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IDENTIFYING THE BEST SYSTEM IN THE PRESENSE OF STOCHASTIC CONSTRAINTS WITH VARYING THRESHOLDS

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ABSTRACT

We consider the problem of finding a system with the best primary performance measure among a finite number of simulated systems in the presence of stochastic constraints on secondary performance measures. When no feasible system exists, the decision maker may be willing to consider looser thresholds. Given that there is no change in the underlying simulated systems, we adopt the concept of green simulation and perform feasibility check across all potential thresholds simultaneously. We propose an indifference-zone procedure that takes multiple threshold values for each constraint based on the user's inputs and returns the best system that is feasible to the most desirable thresholds. We prove that our procedure yields the best system in the most desirable feasible region possible with at least a prespecified probability. Our experimental results show that the proposed procedure performs well with respect to the number of required observations as compared with an existing procedure.

1 INTRODUCTION

We consider the problem of selecting the best or near-best system with respect to a primary performance measure among a finite number of simulated systems while also satisfying constraints on one or more secondary performance measures. When no feasible system exists with respect to a given set of threshold values, the decision maker may be willing to relax the threshold values of some constraints so that a feasible system can be found. Thus, the constraint thresholds may change. We illustrate this problem with an example.

Suppose a decision maker wants to control an inventory level such that the expected profit can be maximized. She considers using an (s, S) inventory policy (namely ordering products to increase the inventory level to S when the inventory level at a review period is below s and placing no order otherwise). Two constraints exist, namely the probability that a shortage occurs between two successive review periods should be less than or equal to $q_1 = 5\%$ and the expected cost per review period should be less than or equal to $q_2 = \$115{,}000$. The decision maker is flexible with the second constraint and would like to relax the threshold to \$120,000 or \$130,000 if no feasible system can be found with $q_2 = \$115{,}000$. If there is still no feasible systems with respect to $q_2 = \$130{,}000$, then the decision maker is willing to raise the threshold q_1 to 6%, still with three possible values for q_2.

Ranking and selection (R&S) procedures aim to identify a system with the best performance among finitely many systems whose performances are estimated by stochastic simulation. Kim and Nelson (2005) and Hong et al. (2015) provide a literature review on R&S. When the problem requires not only selecting the best system with respect to a primary performance measure but also determining the feasibility with respect

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to stochastic constraints on secondary performance measures, it becomes constrained R&S. There are three major approaches to solving constrained R&S, namely the indifference-zone (IZ) approach, the optimal computing budget allocation (OCBA) approach, and the Bayesian approach. Lee et al. (2012), Hunter and Pasupathy (2013), and Pasupathy et al. (2015) propose sampling frameworks that can approximate the optimal computing budget allocation considering stochastic constraints. Xie and Frazier (2013) propose a sequential policy from the Bayesian approach for allocating simulation effort to determine a set of simulated systems that have mean performance exceeding a threshold. For the IZ approach, Batur and Kim (2010) propose a fully sequential procedure that finds a set of feasible systems given multiple constraints. Andradóttir and Kim (2010) propose procedures that select the best system with respect to the primary performance measure among a finite number of simulated systems in the presence of a single stochastic constraint on a secondary performance measure. Healey et al. (2013) apply the concept of dormancy to efficiently solve constrained R&S and Healey et al. (2014) propose procedures to select the best system in the presence of multiple constraints.

For constrained R&S, if each constraint has one fixed threshold value, procedures due to Andradóttir and Kim (2010) or Healey et al. (2014) can be used. When the decision maker is willing to consider more than one threshold value, one may consider iteratively applying those procedures "from scratch" to each threshold. However, this wastes all the information from a previous constrained R&S and becomes computational inefficient. Given the fact that there is no change in the simulation model of each system, a natural idea is to recycle the data for constrained R&S with different thresholds. The idea of recycling simulation observations for computer experiments is proposed in Feng and Staum (2015). However, they focus on estimation rather than comparison. Zhou et al. (2020a) propose a procedure that performs feasibility determination when the decision maker wants to consider multiple threshold values on each constraint. They use the idea of green simulation and perform feasibility determination simultaneously with respect to all thresholds at the same time so that the overall required number of observations is reduced. However, their focus is on feasibility determination rather than on finding the best feasible system.

In this paper, we adopt the concept of recycling simulation observations in the context of constrained R&S when constraint thresholds can change. We provide a fully sequential procedure that returns the best system that also satisfies stochastic constraints with the most desirable thresholds possible in which there is at least one feasible system. We prove that our procedure achieves a desired overall probability of correct selection and also performs well in reducing the required number of observations compared with an existing approach, namely applying a procedure of Andradóttir and Kim (2010) iteratively to each possible set of threshold values.

The rest of the paper is organized as follows: Section 2 provides the background for our problem. Section 3 proposes a procedure and Section 4 discusses the statistical validity of the procedure. In Section 5, we present the numerical results for our procedure and compare its performance with that of an existing procedure. Concluding remarks are provided in Section 6. A more detailed version of this paper is provided by Zhou et al. (2020b).

2 FORMULATION AND NOTATION

In this section, we provide the problem formulation and notation.

We consider k systems whose primary performance measure, as well as s constraints on secondary performance measures, can be estimated through stochastic simulation. Let Θ denote the index set of all possible systems (i.e., $\Theta = \{1, \ldots, k\}$). Let X_{in} be the observation associated with the primary performance measure from replication n of system i, and $Y_{i\ell n}$ be the observation associated with the ℓth stochastic constraint from replication n of system i, where $\ell = 1, \ldots, s$. We also define the expected values of the primary and secondary performance measures for each system $i \in \Theta$ and constraint $\ell = 1, \ldots, s$ as $x_i = \mathrm{E}[X_{in}]$ and $y_{i\ell} = \mathrm{E}[Y_{i\ell n}]$, respectively. Constrained R&S is to select

$$\begin{array}{ll} \arg\max_{i \in \Theta} & x_i \\ \text{s.t.} & y_{i\ell} \leq q_\ell \quad \text{for all } \ell = 1, \ldots, s, \end{array}$$

where q_ℓ denotes the constraint threshold for constraint ℓ.

For a given threshold vector $\boldsymbol{q} = (q_1, \ldots, q_s)$, procedures due to Andradóttir and Kim (2010) can be used to find the best system if there is only one constraint and procedures due to Healey, Andradóttir, and Kim (2014) are suitable if there are multiple constraints. In this paper, we assume that the decision maker has a list of possible threshold values in consideration for each constraint and hopes to select the best system with respect to the most desirable thresholds possible. We let d_ℓ denote the number of distinct threshold values and q_ℓ^m denote the mth distinct threshold value on constraint ℓ, where $m = 1, \ldots, d_\ell$ and $\ell = 1, \ldots, s$. We assume $q_\ell^1 < \ldots < q_\ell^{d_\ell}$, where $\ell = 1, \ldots, s$.

More specifically, instead of inputting a fixed threshold vector $\boldsymbol{q} = (q_1, \ldots, q_s)$, a decision maker inputs an ordered list of vectors of threshold values $\{\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \ldots, \boldsymbol{q}^{(d)}\}$, where d denotes the total number of threshold vectors that she is interested to test. We let $\boldsymbol{q}^{(m)}$ denote the m-th preferable threshold vector and let $q_\ell^{(m)}$ be the corresponding threshold value on constraint ℓ in $\boldsymbol{q}^{(m)}$, where $m = 1, \ldots, d$ and $\ell = 1, \ldots, s$. We assume that $\boldsymbol{q}^{(1)}$ is the most preferable threshold vector and $\boldsymbol{q}^{(d)}$ is the least preferable threshold vector. Then we introduce the threshold index vector $\boldsymbol{I}^{(m)}$ to include the indices of the threshold values that form $\boldsymbol{q}^{(m)}$. Similar to the definition of $q_\ell^{(m)}$, $I_\ell^{(m)}$ represents the corresponding threshold index on constraint ℓ.

Consider the example of selecting the best inventory control policy discussed in Section 1. Then $s = 2, d_1 = 2$ (i.e., two threshold values for the first constraint), $d_2 = 3$ (i.e., three threshold values for the second constraint), $q_1^1 = 5, q_1^2 = 6$, and $q_2^1 = 11500, q_2^2 = 120000, q_2^3 = 130000$. Moreover, we consider the following $d = 6$ threshold vectors

$$\boldsymbol{q}^{(1)} = \begin{bmatrix} 5 \\ 115000 \end{bmatrix}, \quad \boldsymbol{q}^{(2)} = \begin{bmatrix} 5 \\ 120000 \end{bmatrix}, \quad \boldsymbol{q}^{(3)} = \begin{bmatrix} 5 \\ 130000 \end{bmatrix},$$
$$\boldsymbol{q}^{(4)} = \begin{bmatrix} 6 \\ 115000 \end{bmatrix}, \quad \boldsymbol{q}^{(5)} = \begin{bmatrix} 6 \\ 120000 \end{bmatrix}, \text{ and } \boldsymbol{q}^{(6)} = \begin{bmatrix} 6 \\ 130000 \end{bmatrix}.$$

Note that $q_1^{(1)} = q_1^{(2)} = q_1^{(3)} = 5, q_1^{(4)} = q_1^{(5)} = q_\ell^{(6)} = 6$, while $q_2^{(1)} = q_2^{(4)} = 115000, q_2^{(2)} = q_2^{(5)} = 120000$ and $q_2^{(3)} = q_2^{(6)} = 130000$. The threshold index vectors are

$$\boldsymbol{I}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{I}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{I}^{(3)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{I}^{(4)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{I}^{(5)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \text{ and } \boldsymbol{I}^{(6)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

and hence $I_1^{(1)} = I_1^{(2)} = I_1^{(3)} = 1, I_1^{(4)} = I_1^{(5)} = I_1^{(6)} = 2$, while $I_2^{(1)} = I_2^{(4)} = 1, I_2^{(2)} = I_2^{(5)} = 2$, and $I_2^{(3)} = I_2^{(6)} = 3$. A detailed discussion of several ways of setting the input threshold vectors based on the decision maker's desire is included in Zhou et al. (2020b).

To solve the constrained R&S with varying constraint thresholds, we consider two phases: Phase I to identify feasible systems and Phase II to select a system with the largest x_i based on a comparison among feasible systems.

To check the feasibility of each system with respect to constraint ℓ, Andradóttir and Kim (2010) introduce a tolerance level, namely $\epsilon_\ell > 0$, for constraint ℓ, which is a positive real value predefined by the decision maker. This is often interpreted as the amount the decision maker is willing to be off from a given threshold value. Consider a threshold value q_ℓ^m for $m = 1, 2, \ldots, d_\ell$. Any system i with $y_{i\ell} \leq q_\ell^m - \epsilon_\ell$ is considered a desirable system with respect to constraint ℓ and threshold value q_ℓ^m. We let $D_\ell(q_\ell^m)$ denote the set of desirable systems for constraint ℓ and q_ℓ^m. Systems with $y_{i\ell} \geq q_\ell^m + \epsilon_\ell$ are considered as unacceptable systems for constraint ℓ and threshold q_ℓ^m, and are placed in set $U_\ell(q_\ell^m)$. Systems that fall within a tolerance level of q_ℓ^m, which means $q_\ell^m - \epsilon_\ell < y_{i\ell} < q_\ell^m + \epsilon_\ell$, are considered as acceptable systems, placing them in the set $A_\ell(q_\ell^m)$. More specifically,

$$D_\ell(q_\ell^m) = \{i \in \Theta | y_{i\ell} \le q_\ell^m - \epsilon_\ell\};$$
$$U_\ell(q_\ell^m) = \{i \in \Theta | y_{i\ell} \ge q_\ell^m + \epsilon_\ell\}; \text{ and}$$
$$A_\ell(q_\ell^m) = \{i \in \Theta | q_\ell^m - \epsilon_\ell < y_{i\ell} < q_\ell^m + \epsilon_\ell\}.$$

When feasibility check is performed, we let $\mathrm{CD}_{i\ell}(q_\ell^m)$ denote the correct decision event of system i with respect to constraint ℓ, which is defined as declaring system i as feasible if $i \in D_\ell(q_\ell^m)$ and as infeasible if $i \in U_\ell(q_\ell^m)$. Any feasibility decision is considered correct if $i \in A_\ell(q_\ell^m)$. We let $\mathrm{CD}_{i\ell}$ denote the correct decision event across all thresholds on constraint ℓ for system i, i.e., $\mathrm{CD}_{i\ell} = \cap_{m=1}^{d_\ell} \mathrm{CD}_{i\ell}(q_\ell^m)$. A correct decision for Phase I is defined as $\mathrm{CD} = \cap_{i\in\Theta} \cap_{\ell=1}^{s} \mathrm{CD}_{i\ell}$.

To select the best system with respect to the primary performance measure in Phase II, the decision maker needs to choose an indifference-zone parameter δ, which is the smallest absolute difference that the decision maker considers significant. More specifically, any system whose primary performance measure is at least δ smaller (larger) than system i is considered as inferior (superior) to system i. We use CS_i to denote the correct selection between system i and the best system. Let m^* be the smallest m such that $D_\ell(q_\ell^{(m)}) \neq \emptyset$ for all ℓ. If there exists at least one constraint ℓ such that $D_\ell(q_\ell^{(d)}) = \emptyset$, i.e., m^* does not exist, we set $m^* = d+1$ and define $D_\ell(q_\ell^{(d+1)}) = A_\ell(q_\ell^{d+1}) = \emptyset$. We also define $D_\ell(q_\ell^{(0)}) = A_\ell(q_\ell^{(0)}) = \emptyset$ if $m^* = 1$. If $m^* \le d$, then $\boldsymbol{q}^{(m^*)}$ is the most preferable threshold vector possible where at least one desirable system exists. Further, let B denote the set of desirable systems with respect to $\boldsymbol{q}^{(m^*)}$ (i.e., $B = \cap_{\ell=1}^{s} D_\ell(q_\ell^{(m^*)})$) and let b be the index of the best system among the systems in B, so that $x_b \ge x_i$ for $i, b \in B$. Then if $m^* \le d$,

$$\mathrm{CS} = \Bigg\{ \text{select } i \text{ such that either } i \in \cap_{\ell=1}^{s} \left(D_\ell\left(q_\ell^{(m^*)}\right) \cup A_\ell\left(q_\ell^{(m^*)}\right)\right) \text{ and } x_i > x_b - \delta$$
$$\text{or } i \in \cup_{m<m^*} \cap_{\ell=1}^{s} \left(D_\ell\left(q_\ell^{(m)}\right) \cup A_\ell\left(q_\ell^{(m)}\right)\right) \Bigg\}.$$

If $m^* = d+1$, CS is to declare that no feasible systems exist.

Throughout the paper, we let $\mathcal{I}(\cdot)$ be the indicator function and use the additional notation defined below.

$$n_0 \equiv \text{initial sample size for each system } (n_0 \ge 2);$$
$$r_i \equiv \text{number of observations so far for system } i \ (r_i \ge n_0);$$
$$S^2_{X_{ij}}(n_0) \equiv \text{sample variance of } X_{i1} - X_{j1}, \dots, X_{in_0} - X_{jn_0} \text{ between system } i \text{ and } j;$$
$$S^2_{Y_{i\ell}}(n_0) \equiv \text{sample variance of } Y_{i\ell 1}, \dots, Y_{i\ell n_0} \text{ for system } i \text{ and constraint } \ell;$$
$$R(r_i; v, w, z) \equiv \max\left\{0, \frac{(n_0-1)wz}{v} - \frac{v}{2c} r_i\right\} \text{ for } v, w, z \in \mathbb{R}^+ \text{ and } c \in \{1, 2, \dots, \infty\};$$
$$\alpha \equiv \text{overall nominal error for a procedure under consideration;}$$
$$\beta_1 \equiv \text{nominal error of feasibility check for one constraint of one system;}$$
$$\beta_2 \equiv \text{nominal error of comparison between two systems.}$$

3 PROCEDURE

In this section, we provide a procedure that runs Phases I and II simultaneously.

Similar to the findings in Zhou et al. (2020a), for constraint ℓ, if a system is declared feasible with respect to q_ℓ^m, where $m = 1\ldots,d_\ell$, this system is also declared feasible with respect to all thresholds $q_\ell^{m+1},\ldots,q_\ell^{d_\ell}$ with observation recycling. If a system is declared infeasible with respect to threshold q_ℓ^m, then this system is also declared infeasible with respect to $q_\ell^1,\ldots,q_\ell^{m-1}$. This is the main idea in our procedure.

Before starting our procedure, we need a few definitions. Given the decision maker's input of threshold vectors $\boldsymbol{q}^{(1)},\ldots,\boldsymbol{q}^{(d)}$ and threshold vectors, $\boldsymbol{I}^{(1)},\ldots,\boldsymbol{I}^{(d)}$, we use a variable θ to keep track of the threshold vector for which we are trying to determine feasibility. Initially, θ is set to d, which is the index of the least preferable threshold vector.

Then we define four sets as follows:

- P is a set of systems whose feasibility is partially determined. More specifically, systems in P are declared feasible with respect to $\boldsymbol{q}^{(\theta)}$, but not determined yet with respect to $\boldsymbol{q}^{(\theta')}$ for any $1 \le \theta' \le \theta - 1$.
- F is a set of systems whose feasibility is fully determined. More specifically, systems in F are declared feasible with respect to $\boldsymbol{q}^{(\theta)}$ and infeasible with respect to $\boldsymbol{q}^{(1)},\ldots,\boldsymbol{q}^{(\theta-1)}$.
- M is a set of systems whose feasibility with respect to $\boldsymbol{q}^{(\theta)}$ is not determined yet.
- SS_i is a set of systems found to be superior to system i in terms of the primary performance measure.

In addition, the following two notations are needed:

- $Z_{i\ell}^m$ is a binary variable that indicates whether system i is feasible with respect to constraint ℓ for threshold q_ℓ^m. More specifically,

$$Z_{i\ell}^m = \begin{cases} 1, & \text{if system } i \text{ is declared feasible to constraint } \ell \text{ with threshold value } q_\ell^m, \\ 0, & \text{if system } i \text{ is declared infeasible to constraint } \ell \text{ with threshold value } q_\ell^m, \\ 2, & \text{if the feasibility is not determined yet.} \end{cases}$$

- $v_{i\ell}^{\mathrm{UB}}$ and $v_{i\ell}^{\mathrm{LB}}$ are two values that help to determine the feasibility of system i simultaneously with respect to all the threholds on constraint ℓ. A more detailed discussion can be found in Zhou et al. (2020a).

A detailed description of the simultaneously running procedure is shown in Algorithm 1.

4 STATISTICAL VALIDITY

In this section, we discuss how to ensure the statistical validity of Algorithm 1.

To guarantee the statistical validity of $\mathcal{ZAK}+$, we need the following assumptions. First, we assume that for each system i, where $i = 1,\ldots,k$, we have

$$\begin{bmatrix} X_{in} \\ Y_{i1n} \\ \vdots \\ Y_{isn} \end{bmatrix} \overset{iid}{\sim} N_{s+1}\left(\begin{bmatrix} x_i \\ y_{i1} \\ \vdots \\ y_{is} \end{bmatrix}, \Sigma_i \right), \qquad n = 1,2,\ldots,$$

where $\overset{iid}{\sim}$ denotes independent and identically distributed, N_{s+1} denotes $(s+1)$-dimensional multivariate normal, and Σ_i is the $(s+1)\times(s+1)$ covariance matrix of the vector $(X_{in}, Y_{i1n},\ldots,Y_{isn})$.

Normally distributed data is a common assumption used in many R&S procedures due to the fact that it can be justified by the Central Limit Theorem when observations are either within-replication averages

Algorithm 1 $\mathcal{ZAK}$+

[**Setup**:] Choose confidence level $1-\alpha$, tolerance level ϵ_ℓ, indifference-zone parameter δ, threshold vectors $\{\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \ldots, \boldsymbol{q}^{(d)}\}$, and corresponding index vectors $\{\boldsymbol{I}^{(1)}, \boldsymbol{I}^{(2)}, \ldots, \boldsymbol{I}^{(d)}\}$. Set $M = \{1, 2, \ldots, k\}$ and set $SS_i = \emptyset$ and $Z_{i\ell}^m = 2$ for all $i \in M, \ell = 1, \ldots, s$, and $m = 1, \ldots, d_\ell$. Set $F = \emptyset$ and $P = \emptyset$. Set $\theta = d$.

[**Initialization**:]

Obtain n_0 observations from system i, and compute $S^2_{Y_{i\ell}}(n_0)$ and $S^2_{X_{ij}}(n_0)$ for all $i, j \in M$, where $i \neq j$, and $\ell = 1, \ldots, s$.

Set $\eta_1 = \frac{1}{2}((2\beta_1)^{-2/(n_0-1)} - 1)$ and $\eta_2 = \frac{1}{2}((2\beta_2)^{-2/(n_0-1)} - 1)$, where β_1 and β_2 are computed based on the discussion in Section 4.

Set $r_i = n_0$ and $\mathrm{ON}_i = \{1, \ldots, s\}$ for all $i \in M$. Set $\mathrm{ON}_{i\ell} = \{1, \ldots, d_\ell\}$ for all $\ell = 1, \ldots, s$ and all $i \in M$.

Set $v_{i\ell}^{\mathrm{UB}} = \infty$ and $v_{i\ell}^{\mathrm{LB}} = -\infty$ for $i \in M$ and $\ell = 1, \ldots, s$.

[**Comparison**:] For $i, j \in (M \cup P \cup F)$ s.t. $i \neq j, i \notin SS_j, j \notin SS_i$, and

$$\sum_{n=1}^{r_i} X_{i\ell n} \leq \sum_{n=1}^{r_j} X_{j\ell n} - R(r_i; \delta, \eta_2, S^2_{X_{ij}}),$$

If $j \in M$, add j to SS_i.

Else,

- If $i \in (M \cup P)$, add j to SS_i.
- Else, eliminate i from F, delete SS_i.

[**Feasibility Check**:]

for $i \in (M \cup P)$ **do**

for $\ell \in \mathrm{ON}_i$ **do**

$v_{i\ell}^{\mathrm{UB}} = \min(v_{i\ell}^{\mathrm{UB}}, \bar{Y}_{i\ell}(r_i) + R(r_i; \epsilon_\ell, \eta_1, S^2_{Y_{i\ell}}(n_0))/r_i)$.

$v_{i\ell}^{\mathrm{LB}} = \max(v_{i\ell}^{\mathrm{LB}}, \bar{Y}_{i\ell}(r_i) - R(r_i; \epsilon_\ell, \eta_1, S^2_{Y_{i\ell}}(n_0))/r_i)$.

for $m \in \mathrm{ON}_{i\ell}$ **do**

If $v_{i\ell}^{\mathrm{UB}} \leq q_\ell^m$, set $Z_{i\ell}^m = 1$ and $\mathrm{ON}_{i\ell} = \mathrm{ON}_{i\ell} \backslash \{m\}$;

If $v_{i\ell}^{\mathrm{LB}} \geq q_\ell^m$, set $Z_{i\ell}^m = 0$ and $\mathrm{ON}_{i\ell} = \mathrm{ON}_{i\ell} \backslash \{m\}$.

end for

If $\mathrm{ON}_{i\ell} = \emptyset$, set $\mathrm{ON}_i = \mathrm{ON}_i \backslash \{\ell\}$.

end for

If $\prod_{\ell=1}^s Z_{i\ell}^{I_\ell^{(\theta)}} = 0$, eliminate i from M, delete SS_i.

If $\exists$ minimum $\kappa \leq \theta$ s.t. $\prod_{\ell=1}^s Z_{i\ell}^{I_\ell^{(\kappa)}} = 1$,

- If $\kappa < \theta$, set $M = M \cup P, F = \emptyset, P = \emptyset$, and $\theta = \kappa$.
- Move i from M to P.
- For all $j \in F$ with $i \in SS_j$, eliminate j from F, delete SS_j.
- If $\kappa = 1$ or $\sum_{\ell=1}^s Z_{i\ell}^{I_\ell^{(\kappa-1)}} = 0$ for $\kappa > 1$, then move i from P to F. Futhermore, if there exists j such that $j \in P \cup F$ and $j \in SS_i$, then eliminate i from F and delete SS_i.

end for

[**Stopping Condition**:]

If $|M| = 0, |P| = 0$ and $|F| = 1$, then stop and return the system in F as the best system. Else if $|M| = 0, |P| = 0$ and $|F| = 0$, then stop and return no feasible systems exist.

Otherwise, for all $i \in (M \cup P \cup F)$, set $r_i = r_i + 1$, take one additional observation $Y_{i\ell r_i}$, go to [**Comparison**].

or batch means (Law and Kelton 2000). Moreover, the primary and secondary performance measures are usually correlated. When common random numbers (CRN) are introduced in simulating observations from each system, observations between systems are correlated. Our formulation allows correlations between both performance measures and systems.

We then assume that for any $1 \leq t \leq k$ and any subset $\Theta' \subseteq \Theta$ with cardinality t, the feasibility check phase guarantees

$$\begin{cases} \Pr\{\cap_{i\in\Theta'} \cap_{\ell=1}^{s} \mathrm{CD}_{i\ell}\} \geq (1-s\beta_1)^t, & \text{if systems are simulated independently;} \\ \Pr\{\cap_{i\in\Theta'} \cap_{\ell=1}^{s} \mathrm{CD}_{i\ell}\} \geq 1-ts\beta_1, & \text{if systems are simulated under CRN.} \end{cases}$$

Similarly, we assume that for any $1 \leq t \leq k$ and any subset $\Theta' \subseteq \{i \in \{1,\ldots,k\} : x_i \leq x_b - \delta\}$ with cardinality t, the comparison phase guarantees

$$\begin{cases} \Pr\{\cap_{i\in\Theta'} \mathrm{CS}_i\} \geq (1-\beta_2)^t, & \text{if systems are simulated independently;} \\ \Pr\{\cap_{i\in\Theta'} \mathrm{CS}_i\} \geq 1-t\beta_2, & \text{if systems are simulated under CRN.} \end{cases}$$

This means that we assume the feasibility check and the comparison procedures have lower bounds on the probability of correct selection, depending on whether the systems are simulated independently or under CRN. These two assumptions are critical to the proofs of statistical validity of the proposed procedure.

We finally assume that if $m^* \leq d$, then there do not exist any systems that fall in the indifference zone of the primary performance measure. That is, for any system $i \in \cap_{\ell=1}^{s}(D_\ell(q_\ell^{(m^*)}) \cup A_\ell(q_\ell^{(m^*)}))$, where $i \neq b$, we assume $x_i \leq x_b - \delta$.

Based on the assumptions above, one approach is to first decide the choice of $e = s\beta_1/\beta_2$. This is the ratio of (i) the error for a feasibility check with respect to all s constraints for one system to (ii) the error of a comparison between two systems, which should be decided based on the decision maker's idea on whether she wants to allocate more error to feasibility check or comparison. We provide the case when the decision maker chooses $e = 1$ and assume that systems are simulated independently. Let $\beta = s\beta_1 = \beta_2$. Then the value of β can be found by solving the following equation:

$$(1-\beta)^{j_\beta} + (1-2\beta)^{k-j_\beta-1} + (1-\beta) - 2 = 1-\alpha,$$

where j_β is the integer in $\{0,1,\ldots,k-1\}$ that is closest to

$$\frac{\log C + (k-1)\log(1-2\beta)}{\log(1-2\beta)+\log(1-\beta)}, \text{ where } C = \frac{\log(1-2\beta)}{\log(1-\beta)}.$$

The proof can be found in Zhou et al. (2020b).

5 NUMERICAL EXPERIMENTS

In this section, we provide experimental results when there is one constraint ($s = 1$) and two thresholds ($d = 2$) to demonstrate the performance of procedure $\mathcal{ZAK}+$ compared with repeatedly applying procedure $\mathcal{AK}+$ from Andradóttir and Kim (2010). Experiments with $s > 1$ and $d > 2$ are included in Zhou et al. (2020b).

As we have only one constraint, we drop subscript ℓ from all notation, including $y_{i\ell}, q_\ell^m, q_\ell^{(m)}$, and $I_\ell^{(m)}$. We set the thresholds as $q^1 = 0$ and $q^2 = 2\epsilon$, and assume that the decision maker prefers $q^1 = 0$ to $q^2 = 2\epsilon$. That is $q^{(1)} = 0, q^{(2)} = 2\epsilon, I^{(1)} = 1$, and $I^{(2)} = 2$. We apply $\mathcal{AK}+$ to each threshold $\boldsymbol{q}^m$, where $m = 1,2$, in the order *from the most preferable to the least preferable* until a best feasible system is found. We set $y_i = 0$, where $i = 1,\ldots,k$.

The experiment is based on 10,000 macro replications with $\alpha = 0.05, n_0 = 20, \epsilon = 1/\sqrt{n_0}$, and $\delta = 1/\sqrt{n_0}$. We report estimated probability of correct selection and average total number of observations. We set all variances for both primary and secondary performance measures equal to 1 and consider independent systems in terms of the primary performance measure with no correlation between the secondary performance measure.

We consider two experimental settings: (i) there exist feasible systems with respect to $q^{(1)}$; (ii) there do not exist feasible systems with respect to $q^{(1)}$, but there exist feasible systems with respect to $q^{(2)}$. We consider the number of systems $k \in \{6, 15, 30, 99\}$ The mean configurations are shown as follows.

- Case (i):

$$x_i = \begin{cases} 0, & i = 1, \dots, k/3 - 1, \\ \delta, & i = k/3, \\ (i + 1 - k/3)\delta, & i = k/3 + 1, \dots, k \end{cases} \quad \text{and } y_i = \begin{cases} -\epsilon, & i = 1, \dots, k/3, \\ \epsilon, & i = k/3 + 1, \dots, 2k/3, \\ 3\epsilon, & i = 2k/3 + 1, \dots, k. \end{cases}$$

- Case (ii):

$$x_i = \begin{cases} 0, & i = 1, \dots, 2k/3 - 1, \\ \delta, & i = 2k/3, \\ (i + 1 - 2k/3)\delta, & i = 2k/3 + 1, \dots, k, \end{cases} \quad \text{and } y_i = \begin{cases} \epsilon, & i = 1, \dots, 2k/3, \\ 3\epsilon, & i = 2k/3 + 1, \dots, k. \end{cases}$$

For both settings, the infeasible systems all have superior performance to the feasible systems, and all feasible systems that are not the best system are exactly δ worse in the primary performance measure compared with the best system. This becomes challenging for the purpose of selecting the best system. The experimental results are shown in Table 1.

Table 1: Average number of observations and estimated PCS (reported in parentheses) for multiple systems with a single constraint and two thresholds.

k	Case (i)		Case (ii)	
	$\mathcal{ZAK}+$	$\mathcal{AK}+$	$\mathcal{ZAK}+$	$\mathcal{AK}+$
6	658.702	659.92	882.698	1215.616
	(0.976)	(0.984)	(0.962)	(0.972)
15	1916.463	1850.600	2667.507	3538.856
	(0.981)	(0.983)	(0.962)	(0.963)
30	4275.535	4216.827	5972.886	8082.279
	(0.982)	(0.984)	(0.964)	(0.965)
99	16593.290	16488.280	23320.210	32173.26
	(0.984)	(0.984)	(0.964)	(0.963)

We see that both procedures are statistical valid. $\mathcal{ZAK}+$ performs similar or slightly worse compared to $\mathcal{AK}+$ in case (i), but performs better in case (ii). As $\mathcal{AK}+$ needs to be applied once in case (i) but twice in case (ii), it is expected that $\mathcal{AK}+$ would require much more observations in case (ii) compared with case (i). As the number of times $\mathcal{AK}+$ needs to be applied may increase when number of thresholds increases, $\mathcal{ZAK}+$ is expected to perform better than $\mathcal{AK}+$ with larger number of thresholds.

6 CONCLUSION

We consider constrained R&S with varying threshold values and propose a statistical valid procedure that selects the best system with respect to a primary performance measure while also satisfying constraints on secondary performance measures with respect to the most preferable threshold vector possible. Our experimental results show that the proposed procedure performs well in reducing the number of required observations.

REFERENCES

Andradóttir, S., and S.-H. Kim. 2010. "Fully Sequential Procedures for Comparing Constrained Systems via Simulation". *Naval Research Logistics* 57(5):403–421.

Batur, D., and S.-H. Kim. 2010. "Finding Feasible Systems in the Presence of Constraints on Multiple Performance Measures". *ACM Transactions on Modeling and Computer Simulation* 20(13).

Feng, M., and J. Staum. 2015. "Green Simulation Designs for Repeated Experiments". In *Proceedings of the 2015 Winter Simulation Conference*, edited by K.-H. Bae, B. Feng, S. Kim, S. Lazarova-Molnar, Z. Zheng, T. Roeder, and R. thiesing, 403–413. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.

Healey, C., S. Andradóttir, and S.-H. Kim. 2013. "Efficient Comparison of Constrained Systems using Dormancy". *European Journal of Operations Research* 224:340–352.

Healey, C., S. Andradóttir, and S.-H. Kim. 2014. "Selection Procedures for Simulations with Multiple Constraints under Independent and Correlated Sampling". *ACM Transactions on Modeling and Computer Simulation* 24 (3)(14).

Hong, L. J., B. L. Nelson, and J. Xu. 2015. "Discrete Optimization via Simulation". In *Handbook of Simulation Optimization. International Series in Operations Research & Management Science*, edited by M. Fu, Volume 216, 9–44. NY: Springer.

Hunter, S. R., and R. Pasupathy. 2013. "Optimal Sampling Laws for Stochastically Constrained Ranking and Selection under Independent or Common Random Numbers". *INFORMS Journal on Computing* 25(3):527–542.

Kim, S.-H., and B. Nelson. 2005. "Selecting the Best System". In *Handbooks in Operations Research and Management Science: Simulation*, edited by S. G. Henderson and B. L. Nelson, Chapter 17, 501–534. Oxford, UK: Elsevier.

Law, A., and D. Kelton. 2000. *Simulation Modeling and Analysis*. New York: Academic Press.

Lee, L. H., N. A. Pujowidianto, C.-H. C. L. W. Li, and C. M. Yap. 2012. "Approximation Simulation Budget Allocation for Selecting the Best Design in the Presence of Stochastic Constraints". *IEEE Transactions on Automatic Control* 57:2940–2945.

Pasupathy, R., S. R. Hunter, N. A. Pujowidianto, L. H. Lee, and C. H. Chen. 2015. "Stochastically Constrained Ranking and Selection via SCORE". *ACM Transactions on Modeling and Computer Simulation* 25(1).

Xie, J., and P. Frazier. 2013. "Sequential Bayes-optimal Policies for Multiple Comparisons with a Known Standard". *Operations Research* 61:1069–1257.

Zhou, Y., S. Andradóttir, and S.-H. Kim. 2020a. "Finding Feasible Systems Using Recycled Observations When Constraint Thresholds Change". Technical report, H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA.

Zhou, Y., S. Andradóttir, and S.-H. Kim. 2020b. "Selection of the Best in the Presence of Subjective Stochastic Constraints". Technical report, H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA.

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