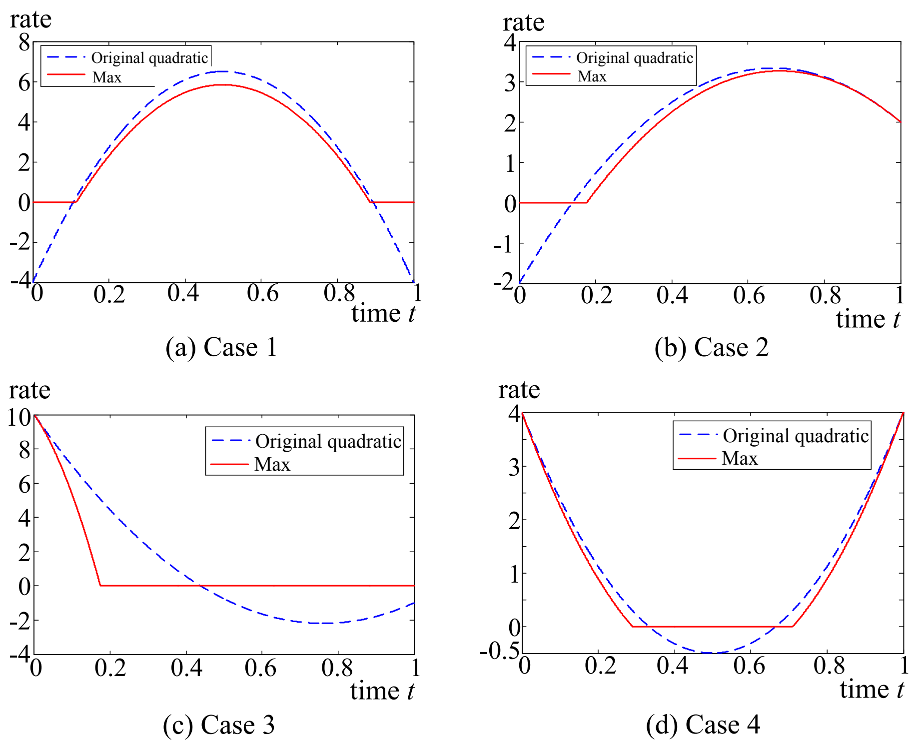


Figure 1: Plots of the original quadratic and max functions for the four cases

Figure 1 illustrates the four cases in Subfigures (a) to (d). In each subfigure, the original quadratic function is plotted as the dashed curve and the updated max function is plotted as the solid curve. For Case 1, shown in Figure 1(a), the original quadratic function is $-42x^2 + 42x - 4$ and the updated max function is $\max\{0, -39.3641x^2 + 39.3641x - 4\}$. For Case 2, shown in Figure 1(b), the original quadratic function is $-12x^2 + 16x - 2$ and the updated max function is $\max\{0, -12.6757x^2 + 17.3513x - 2.6757\}$. For Case

3, shown in Figure 1(c), the original quadratic function is $21x^2 - 32x + 10$ and the updated max function is $\max\{0, -145.5505x^2 - 32x + 10\}$. For Case 4, shown in Figure 1(d), the original quadratic function is $18x^2 - 18x + 4$ and the updated max function is $\max\{0, 19.4574x^2 - 19.4574x + 4\}$. In Cases 1, 2, and 4, the difference between the values of a and a^+ is small. Case 3, however, is interesting in that the values differ greatly; in fact, the signs of a and a^+ differ so the original quadratic function is convex but the updated max function is concave.

We solve for the updated coefficients (a^+, b^+, c^+) using the secant method (Conte and deBoor 1980). The one-dimensional search seeks the value of a^+ that returns the specified integral value λ. The search is one dimensional because (for every case) specifying a value for a^+ implies the values of b^+ and c^+. The search uses a as the initial guess. Practical convergence requires only a few iterations.

4.2 Linking multiple intervals

The single-interval coefficient updating of Section 4.1 never changes nonnegative endpoint rates, so continuity of the rate function τ^+ is unaffected. Nevertheless, some care must be taken about the order of updating, because in Case 4 the values of the endpoint first derivatives g_0 and g_1 change. To maintain first-derivative continuity, first update Case 4 intervals, then Case 2 and 3 intervals, and finally Case 1 intervals.

5 NUMERICAL EXAMPLES

We use four examples to illustrate piecewise-quadratic rate-functions across multiple intervals. The first example uses real-world call-center data, where negativity is not an issue, from Nicol and Leemis (2014a). For the other three examples, we intentionally create extreme cases with negative rates in the original fitted piecewise-quadratic function. For all four examples, we compare the piecewise-quadratic max rate function with I-SMOOTH after seven iterations, which results in essentially the asymptotic I-SMOOTH function.

Each example is illustrated in a figure, with three subfigures showing (a) the given rates, (b) the fitted max function (and, if different, the original piecewise-quadratic function), and (c) or (d) the fitted piecewise-constant rate function computed via I-SMOOTH, respectively. For Example 1 only, Figure 2(c) also shows the fitted piecewise-linear rate function via Algorithm 2 in Nicol and Leemis (2014a).

In Example 1, the call-center data from Nicol and Leemis (2014a), the thirteen rates are $\lambda_1 = 68.6$, $\lambda_2 = 126.0$, $\lambda_3 = 140.2$, $\lambda_4 = 139.4$, $\lambda_5 = 125.2$, $\lambda_6 = 115.0$, $\lambda_7 = 126.8$, $\lambda_8 = 140.2$, $\lambda_9 = 140.0$, $\lambda_{10} = 119.4$, $\lambda_{11} = 100.6$, $\lambda_{12} = 70.4$, and $\lambda_{13} = 70.2$, as plotted in Subfigure 2(a). Subfigures 2 (b) to (d) show the fitted piecewise-quadratic, piecewise-linear from Algorithm 2 of Nicol and Leemis (2014), and I-SMOOTH (after seven iterations, so $13 \times 2^7 = 1664$ intervals) piecewise-constant rate functions, respectively.

Maybe the most surprising observation is the similarity between the piecewise-quadratic and I-SMOOTH rate functions. Differences are quite minor, such as in the first two peak rates around time 2 and time 4. Although not shown here, after the first two or three iterations, the I-SMOOTH visual results change little.

Example 2, based on the six rates 12, 1, 0.2, 1, 11, and 1, is illustrated in Figure 3. The substantial variation in the rates, with some being close to zero, forces negative values in the fitted piecewise-quadratic function. As in Example 1, subfigures (a), (b), and (c) show the given rates, the piecewise-quadratic rates and the I-SMOOTH (after seven iterations). Subfigure (b), however, now shows two functions, both the original piecewise-quadratic function (dashed curve), which ignores negativity, and the updated rate function (solid curve), which is the greater of zero and the updated quadratic function. Subfigure (b) shows that the updated quadratics are lower than the original quadratics, which occurs because taking the maximum of zero and the original quadratic always increases the integral.

Example 2 has been constructed to illustrate all four cases of Section 4.1. In particular, intervals 1 through 6 illustrate cases 0, 3, 1, 2, 0, 4 respectively. Here, we have used Case 0 to refer to intervals with no negative rates and therefore no updated rate function. Despite negativity being a dominant issue in this example, the differences between values of a_i and a_i^+ are not great.

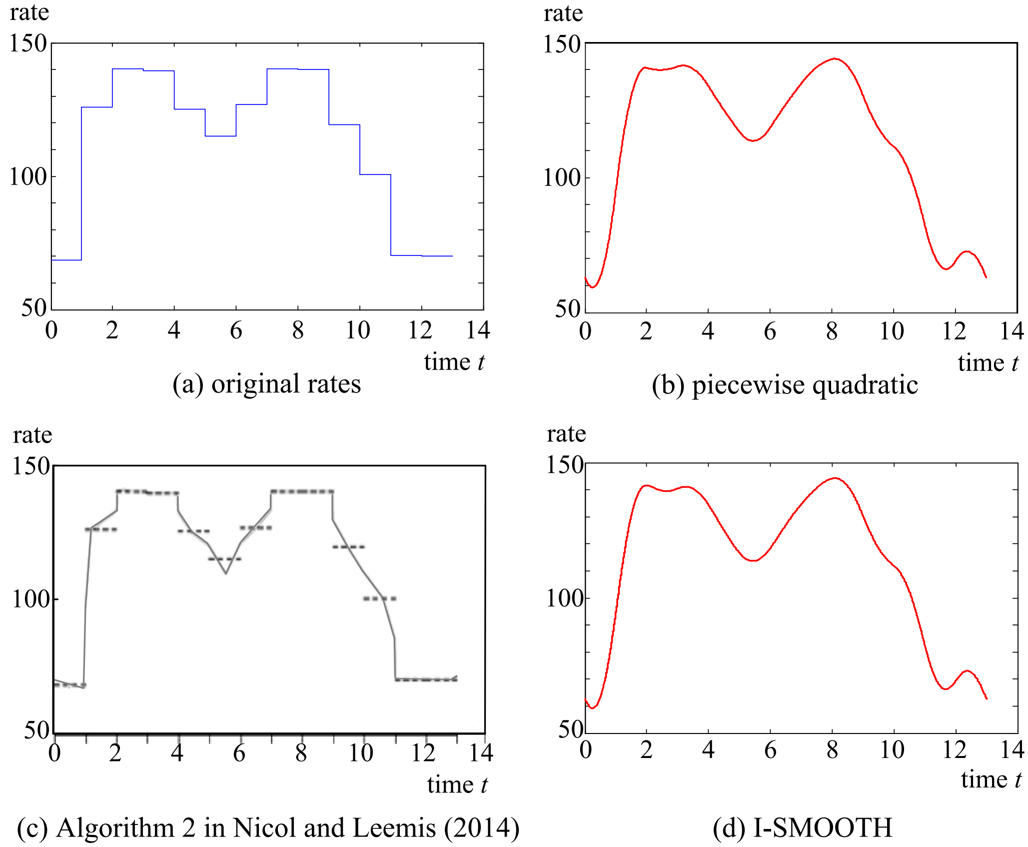


Figure 2: Example 1. Based on the call-center data from Nicol and Leemis (2014a), plots of the original rates, piecewise-quadratic rates, piecewise-linear rates from Nicol and Leemis (2014a), and I-SMOOTH (after seven iterations) piecewise-constant rates in subfigures (a) to (d), respectively.

Examples 3 and 4, also compare I-SMOOTH to taking the maximum of zero and a piecewise-quadratic function. Figures 4 and 5 are the analogous to Example 2. The resulting rate functions again are similar. In Example 3, however, the max function spreads the nonnegative rates over a longer time interval. The reverse is true in Example 4.

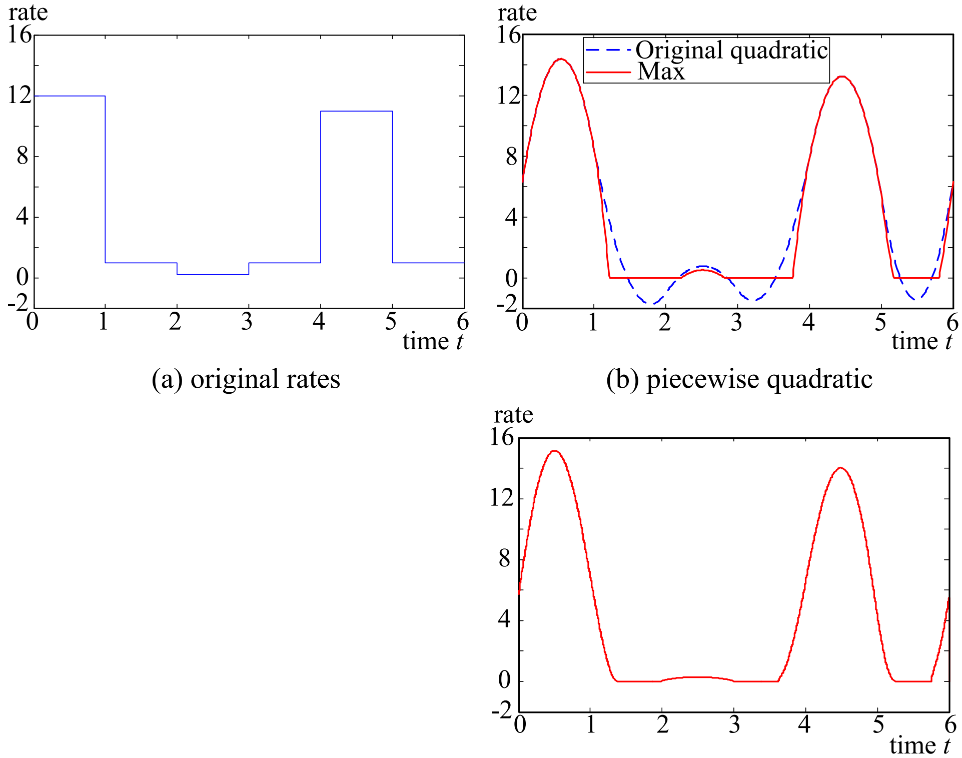
6 POISSON PROCESS GENERATION

Piecewise-quadratic rate functions have the disadvantage, compared to piecewise-linear or piecewise-constant rate functions, that random-variate generation is more complicated and slower. If thinning is used then the penalty in complication and speed is minimal. To use the inverse transformation of the cumulative rate function, however, involves inverting a cubic function.

We state here the inverse-transformation logic to generate the next time, t^*, given the previous time, t_p. In general, given any nonnegative Poisson rate function τ and time t_p, the unique time t^* of the next event satisfies $\int_{t_p}^{t^*} \tau(t)dt = y$, where y is a mean-one independent exponential random variate. Klein and Roberts (1984) state this approach for piecewise-linear rate functions.

In the logic below, we specialize the inverse-cdf algorithm to generate, given the previous-event time t_p, the next-event time t^* from the cyclic piecewise-quadratic rate function $\tau(t,\underline{a},\underline{b},\underline{c})$.

If there are negative rates, the rate function becomes $\tau^+(t,\underline{a}^+,\underline{b}^+,\underline{c}^+) = \max\{0, a_i^+ x^2 + b_i^+ x + c_i^+\}$. When considering τ^+ rather than τ, the primary difficulty remains the same: a cubic function needs to be inverted. The integral of τ is piecewise cubic. In Step 5 below, the closed-form inversion logic (Abramowitz



(a) original rates (b) piecewise quadratic (c) I-SMOOTH

Figure 3: Example 2. Given the six rates 12, 1, 0.2, 1, 11, and 1, plots of (a) the original rates, (b) the piecewise-quadratic and corresponding max with zero, and (c) I-SMOOTH (after seven iterations) rates.

and Stegun, 1972, page 17) is embedded in the function `cubic0`. Because rates are nonnegative, there is a unique positive real root. Here we ignore negativity, which would require additional bookkeeping.

`Given`: The previous time, t_p, and the rate-function parameters (a_i, b_i, c_i) for $i = 1, 2, \ldots, k$.
`Generate`: The next time, t^*.
`Logic`:

1. `Generate an exponential random variate,` y`, with mean one.`
   (a) `Generate an independent` $\mathrm{U}(0,1)$ `random number,` u.
   (b) `Set` $y = -\ln(1-u)$.
2. `Compute the location, within the cycle, of the previous time,` t_p.
   (a) `The number of previous cycles:` $j = \lfloor t_p/k \rfloor$.
   (b) `The interval number:` $i = \max\{1, \lceil t_p - j \times k \rceil\}$.
   (c) `The previous time within its interval:` $x = t_p - j \times k - (i-1)$.
3. `Compute, for interval` i`, the rate integral and the integrals to the left of and right of` t_p.
   (a) $s = a_i/3 + b_i/2 + c_i$
   (b) $s_l = x \times (x \times (x \times (a_i/3) + (b_i/2)) + c_i)$
   (c) $s_r = s - s_l$
4. `Search for the interval` i `that contains the next event time,` t^*.
   `If` $y \le s_r$ then
      $y = s_l + y$
   `Else`

(a) $s = s_r$
(b) `While` $y > s$
  i. `Compute remaining exponential area:` $y = y - s$.
  ii. `Move to the next interval:` $i = i + 1$. `If` $i > k$ `then` $j = j + 1$, $i = 1$.
  iii. `Compute the area of interval` i: $s = a_i/3 + b_i/2 + c_i$.
`EndWhile`
`EndIf`

5. `Solve for time` x `in interval` i; `return the corresponding event time` t^*.
   (a) `call` cubic0(a_i, b_i, c_i, y, x)
   (b) $t^* = j \times k + (i-1) + x$
   (c) `return` t^*

7 DISCUSSION

We model nonhomogeneous Poisson rate functions using the maximum of zero and a piecewise-quadratic function. This suggestion provides an alternative, or a post processor, to the piecewise-constant rate function obtained with the authors' previous algorithm I-SMOOTH. I-SMOOTH has the advantage of minimizing a stated objective function, but the disadvantage of doubling the number of intervals, and therefore doubling the number of coefficients, at each iteration. In our examples, the obtained rate functions are similar.

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APPENDIX: FITTING WITH GIVEN ENDPOINT RATES

Another approach to fitting a piecewise-quadratic function is to provide desirable end-point rates between all adjacent intervals. Various heuristics for such values are reasonable. For example, Nicol and Leemis (2014a) use the average of the adjacent rates. Or maybe the asymptotic I-SMOOTH rates could be found.

Assume that the desirable end-points rates are known; call them $\tau_1, \tau_2, \ldots, \tau_k = \tau_0$. For interval i, our goal is to fit a quadratic function $q(t) = a_i t^2 + b_i t + c_i$ to satisfy three conditions: $q(i-1) = \tau_{i-1}$, $q(i) = \tau_i$, and $\int_{i-1}^{i} q(t)dt = \lambda_i$. Fitting is easy because the quadratic coefficients (a_i, b_i, c_i) are independent of other intervals' coefficient values.

Lemma 1 provides the coefficients for x in the unit interval.

Lemma 1 The unique quadratic function $q(x) = ax^2 + bx + c$ that satisfies $\int_0^1 q(x)dx = \alpha$, $q(0) = \beta$, $q(1) = \gamma$ is $a = 3[\beta + \gamma - 2\alpha]$, $b = 2[3\alpha - 2\beta - \gamma]$ and $c = \beta$.

To apply Lemma 1 to interval i, let $x = t - i + 1$, the fractional part of time t. Then we fit $q(x) = a_i x^2 + b_i x + c_i$ to satisfy $q(0) = \tau_{i-1}$, $q(1) = \tau_i$, and $\int_0^1 q(x)dx = \lambda_i$.

Result 1 For interval i, the unique quadratic function that satisfies $q(0) = \tau_{i-1}$, $q(1) = \tau_i$, and $\int_0^1 q(x)dx = \lambda_i$ is $c_i = \tau_{i-1}$, $b_i = 2[3\lambda_i - \tau_i - 2\tau_{i-1}]$, and $a_i = 3[\tau_{i-1} + \tau_i - 2\lambda_i]$.

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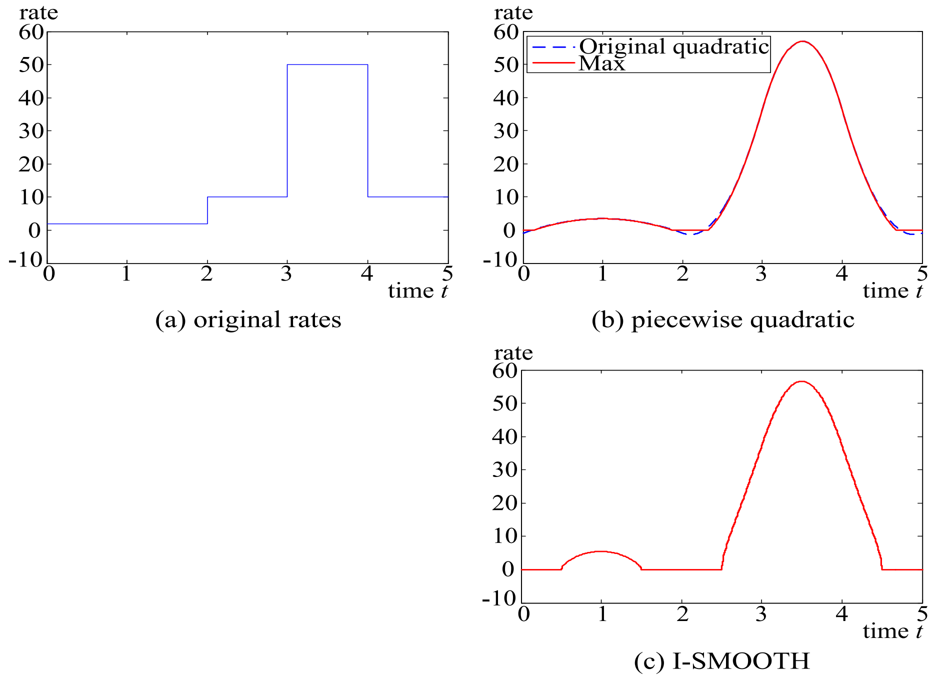

Figure 4: Example 3. The original rates (2, 2, 10, 50, 10) and fitted rate functions by the piecewise-quadratic and I-SMOOTH (after seven iterations) methods in subfigures (a) to (c), respectively.

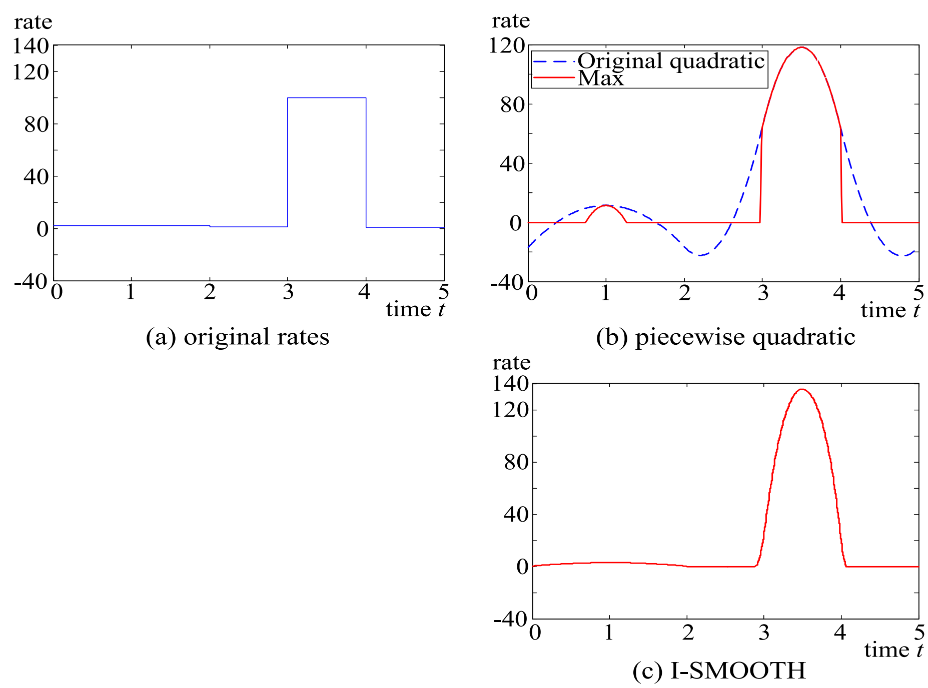

Figure 5: Example 4. The original rates (2,2,1,100,0.5) and fitted rate functions by the piecewise-quadratic and I-SMOOTH (after seven iterations) methods in subfigures (a) to (c), respectively.