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AN OPTIMAL OPPORTUNITY COST SELECTION PROCEDURE FOR A FIXED NUMBER OF DESIGNS

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ABSTRACT

The expected opportunity cost is an important quality measure for the selection for the best simulated design among a set of design alternatives. It takes the case of incorrect selection into consideration and is particularly useful for risk-neutral decision makers. In this paper, we characterize the optimal selection rule which minimizes the expected opportunity cost by controlling the number of simulation replications allocated to each design. The observation noise of each design is allowed to have a general distribution. A comparison with other selection procedures in the numerical experiments shows the higher efficiency of the proposed method.

1 INTRODUCTION

We consider the problem of selecting the best simulated design from a finite set of design alternatives, given that the performance of each design has to be evaluated via simulation experiments. The advantage of simulation is that it can analyze and evaluate complex systems where analytical solutions are not available. However, it is usually computationally expensive (Law and Kelton 2000), because the ultimate accuracy of the estimate for the mean of a design cannot improve any faster than $O(1/\sqrt{n})$, the result of averaging independent and identically distributed (i.i.d.) observation noise, where n is the number of simulation replications. This explains the increasing popularity of research for efficient selection procedures. A comprehensive review and comparison of the selection procedures are in Branke et al. (2007).

Due to the randomness of simulation output, the best design cannot be selected with certainty, and there are two measures to assess the quality of the selection for the best design. The predominant measure is the alignment probability or the probability of correct selection ($P\{CS\}$) for the best design. The indifference-zone (IZ) approach for the selection problem aims at allocating the simulation budget to provide a guaranteed lower bound for the probability of correct selection (Dudewicz and Dalal 1975, Rinott 1978, Kim and Nelson 2001, Nelson et al. 2001). The optimal computing budget allocation (OCBA) allocates the samples sequentially in order to maximize an approximation of $P\{CS\}$ under a budget constraint (Chen et al. 2000, Chen et al. 2003, Chen et al. 2008, Chen and Lee 2011). The large-deviations (LD) approach (Glynn and Juneja 2004, Hunter and Pasupathy 2013) relaxes the normal assumption for the observation noise in IZ and OCBA by allowing the noise to have a general light-tailed distribution.

The other important measure is the expected opportunity cost ($E[OC]$) of a potentially incorrect selection. $E[OC]$ is more with practical concerns in business, engineering, and other applications where design performance represents economic value and is particularly useful for risk-neutral decision makers (Chick and Wu 2005). The existing research works on $E[OC]$ selection problems basically assume normal observation noise for the performance of each design. The expected value of information (EVI) procedure allocates samples to maximize the EVI that will be obtained from sampling in two stages or sequentially using predictive distributions of further samples (Chick and Inoue 2001a, Chick and Inoue 2001b). An OCBA type of greedy selection procedure was proposed by He et al. (2007) to reduce an upper bound of $E[OC]$. However, the $E[OC]$ selection with a general observation noise distribution was rarely explored.

In this paper, we characterize the optimal selection rule when solving the selection problem on a finite set of simulated designs. The quality of the selection is assessed by the $E[OC]$ and the observation noise of each design is allowed to have a general distribution. By approximating the $E[OC]$ by an upper bound of it, we derive the optimality conditions of the selection problem and design a corresponding sequential selection procedure for the implementation purpose. The higher efficiency of the proposed selection approach is demonstrated in the numerical experiments.

The organization of the paper is as follows. In the next section, the formulation of the selection problem is provided together with the assumptions made. Section 3 derives the optimal selection rule and designs a sequential selection algorithm for the selection problem formulated. The performance of the proposed selection method is illustrated in numerical experiments in Section 4. Section 5 concludes the paper.

2 PROBLEM FORMULATION

We consider the problem of selecting the best design from a given set of k designs. The total simulation budget is n. Let J_i and σ_i^2 be the mean and variance of design i, and n_i be the number of simulation replications to design i, $i = 1,2,...,k$. Let α_i be the proportion of the total simulation budget devoted to design i. $L(x_i, \omega_{ij})$ is the output of the j-th simulation replication for design i, where ω_{ij} is the random noise, and $\bar{J}_i$ is the sample mean of design i. That is, $\bar{J}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} L(x_i, \omega_{ij})$. We assume that the simulation output samples $L(x_i, \omega_{ij})$ are independent from replication to replication as well as independent across different designs. And, without loss of generality, we assume that the best design is the design with the minimal mean. The real best design t and the observed best design b are defined as $t = \arg\min_i J_i$ and $b = \arg\min_i \bar{J}_i$.

Let $\Lambda_i(\theta) = \log E[\exp(\theta L(x_i, \omega_{ij}))]$ denote the log-moment generating function of $L(x_i, \omega_{ij})$, $i = 1,2,...,k$, and let $I_i(\cdot)$ denote the Fenchel-Legendre transform of Λ_i, i.e., $I_i(x) = \sup_{\theta\in\mathbb{R}}(\theta x - \Lambda_i(\theta))$. For any set A, let A° be its interior and for any function $f(\cdot)$, let $f'(x)$ denote the derivative of f at x. The effective domain of Λ_i is $\mathscr{D}_{\Lambda_i} = \{\theta \in \mathbb{R} : \Lambda_i(\theta) < \infty\}$ and $\mathscr{F}_i = \{\Lambda_i'(\theta) : \theta \in \mathscr{D}_{\Lambda_i}^\circ\}$. Let $J_{max} = \max_{i\in\{1,2,...,k\}} J_i$. We further assume that the interval $[J_t, J_{max}] \subset \bigcap_{i=1}^k \mathscr{F}_i^\circ$. Loosely speaking, this assumption ensures that each sample mean $\bar{J}_i$ can take any value in the interval $[J_t, J_{max}]$. In particular, it ensures that $P(\bar{J}_i < \bar{J}_t) > 0$ for $i = 1,2,...,k$ and $i \neq t$.

The main idea of this research is to control the proportion α_i of the total simulation budget to each design i so that the $E[OC]$ of the selection for the best design can be minimized. This selection problem is difficult in that $E[OC]$ cannot be analytically expressed. Although $E[OC]$ can be estimated via Monte Carlo simulation, it is very time-consuming. To evaluate $E[OC]$ in a relatively fast and inexpensive way, we present an approximation and also an upper bound of $E[OC]$, which is the key for the proposed selection strategy.

We let the random vector $\boldsymbol{Z}_{t,i}^{(n)} = (\bar{J}_t, \bar{J}_i)$ and the moment generating function of $\boldsymbol{Z}_{t,i}^{(n)}$ be $M_{t,i}^{(n)}(\boldsymbol{\theta}) = E[\exp(\langle\boldsymbol{\theta}, \boldsymbol{Z}_{t,i}^{(n)}\rangle)]$, where $\langle\cdot,\cdot\rangle$ denotes the inner product and $\boldsymbol{\theta} \in \mathbb{R}^2$. It was shown in Dembo and Zeitouni (1998) that the rate function $I_{t,i}(\boldsymbol{x})$ of $\boldsymbol{Z}_{t,i}^{(n)}$ is,

$$I_{t,i}(\boldsymbol{x}) = \sup_{\boldsymbol{\theta}}(\langle\boldsymbol{\theta}, \boldsymbol{x}\rangle - \frac{1}{n}\log M_{t,i}^{(n)}(n\boldsymbol{\theta})). \quad (1)$$

If we define set $B = \{(y_1, y_2) : y_1 \geq y_2 \text{ and } y_1, y_2 \in \mathbb{R}\}$ and function $G_{t,i}(\alpha_t, \alpha_i) = \inf_{\boldsymbol{x}\in B} I_{t,i}(\boldsymbol{x})$, it was demonstrated in Glynn and Juneja (2004) that

$$G_{t,i}(\alpha_t, \alpha_i) = \alpha_t I_t(x(\alpha_t, \alpha_i)) + \alpha_i I_i(x(\alpha_t, \alpha_i)), \quad (2)$$

where $x(\alpha_t, \alpha_i)$ is the unique solution to $\alpha_t I_t'(x) + \alpha_i I_i'(x) = 0$.

We let $\delta_{i,j} = J_i - J_j$ for $i \neq j$, and an upper bound of $E[OC]$ can be obtained by,

$$E[OC] = E[J_b - J_t] = \sum_{i=1, i\neq t}^{k} \delta_{i,t} P(b=i) \leq \sum_{i=1, i\neq t}^{k} \delta_{i,t} P(\bar{J}_t \geq \bar{J}_i) \leq \sum_{i=1, i\neq t}^{k} \delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i)) \equiv AEOC. \tag{3}$$

More details of the derivation of (3) are available in Gao and Shi (2014). We call $\sum_{i=1,i\neq t}^{k} \delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i))$ *approximated expected opportunity cost* (AEOC). It can be shown in Gao and Shi (2014) that AEOC is not only an upper bound of $E[OC]$, but also a good approximation for $E[OC]$. Thus, in this research, we consider the following selection problem:

$$\begin{aligned} & \min AEOC \\ s.t. \quad & \sum_{i=1}^{k} \alpha_i = 1, \\ & \alpha_i > 0, \; i = 1, 2, ..., k. \end{aligned} \tag{4}$$

3 OPTIMAL SELECTION RULE

To obtain the optimality conditions of the selection problem (4), we have the following lemma. The proof of it can be found in Gao and Shi (2014).

Lemma 1 The function AEOC is convex and as a result, problem (4) is a convex optimization problem.

With the convexity result in Lemma 1, the solution satisfying the Karush-Kuhn-Tucker (KKT) conditions (Walker 1999) is the optimal solution to problem (4).

Theorem 1 The selection rule minimizing problem (4) satisfies the following optimality conditions,

$$\sum_{i=1, i\neq t}^{k} \frac{I_t(x(\alpha_t, \alpha_i))}{I_i(x(\alpha_t, \alpha_i))} = 1, \tag{5}$$

$$\delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i)) I_i(x(\alpha_t, \alpha_i)) = \delta_{j,t} \exp(-nG_{t,j}(\alpha_t, \alpha_j)) I_j(x(\alpha_t, \alpha_j)), \text{ for } i \neq j \neq t. \tag{6}$$

Proof. Let γ be the Lagrange multiplier. The KKT conditions of problem (4) are stated as follows:

$$-n\delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i)) I_i(x(\alpha_t, \alpha_i)) = \gamma, \; i = 1, 2, ..., k \text{ and } i \neq t, \tag{7}$$

$$-n \sum_{i=1, i\neq t}^{k} \delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i)) I_t(x(\alpha_t, \alpha_i)) = \gamma, \tag{8}$$

By (7), for $i = 1, 2, ..., k$ and $i \neq t$,

$$\frac{\gamma}{I_i(x(\alpha_t, \alpha_i))} = -n\delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i)). \tag{9}$$

Taking (9) into (8), condition (5) follows.

Condition (6) can be derived by applying (7) for design i and j with $i \neq j \neq t$. □

Theorem 1 describes the optimal status of the selection rule for minimizing the $E[OC]$. To design a selection algorithm for implementation based on the optimality conditions (5) and (6), we first obtain some intuitions of them. In this discussion, we temporarily ignore the constraint $\sum_{i=1}^{k} \alpha_i = 1$. That is, when there is a change in the value of α_i, we assume that α_j for $j \neq i$ is not influenced. For simplicity, we denote $U = \sum_{i=1,i\neq t}^{k} \frac{I_t(x(\alpha_t,\alpha_i))}{I_i(x(\alpha_t,\alpha_i))}$ and $V_i = \delta_{i,t} \exp(-nG_{t,i}(\alpha_t, \alpha_i)) I_i(x(\alpha_t, \alpha_i))$, $i = 1, 2, ..., k$ and $i \neq t$.

Equation (5) requires that $U = 1$. Since $\frac{\partial U}{\partial \alpha_t} < 0$ and $\frac{\partial U}{\partial \alpha_i} > 0$ for $i = 1,2,...,k$ and $i \neq t$ (Gao and Shi 2014), if for an allocation, $U > 1$, additional simulation replications should be allocated to design t such that α_t is increased and U is decreased. On the other hand, if for an allocation, $U < 1$, additional simulation replications should be allocated to the design set $\{i : 1 \leq i \leq k \text{ and } i \neq t\}$ such that some α_i is increased and U is increased. Therefore, condition (5) keeps a general balance between the simulation budget devoted to the best design and non-best designs.

Equation (6) requires that $V_i = V_j$ for $i \neq j \neq t$. Since $\frac{\partial V_i}{\partial \alpha_i} < 0$ for $i \neq t$ (Gao and Shi 2014), if for an allocation, $V_i > V_j$ for all $j \neq i \neq t$, additional simulation replications should be allocated to design i to reduce V_i. Thus, condition (6) keeps an appropriate ratio for the simulation budget allocated to each non-best design.

Based on the discussion above, we design the optimal $E[OC]$ allocation (OEA) procedure. Note that in practice, t, $I_i(x)$, $G_{t,i}(\alpha_t, \alpha_i)$ and $x(\alpha_t, \alpha_i)$ in (5) and (6) are unknown to us and need to be estimated using the sample means and variances.

OEA Algorithm

1. Specify the total simulation budget n, the initial simulation replication number n^0 and the incremental budget Δn_0. Iteration counter $r \leftarrow 0$. Perform n^0 simulation replications to all designs. $m_i = n^0$ and $\hat{\alpha}_i = m_i/n$ for $i = 1,2,...,k$. $m^r = \sum_{i=1}^{k} m_i$.
2. Calculate sample mean $\bar{J}_i = \frac{1}{m_i}\sum_{j=1}^{m_i} L(x_i, \omega_{ij})$ and sample variance $S_i^2 = \frac{1}{m_i-1}\sum_{j=1}^{m_i}(L(x_i, \omega_{ij}) - \bar{J}_i)^2$ for $i = 1,2,...,k$.
3. If $m^r = n$, stop. Otherwise,
   a. Find $\hat{t} = \arg\min_i \bar{J}_i$.
   b. Obtain the estimate $\hat{U}^m$ and $\hat{V}_i^m$ for $i = 1,2,...,k$ and $i \neq \hat{t}$.
   c. If $\hat{U}^m > 1$, $i^r = \hat{t}$. Otherwise, $i^r = \arg\max_{i \neq \hat{t}} \hat{V}_i^m$.
   d. Perform $\Delta n = \min\{\Delta n_0, n - m^r\}$ additional replications for design i^r. Update the sample mean and variance for this design.
   e. $m_{i^r} = m_{i^r} + \Delta n$ and $\hat{\alpha}_{i^r} = m_{i^r}/n$.
   f. $m^{r+1} = \sum_{i=1}^{k} m_i$.
   g. $r \leftarrow r + 1$.

In the algorithm, we first allocate an equal number of simulation replications to each design and obtain the initial estimate for the mean and variance of each design. Then, we iteratively provide an incremental budget and allocate it to either the best design or one of the non-best designs based on the estimated U and V_i. The procedure is terminated when the total simulation budget is consumed. At this point, the sum of α_i for $i = 1,2,...,k$ is 1.

Note that the algorithm is easy to implement when the distribution of the observation noise is known or assumed. In the absence of such knowledge or assumption for the noise distribution, the underlying rate function can be estimated through the Fenchel-Legendre transform introduced in Section 2.

4 NUMERICAL EXPERIMENTS

In the numerical experiments, the performance of the proposed *OEA* is compared with 3 other selection procedures:

- *Equal Allocation*: This is the simplest way to conduct simulation experiments and has been widely applied. The total simulation budget is equally allocated to each design, so that all the designs are simulated equally often.
- *LL Allocation*: The *LL* procedure is a two-stage procedure designed to improve the expected value of information of additional samples for the opportunity cost under the normal noise assumption.

In this comparison, we use $LL(S)$, the sequential version of LL. For more details of LL and $LL(S)$, see Chick and Inoue (2001b).

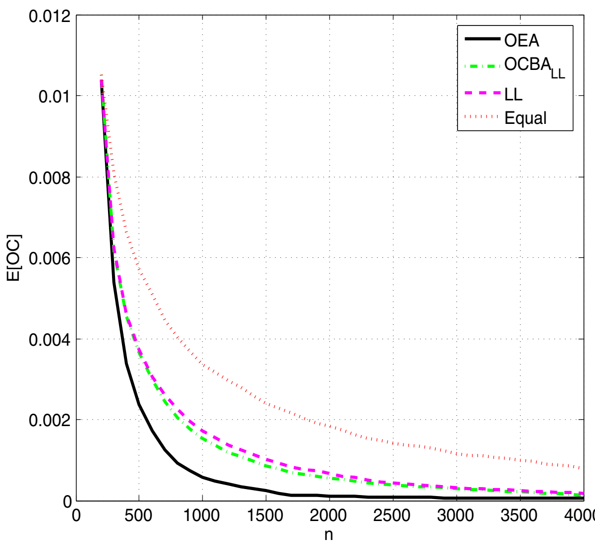
- $OCBA_{LL}$ *Allocation*: The $OCBA_{LL}$ procedure is an variation of *OCBA* which sequentially allocates the simulation budget to reduce $E[OC]$ with normal observation noise (He et al. 2007). $OCBA_{LL}$ specifies a number $s \leq k$. At each iteration, it provides an incremental budget and allocate it to the s designs in the set of the k designs which can reduce $E[OC]$ the most.

In order to compare the performance of the 4 approaches, 2 typical non-normal selection problems are considered.

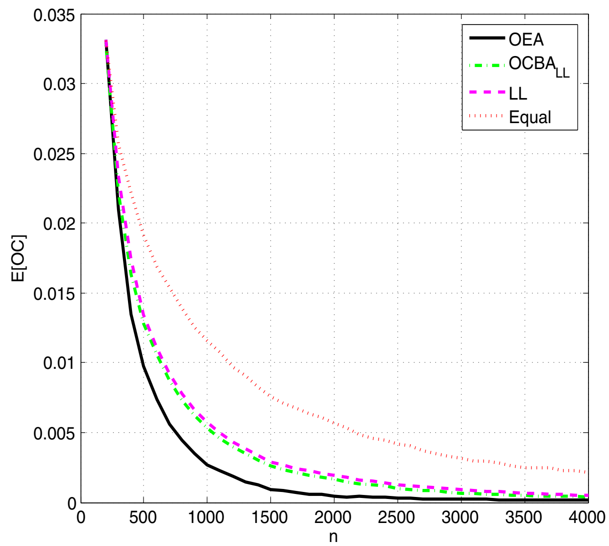
The first is an exponential example. It has 10 designs. Design i has exponential distribution with parameter $\lambda_i = 11 - i$ for $i = 1, 2, ..., 10$. Consequently, J_i is $1/(11-i)$.

The second is a Bernoulli example. It has 10 designs. Design i has Bernoulli distribution with success probability $p_i = 0.15 + 0.05i$ and $J_i = 0.15 + 0.05i$ for $i = 1, 2, ..., 10$.

Although LL and $OCBA_{LL}$ are designed for normal observation noise, their frameworks can be can be adequately applied for non-normal cases. For *OEA*, LL and $OCBA_{LL}$, we perform 20 initial replications for each design and the incremental budget is 5. The estimate of the $E[OC]$ is based on the average of 8000 independent applications of each procedure to the problem. The comparison of the 4 approaches is reported in Figure 1.



(a) Exponential example



(b) Bernoulli example

Figure 1: $E[OC]$ comparison of the 4 selection procedures.

We can see that for both examples, *OEA* performs the best. It leads to lower $E[OC]$ with different simulation budgets. LL and $OCBA_{LL}$ have close performance and are inferior to *OEA*. This is not surprising because these two methods are developed in the normal context and may not fit into the non-normal structure of each design so well as *OEA* does. The equal allocation does not involve any efficient mechanism for $E[OC]$ reduction and has the worst performance.

5 CONCLUSIONS

Traditional selection approaches focus on the probability of correct selection when evaluating the quality of a selection for the best design. The $E[OC]$ differs from the previous approaches in that it penalizes particularly bad choices more than the slightly incorrect selection. This makes $E[OC]$ especially useful for risk-neutral decision makers.

In the paper, we adopt the $E[OC]$ as the quality measure and aim at identifying the optimal selection rule. This is accomplished by developing an upper bound, and also a reasonable approximation for the $E[OC]$ which can be calculated in a fast and inexpensive way. We derive the optimality conditions for the selection problem and a corresponding sequential selection algorithm is designed for implementation. The numerical testing shows that the proposed procedure is more efficient than the compared methods.

There are two future research directions. First, the correlation of the simulation output samples between different replications of the same design and across different designs can be considered. In addition, we may also consider the selection problem in presence of stochastic constraints, which can better reflect the simulation practice.

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