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REMOVING THE INHERENT PARADOX OF THE BUFFON'S NEEDLE MONTE CARLO SIMULATION USING FIXED-POINT ITERATION METHOD

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ABSTRACT

In teaching simulation, the Buffon's needle is a popular experiment to use for designing a Monte Carlo simulation to approximate the number π. Simulating the Buffon's needle experiment is a perfect example for demonstrating the beauty of a Monte Carlo simulation in a classroom. However, there is a common misconception concerning the Buffon's needle simulation. Erroneously, the simulation of the needle drop cannot be used to evaluate π. We have to simulate the needle's angle from an uniform $\left(0, \frac{\pi}{2}\right)$ distribution. It is self-referential in theory, since it requires the number π as the input value to approximate π. In this study, we propose a new method using the fixed-point iteration to remove the inherent paradox of the Buffon's needle simulation. A new algorithm with Python implementation is proposed. The simulation outputs indicate that our new method is as good as if we use the true π value as an input.

1 INTRODUCTION

Buffon's needle was the earliest problem in geometric probability to be introduced and solved by Buffon (Buffon 1777). In teaching simulation, it is a very popular experiment to use for designing a Monte Carlo simulation to approximating the π value. No doubt, simulating the Buffon's needle experiment is perfect example for demonstrating the power and beauty of the Monte Carlo simulation in a classroom. However many researchers do not realize that there is a common misconception about the Buffon's needle simulation (Siniksaran 2008; Wicklin 2014; Weisstein 2014; Xie 2013). In the Buffon's needle simulation model, people erroneously use the true π value to simulate the needle's angles with lines. These angles have to be generated from a uniform distribution over an interval $\left(0, \frac{\pi}{2}\right)$. In theory, we cannot simulate the Buffon's needle experiment to approximate the π value. It does required the true π value as the input value to approximate π value.

A new method is proposed for simulating the Buffon's needle experiment without using the true π value as the input. Instead, we use an estimated π value as the input. Our method is based on the convergence theory of the Fixed-Point iteration method. Of course, there are numbers of methods for estimating π (Beckmann 1974) that could be used here. The Fixed-Point iteration is a common numerical method for root funding. In teaching Numerical Analysis, Senior Seminar, and Undergraduate Research, we use the Buffon's needle experiment as an example to introduce the Fixed-Point iteration method . The Buffon's needle simulation model can be fitted into the Fixed-Point iteration model. In this study, we construct and derive a convergence interval for the initial π value selection in order to guarantee our simulation output π values converge to the true π value.

A Python implementation program of the new method is proposed. The simulation results show that the new method is as good as if the true π value is used as the input.

One surprising result is that we can use any arbitrary initial π values as the input for our simulation experiment. The output performance is the same as that of others. Our conjecture is that our simulation output π values converge to the true π value for any arbitrary initial π values except zero value. We will continue to investigate this issue in our future study.

2 BUFFON'S NEEDLE PROBLEM

In this section, we introduce the Buffon's needle problem. The solution of this problem is derived through basic probability and elementary calculus. Using this theoretical result, we design the Buffon's needle Monte Carlo simulation. A simulation algorithm is proposed.

2.1 Experiment

The Buffon's needle problem was first posed by the French mathematician Georges-Louis Leclerc, Comte de Buffon (Buffon 1733):

> "A large plane area is ruled with equidistant parallel lines, the distance between two consecutive lines of the series being a. A thin needle of length $l < a$. is tossed randomly onto the plane. What is the probability that the needle will intersect one of the lines?"

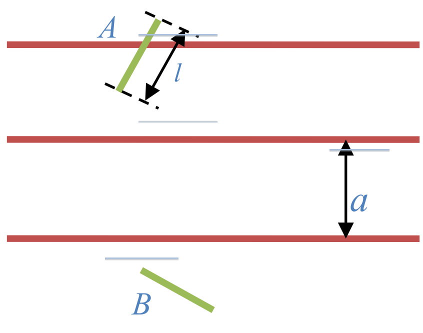

Figure 1: Needle A intersects a line and Needle B does not.

This question became known as the famous Buffon's Needle problem (Buffon 1777; Burton 2007). It can be solved using integral calculus. Based on this analytical solution, the Buffon's needle experiment can be used to approximate the number π.

2.2 Integral Calculus Solution

For a given needle of length l, we model the dropping of the needle on the ruled plane with parallel lines a units apart as follows. Let x be the distance from the center of the needle to the nearest line and θ be the acute angle between the needle and the lines.

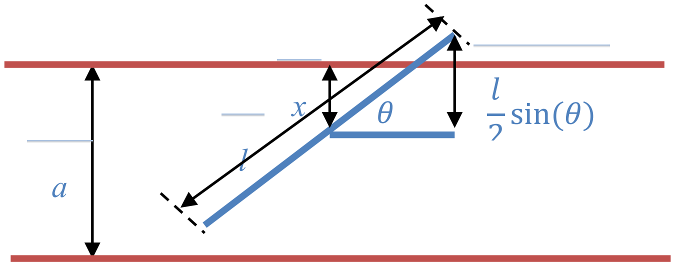

Figure 2: x is the distance from the needle center to the nearest line and θ is the acute angle between the needle and the line.

Based on this model setting, x is a uniform random variable over the interval $\left(0, \frac{a}{2}\right)$ with the probability density function:

$$f_x(x) = \begin{cases} \frac{2}{a}, & 0 \leq x \leq \frac{a}{2} \\ 0, & \text{otherwise} \end{cases}$$

and θ is a uniform random variable over the interval $\left(0, \frac{\pi}{2}\right)$ with the probability density function:

$$f_\theta(x) = \begin{cases} \frac{2}{\pi}, & 0 \leq x \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

The two random variables, x and θ, are independent. Therefore the joint probability density function of (x, θ) is

$$f(x, \theta) = \begin{cases} \frac{4}{a\,\pi}, & 0 \leq x \leq \frac{a}{2} \text{ and } 0 \leq \theta \leq \frac{\pi}{2} \\ 0, & \text{othewise} \end{cases}$$

The original Buffon's Needle problem is only posed for the short needle case ($l < a$). A needle intersects a line if

$$x \leq \frac{l}{2}\sin(\theta).$$

The probability that the needle will intersect a line is

$$p = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin(\theta)} f(x, \theta) dx\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin(\theta)} \frac{4}{a\pi}\, dx\, d\theta = \frac{2l}{a\pi}. \tag{1}$$

For the long needle case ($l \geq a$), a needle intersects a line if

$$x \leq \min\left(\frac{l}{2}\sin(\theta), \frac{a}{2}\right).$$

The probability that the needle will intersect a line is

$$p = \int_0^{\frac{\pi}{2}} \int_0^{\min\left(\frac{l}{2}\sin(\theta), \frac{a}{2}\right).} f(x, \theta) dx\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\min\left(\frac{l}{2}\sin(\theta), \frac{a}{2}\right)} \frac{4}{a\pi}\, dx\, d\theta$$

$$= \left[\int_0^{\frac{\pi}{2}}\int_0^{arc\sin\left(\frac{a}{l}\right)} + \int_0^{\frac{\pi}{2}}\int_0^{\frac{a}{2}}\right]\frac{4}{a\pi}\,dx\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{arc\sin\left(\frac{a}{l}\right)}\frac{4}{a\pi}\,dxd\theta + \frac{a}{2}\left[\frac{\pi}{2} - arc\sin\left(\frac{a}{l}\right)\right]$$

$$= \frac{2}{\pi}\,arc\cos\left(\frac{a}{l}\right) + \frac{2l}{a\pi}\left(1 - \sqrt{1 - \left(\frac{a}{l}\right)^2}\right).$$

Combining both short and long needle cases, the theoretical solution for the Buffon's needle problem is

$$p = \begin{cases} \dfrac{2l}{\pi a}, & \text{if } l < a \\ \dfrac{2}{\pi}\,arc\cos\left(\dfrac{a}{l}\right) + \dfrac{2l}{a\pi}\left(1 - \sqrt{1 - \left(\dfrac{a}{l}\right)^2}\right), & \text{if } l \geq a \end{cases}. \qquad (2)$$

2.3 Estimating π

From the equation (2) in the subsection 2.2, the theoretical solution for the short needle case $(l < a)$ is

$$p = \frac{2l}{\pi a}. \qquad (3)$$

Solve equation (3) for π, we have

$$\pi = \frac{2l}{pa}.$$

The *p* value can be estimated through the Buffon's needle experiment. If the experiment contains *n* needles and there are *m* needles intersect lines, then the probability *p* can be estimated by the proportion:

$$\hat{p}_n = \frac{m}{n}.$$

The π value can be directly estimated by

$$\hat{\pi}_n = \frac{2nl}{ma}.$$

2.4 Monte Carlo Simulation Design

The Buffon's needle experiment can be implemented through Monte Carlo simulation. We generate *n* independent needle tosses to estimate the probability that a needle intersects a line. We summarize our simulation design into the following algorithm:

Algorithm 1

```
m = 0;
do i = 1 to n;
    generate x_i ~ uniform(0, a/2);
    generate θ_i ~ uniform(0, π/2);
    if x_i < min((l/2)sin(θ), a/2):
        m = m + 1
    end if
```

```
end do
print  p̂n = m/n;
print  π̂n = 2nl/ma.
```

Here x_i is the distance from the center of the i^{th} needle to a nearest line, and θ_i is the acute angle between the i^{th} needle and the lines.

3 PARADOX

From Algorithm 1, we can see that Monte Carlo simulation of Buffon's needle experiment is simple and easy to implement. It is a perfect example that demonstrates the beauty of Monte Carlo simulation in a classroom. However, there is a common misconception concerning the Buffon's needle simulation. Does it work? Erroneously (Wicklin 2014; Wikipedia 2014) , the simulation of the needle drop cannot be used to evaluate π. Why? We have to simulate the needle's angle from an uniform $\left(0, \frac{\pi}{2}\right)$ distribution. It is self-referential in theory, since it requires number π as an input value to approximate π. The following SAS code program is from the paper "Simulation of Buffon's needle in SAS" from SAS company website (Wicklin 2014):

```
Program Start
/* Buffon's Needle */
   proc iml;
   call randseed(123);
   pi = constant("pi");
   N = 1000;
   z = j(N,2); /* allocate space for (x,y) in unit square */
   call randgen(z,"Uniform"); /* fill with random U(0,1) */
   theta = pi*z[,1]; /* theta ~ U(0, pi) */
   y = z[,2] / 2; /* y ~ U(0, 0.5) */
   P = sum(y < sin(theta)/2) / N; /* proportion of intersections */
   piEst = 2/P;
   print P piEst;
Program End
```

In this SAS program, the SAS system constant pi value is called (used) as an input for the Buffon's needle simulation for estimating the number π.

4 REMOVING THE INHERENT PARADOX

In this section, we briefly introduce the Fixed-Point iteration method and discuss its convergence theorem and algorithm. The Buffon's needle simulation model can be fitted into the Fixed-Point iteration model. A convergence interval for selecting the initial input π value is derived for using the Fixed-point theory. A new algorithm is proposed. Its Python implementation program is introduced. At the end of this section, we provide a simulation output analysis. A conjecture is proposed.

4.1 The Fixed-Point Iteration Method

In order to solve the equation $f(x) = 0$ directly, the Fixed-Point iteration is based upon changing the equation $f(x) = 0$ into the form $g(x) = x$, and it solves $g(x) = x$ instead. Details of this method can be found in any Numerical Analysis book (Burden, Faires, and Reynolds 1981).

Definition 1 If $g(x)$ is defined on $[a, b]$ and $g(p) = p$ for some $p \in [a, b]$, then the function $g(x)$ is said to have the **fixed point** p on $[a, b]$.

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point. It can be easily proved using the Intermediate Value Theorem from Calculus.

Theorem 1 If $g(x)$ is continuous on $[a,b]$ and $g(x) \in [a,b]$, then $g(x)$ has a fixed point in $[a,b]$. Further, suppose $g'(x)$ exists on (a,b), and

$$|g'(x)| \leq k < 1 \qquad \text{for all } x \in (a,b). \tag{4}$$

Then $g(x)$ has a unique fixed point p in $[a,b]$.

From Theorem 1, we have the following Algorithm.

Algorithm 2 Start from any point initial approximation $p_0 \in [a,b]$ and consider the recursive process

$$p_{n+1} = g(p_n), \quad n = 0, 1, 2, \dots$$

If conditions from Theorem 1 are satisfied, p_n converges to a unique fixed point. Moreover, when n is large, p_n can be used to the solution of $g(x) = x$.

4.2 Modeling the Fixed-Point Equation

We let $l = a = 2$, then equation (1) becomes,

$$p = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin(\theta)} f(x,\theta) dx\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\sin(\theta)} \frac{2}{\pi}\, dx\, d\theta = \frac{2}{\pi}. \tag{5}$$

Since $p = \frac{2}{\pi}$, equation (5) is equivalent to the following equation,

$$p = \int_0^{\frac{1}{p}} \sin(\theta)\, p\, dx = p\left(1 - \cos\left(\frac{1}{p}\right)\right).$$

This implies

$$g(p) = p\left(1 - \cos\left(\frac{1}{p}\right)\right).$$

Consider its 1st order derivative

$$g'(p) = 1 - \cos\left(\frac{1}{p}\right) - \frac{1}{p}\sin\left(\frac{1}{p}\right).$$

We need to find the maximum value of $g'(p)$. Consider its 2nd order derivative,

$$g''(p) = \frac{1}{p^3}\cos\left(\frac{1}{p}\right).$$

Let $g''(x) = 0$, it implies $p = \frac{2}{\pi}$, when p is in a reasonable neighborhood of $\frac{2}{\pi}$. Then we have,

$$\max|g'(p)| = \left|g'\left(\frac{\pi}{2}\right)\right| = \left|1 - \frac{\pi}{2}\right| < 1.$$

Let's derive the reasonable neighborhood for p to guarantee that the Fixed-Point iteration converges to the solution. Consider the following figure:

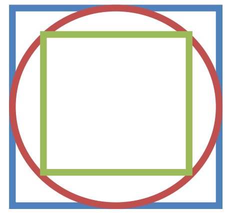

Figure 3: The unit circle is dominated by two squares.

From Figure 3, we know that the area of the unit circle is π, the area of the big circle is 4, and the area of the small square is 2. Then we have the following inequalities,

$$2 < \pi < 4$$

This implies

$$0.4 < p < 1 \quad \left(\text{since } p = \frac{2}{\pi}\right).$$

The interval $[0.4,\ 1]$ is the reasonable neighborhood of $\frac{2}{\pi}$. Therefore

$$|g'(p)| < \left|1 - \frac{\pi}{2}\right| < 1, \quad \text{for all } p \in [0.4, 1].$$

It is easy to check that $g(p)$ is a decreasing function and $g(p) \in [0.4, 1]$ for all $p \in [0.4, 1]$. In summary, our function $g(p)$ satisfies all conditions from Theorem 1. By Theorem 1, we conclude that there is a unique fixed point (solution) in $[0.4, 1]$. In theory, we know that fixed point is $\frac{2}{\pi}$.

4.3 Removing the Paradox

Now we are ready to propose a new method to remove the inherent paradox of the Buffon's needle simulation. We modify our algorithm 1 in two parts. In part I, we use an approximated π value as the simulation input instead of the true π value. In part II, we consider Algorithm 1 is an inner loop for simulating π value. An out loop is added to run the Fixed-Point iterations to improve the quality of the approximated π value for the inner loop input. In theory, if an initial π value is selected from interval $[2, 4]$, our simulation output estimated π value $\hat{\pi}_n$ converges to the true π value, due to the Fixed Point Theory. We summarize our new method into the following algorithm.

Algorithm 1

```
input:
   pi_initial ∈ [2,4];
   k = the number of the Fixed-Point iterations;
   n = the number of the Buffon's needles
do j = 1 to k;
   p = 2/pi_initail
   m = 0;
   do i = 1 to n;
      generate x_i ~ uniform (0, a/2);
      generate θ_i ~ uniform (0, 1/p);
      if x_i < min((1/2)sin(θ), a/2):
         m = m + 1
      end if
   end do
   p = m/n; # updating the estimated π value
```

```
end do
print  p̂_n = p;
print  π̂_n = 2l/(pa).
```

4.4 Python implementation and Output Analysis

Our new algorithm can be easily implemented in any computer language. We provide a Python implementation program here.

```
Program Start
   import random, math

   def main():
       pi_initial = float(input("Enter an initial value for Pi from [2, 4]:  "))
       p = 2/pi_initial
       n = int(input("Enter the number of  fixed point iterations:  "))
       m = int(input("Enter the number of Buffon needles:   "))
       for i in range(n):
           sum = 0
           for j in range(m):
               x = random.random()
               theta = random.random()/p
               if x < math.sin(theta):
                   sum+=1
           p_hat=sum/m
           p = p_hat
       print("The estimated Pi value is: ", 2/p)

   main()
Program End
```

We summarize our simulation outputs into the following Table 1.

Table 1: Simulation Outputs

Initial π Value	The Number of Fixed-Point Iterations	The Number of The Buffon's Needles	The Estimated π Values
2.0	100	10000	3.1177
3.0	100	10000	3.1392
4.0	100	10000	3.1412
2.0	1000	10000	3.1109
3.0	1000	10000	3.1586
4.0	1000	10000	3.1995
20	1000	10000	3.1711
200	1000	10000	3.1422
-150	1000	10000	3.1486
-7	1000	10000	3.1276
3.14156	1000	10000	3.1626
3.14156	1000	10000	3.1240
3.14156	1000	10000	3.1177
3.14156	1000	10000	3.1271

For the last four simulation outputs, we use the true π value 3.14156 as the initial input π value. Overall, our simulation outputs are as good the last four outputs. In theory, our results cannot be better than the last four outputs. For the first six simulation outputs, all initial π values are selected from the Fixed-Point iteration convergence interval $[2,4]$. All results are as we expected.

We have tried a few initial π values outside of the convergence interval. For the initial π values 20, 200, -150, and -7, all outputs are as good as others. We propose a conjecture here: our new method works for any arbitrary initial π values. It is part of our future research objectives. As an additional note, we certainly cannot use zero as the initial π value, since $p = \frac{2}{\pi}$ is the input of our simulation model and denominator cannot be zero. One possible way to disassociate the selection of a starting point complete from π is to consider the periodical property of the sine function. The period of sine function is 2π. That explains why the initial π values can be negative, very small, or very large.

5 CONCLUSIONS

In theory, the Buffon's needle Monte Carlo simulation is not functional for approximating the number π. The simulation model of the Buffon's needle experiment requires the true π value to evaluate π value. This is a self-referential paradox. We propose a new simulation model for the Buffon's needle experiment. This new method does not require the true π value as the simulation input. The simulation outputs from our Python implementation program indicate that the new method is doing as well as if the true π value is used as the simulation input. In addition, we realize that the new method works for any arbitrary initial π values. As a conjecture, we will continue to work on the convergence theory for the new method.

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