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SENSITIVITY ANALYSIS OF ARC CRITICALITIES IN STOCHASTIC ACTIVITY NETWORKS

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ABSTRACT

Using Monte Carlo simulation, this paper proposes a new algorithm for estimating the arc criticalities of stochastic activity networks. The algorithm is based on the following result: given the length of all arcs in a network except for the one arc of interest, which is on the critical path (longest path) if and only if its length is greater than a threshold. Therefore, the new algorithm is named Threshold Arc Criticality (TAC). By applying Infinitesimal Perturbation Analysis (IPA) to TAC, an unbiased estimator of the stochastic derivative of the arc criticalities with respect to parameters of arc length distributions can be derived. With a valid estimator of stochastic derivative of arc criticalities, sensitivity analysis of arc criticalities is carried out via simulation of a small test network.

1 INTRODUCTION

A stochastic activity network is a directed acyclic network where arcs can represent tasks needed to complete a large project, arcs' lengths can represent the time required for completing the corresponding tasks, and nodes can represent events. The directions of arcs indicate the precedence relationships between different tasks - the tail of an arc must be completed before the start of its head node. And a node can start only if all tails of incoming arcs are completed. A path is a route that starts at the source node and stops at the end node of the network. The project completion time is the length of the longest path. And the path with longest length is called the critical path. Arc criticality is defined to be the probability that an arc is on the critical path.

Much of existing research focuses on estimating the distribution of the total project completion time. Different techniques of conditional Monte Carlo sampling are used for estimating the distribution of the total completion times, for instance, the Uniform Cutset Approach (Sigal et al. 1979). One of the goals of project managers is to minimize the total project completion time, so a linear program is introduced to minimize the expectation of the project completion time with limited budget in Bowman (1994). Commonly used techniques for taking stochastic derivatives are Infinitesimal Perturbation Analysis (IPA) and Likelihood Ratio method (LR) (L'Ecuyer 1990). In Fu (2006), IPA, LR and Smooth Perturbation Analysis (SPA) derivatives of expected project completion time are introduced.

We briefly review previous research relevant to arc criticality estimation. Bowman (1995) introduced node release time approach of a new arc criticality estimator with lower variance. Cho and Yum (2004) proposed applying logistic regression to arc criticality. In order to find the IPA estimator of the derivative of arc criticality with respect to parameters of arc length distributions, a continuous integrand of the estimator of arc criticality is necessary. The CAC estimator (Bowman 1995) satisfies the necessary condition. However, section 2 of this paper indicates the challenge of simulating the IPA estimator of the CAC estimator, where the commonly used version of arc criticality estimator given by equation (1) has a discontinuous integrand that makes IPA not applicable. Therefore, for (1), only the LR estimator is applicable. Since the LR estimator has larger variance than IPA, we prefer to find an IPA estimator of the derivatives of arc criticalities. In this paper, we propose a new arc criticality estimator with continuous integrand and derive its IPA estimator.

In section 2, we present the notation and problem setting. In section 3, previous arc criticality estimators and the challenge in deriving stochastic gradient estimators are presented. A new arc criticality estimator called Threshold Arc Criticality (TAC) is derived, and the IPA estimator of TAC is also presented. In section 4, an example is presented, and simulation results are given. In section 5, an application of the derivative estimator is presented. Section 6 contains conclusions and future research.

2 PROBLEM SETTING AND NOTATION

Assume the stochastic network has m arcs and n nodes. In the stochastic activity network, the completion times of different tasks are assumed to be independent with known distributions. We also assume throughout that the critical path denotes the path with the longest length. We define:

$X_i =$ length of arc i (with realizations x_i), $\{X_i\}_{i=1}^m$ are independent,

$F_i(x) = \Pr(X_i \leq x)$ = probability distribution function for arc i,

$\bar{F}_i(x) = 1 - F_i(x)$,

$f_i(x) =$ probability density function for arc i,

$C_a(i) = \Pr\{$arc i is on the critical path$\}$ = criticality of arc i,

$I\{\cdot\} =$ indicator function,

$\mathscr{P}$ = set of all paths,

$|P| =$ length of path P,

$\mathscr{P}_i = \{P \in \mathscr{P} | i \in P\}$ = set of paths containing arc i,

$\mathscr{P}_{i-} = \{P \in \mathscr{P} | i \notin P\}$ = set of paths not containing arc i,

$P^* = \arg\max_{P\in\mathscr{P}} |P| =$ set of arcs on the longest path corresponding to path set $\mathscr{P}$,

$P_i^* = \arg\max_{P\in\mathscr{P}_i} |P| =$ set of arcs on the longest path corresponding to path set $\mathscr{P}_i$,

$P_{i-}^* = \arg\max_{P\in\mathscr{P}_{i-}} |P| =$ set of arcs on the longest path corresponding to path set $\mathscr{P}_{i-}$,

$\|\cdot\|$ is an operator that calculates the length of the longest path for a given set of paths,

From the above definition, we have that $\|\mathscr{P}\| = |P^*|, \|\mathscr{P}_i\| = |P_i^*|, \|\mathscr{P}_{i-}\| = |P_{i-}^*|$,

$\|\cdot\|^i$ is an operator that calculates the length of the longest path for a given set of paths under the condition that arc i has length 0,

From the above definition, we have that $\|\mathscr{P}_i\|^i = |P_i^*|\Big|_{X_i=0}$.

The following example illustrates the notation:
Figure 1 depicts a stochastic network with 4 nodes and 5 arcs, where each node represents an activity in a project and each arc represents the length of the time requested to complete an activity. Then,

$$\mathscr{P} = \{(1,3,5),(2,5),(1,4)\}, \mathscr{P}_1 = \{(1,3,5),(1,4)\}, \mathscr{P}_{1-} = \{(2,5)\}.$$

For a realization of $\{X_i\}_{i=1}^5$ with $x_1 = 5$, $x_2 = 10$, $x_3 = 6$, $x_4 = 3$, $x_5 = 1$. We have,

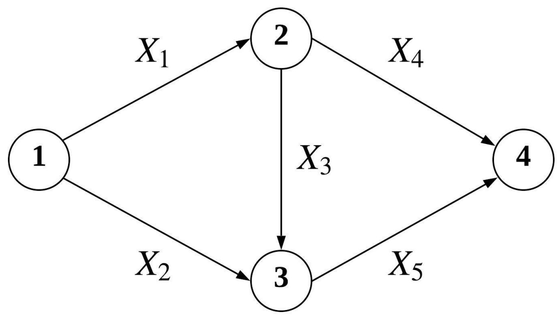


Figure 1: A small stochastic network

$$P^* = (1,3,5), P_1^* = (1,3,5), P_{1-}^* = (2,5)$$
$$\|\mathscr{P}\| = max\{x_1 + x_3 + x_5, x_2 + x_5, x_1 + x_4\} = 12$$
$$\|\mathscr{P}\|^1 = max\{0 + x_3 + x_5, x_2 + x_5, 0 + x_4\} = 11$$
$$\|\mathscr{P}_1\|^1 = max\{0 + x_3 + x_5, 0 + x_4\} = 0 + x_3 + x_5 = 7$$

3 ARC CRITICALITIES

3.1 Previous Work on Arc Criticalities

The commonly used version of arc criticality is:

$$C_a(i) = \int_{\mathbb{R}^m} \cdots \int I\{\|\mathscr{P}_i\| = \|\mathscr{P}\|\} \times \prod_{j=1}^{m} f_j(x_j)\, dx_1 \ldots dx_m. \tag{1}$$

Here, we name equation (1) the Indicator Arc Criticality (IAC).
The node release time version (CAC) of arc criticality (Bowman 1995) is:

$$C_a(i) = \int_{\mathbb{R}^n} \cdots \int C_a(i|T_1 = t_1, ..., T_n = t_n) \times f_{T_1,...,T_n}(t_1, ..., t_n)\, dt_1 \ldots dt_n, \tag{2}$$

where

$$C_a(i|T_1 = t_1, ..., T_n = t_n) = \frac{f_i(t_* - t_i) \prod_{j=1, j\neq i}^{q} F_j(t_* - t_j)}{\sum_{k=1}^{q} f_k(t_* - t_k) \prod_{j=1, j\neq k}^{q} F_j(t_* - t_j)} \times C_n(*|T_1 = t_1, ..., T_n = t_n) \tag{3}$$

and

$$C_n(*|T_1 = t_1, ..., T_n = t_n) = \sum_{j=q+1}^{q+s} C_a(j|T_1 = t_1, ..., T_n = t_n). \tag{4}$$

The node release time of a given node is defined to be the earliest time epoch that the given node can start its activity. In the above notation, T_i refers to the node release time of node i. Thus, the node release time of the last node is the length of the longest path. For example, in Figure 1, we have $T_1 = 0$, $T_2 = X_1$, $T_3 = max\{X_1 + X_3, X_2\} = max\{T_2 + X_3, X_2\}$, $T_4 = max\{T_2 + X_4, T_3 + X_5\}$. For estimating the CAC (Conditional Arc Criticality) (Bowman 1995), in each simulation, we need to compute the integrand (3). Figure 2 illustrates equation (3): arc i is on the critical path if and only if (a) Arc i is on the longest path to its head node, and (b) An arc emanating from arc i's head node is on the longest path in the network (Bowman 1995). Therefore, the first part of (3) as a fraction is the likelihood that arc i is on the longest

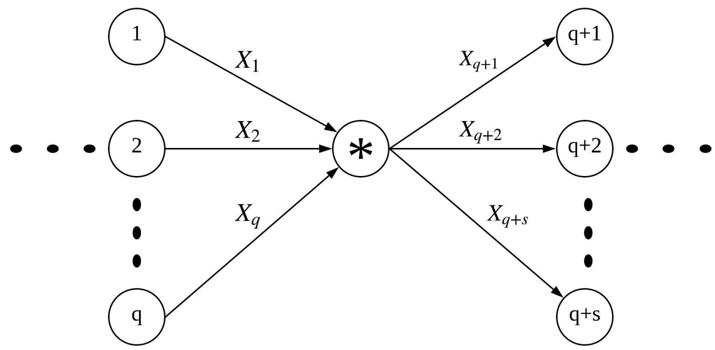


Figure 2: Focused network view (Bowman 1995)

path to node $\star$, and the second part of (3) is the probability that at least one of the emanating arcs of node $\star$ is on the longest path of the network, which can be computed recursively using equation (4). For more details about the arc criticality using node release times, please refer to Bowman (1995).

IPA is not applicable to IAC because the integrand is an indicator function, which is not continuous. Only LR is applicable to IAC. Equation (2) has a continuous integrand, so IPA is applicable. However, the explicit formula for the joint density function of release node times is not readily available, so LR is not applicable to (2). In conclusion, current versions of arc criticality are not amenable for applying either IPA or the LR method. And since IPA estimates usually have lower variance than LR estimator, we prefer IPA than LR when IPA is applicable. Although IPA is applicable to CAC, the IPA estimator for CAC is very complicated. Therefore, we look forward to finding a new arc criticality estimator whose IPA estimator is easy to implement, which will be introduced in section 3.2.

3.2 Threshold Arc Criticalities (TAC)

Before introducing the new version of the arc criticality estimator, we first present a lemma on which it is based.

Lemma 1 $C_a(i|X_j = x_j, 1 \le j \le m, j \ne i) = \Pr(X_i \ge m_i)$, where $m_i = max(\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i, 0)\Big|_{X_j = x_j, 1\le j\le m, j\ne i}$ and it is a continuous function of $\{x_j\}_{1\le j\le m, j\ne i}$.

Proof.

$$
\begin{aligned}
C_a(i|X_j = x_j, 1 \le j \le m, j \ne i) &= \Pr(\text{arc } i \text{ is on the critical path} \mid X_j = x_j, 1 \le j \le m, j \ne i) \\
&= \Pr(\|\mathscr{P}_i\| = \|\mathscr{P}\| \Big| X_j = x_j, 1 \le j \le m, j \ne i) \\
&= \Pr(\|\mathscr{P}_i\| = max(\|\mathscr{P}_i\|, \|\mathscr{P}_{i-}\|) \Big| X_j = x_j, 1 \le j \le m, j \ne i) \qquad (5) \\
&= \Pr(\|\mathscr{P}_i\| \ge \|\mathscr{P}_{i-}\| \Big| X_j = x_j, 1 \le j \le m, j \ne i) \qquad (6) \\
&= \Pr(X_i \ge max(\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i, 0) \Big| X_j = x_j, 1 \le j \le m, j \ne i) \\
&= \Pr(X_i \ge m_i).
\end{aligned}
$$

□

The justification going from (5) to (6) is because for given the arc length of all arcs except arc i, arc i is on the critical path if and only if the longest path length of paths including arc i is greater than or equal to the longest path length of paths excluding arc i. Equation (6) also appears in Dodin and Elmaghraby

(1985) for estimating the Arc Criticality Index. And the conditional event of (6) happens if and only if the length of arc i is greater than or equal to $max(\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i, 0)$.

From Lemma 1, a new expression for arc criticality is given by:

$$
\begin{aligned}
C_a(i) &= \int \cdots \int_{\mathbb{R}^{m-1}} C_a(i | X_j = x_j, 1 \le j \le m, j \ne i) \times \prod_{\substack{j=1 \\ j \ne i}}^{m} f_j(x_j)\, dx_j \\
&= \int \cdots \int_{\mathbb{R}^{m-1}} \bar{F}_i(m_i) \times \prod_{\substack{j=1 \\ j \ne i}}^{m} f_j(x_j)\, dx_j \qquad (7) \\
&= \int_{-\infty}^{\infty} \bar{F}_i(M_i) \times f_{M_i}(x) dx \qquad (8) \\
&= \mathbb{E}_{M_i}(\bar{F}_i(M_i))
\end{aligned}
$$

where

$$M_i = max(\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i, 0).$$

Each simulation first simulates the length of all arcs except for arc i, then calculates m_i, which is a continuous function of simulated arc lengths, and finally calculates the arc criticality estimator $\bar{F}_i(m_i)$. Here, M_i is a continuous function of $\{X_j\}_{1 \le j \le m, j \ne i}$, i.e., a random variable, and m_i is the realization of M_i. In (8), $f_{M_i}(x)$ is the probability density function of M_i. The new version of arc criticality (8) is named Threshold Arc Criticality (TAC), because m_i plays the role of a threshold.

3.3 Stochastic Derivative Estimators of Arc Criticalities

In equation (7), the estimator has a continuous integrand $\bar{F}_i(m_i)$, and the corresponding probability density function $\prod_{\substack{j=1 \\ j \ne i}}^{m} f_i(x_i)$ is also in explicit form. Therefore, the new arc criticality estimator solves the two issues mentioned in section 3.1. As a result, IPA can be used for estimating the derivative of arc criticality. In this section, θ_i is a distributional parameter of arc i.

3.3.1 First Derivatives

The LR estimator for $\frac{\partial C_a(i)}{\partial \theta_j}$ derived from (1) is given by:

$$I\{\|\mathscr{P}_i\| = \|\mathscr{P}\|\} \times \frac{\partial log(f_j(X_j))}{\partial \theta_j} = I\{\|\mathscr{P}_i\| = \|\mathscr{P}\|\} \times \frac{\frac{\partial f_j(X_j)}{\partial \theta_j}}{f_j(X_j)}. \qquad (9)$$

The following IPA estimators are derived from (8):
The IPA estimator for $\frac{\partial C_a(i)}{\partial \theta_i}$ is given by:

$$-\frac{\partial F_i(M_i)}{\partial \theta_i}. \qquad (10)$$

The IPA estimator for $\frac{\partial C_a(i)}{\partial \theta_j}$ when $j \neq i$ is given by:

$$-\frac{\partial F_i(M_i)}{\partial \theta_j} = -\frac{dF_i(M_i)}{dM_i}\frac{\partial M_i}{\partial X_j}\frac{\partial X_j}{\partial \theta_j}$$
$$= \frac{dF_i(M_i)}{dM_i} \times I\{\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i \geq 0\} \times (I\{j \in P^*_{i-}\} - I\{j \in P^*_i\}) \times \left(\frac{\frac{\partial F_j(X_j)}{\partial \theta_j}}{\frac{\partial F_j(X_j)}{\partial X_j}}\right) \tag{11}$$

where

$$\frac{\partial M_i}{\partial X_j} = I\{\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i \geq 0\} \times (I\{j \in P^*_{i-}\} - I\{j \in P^*_i\}), \frac{\partial X_j}{\partial \theta_j} = -\frac{\frac{\partial F_j(X_j)}{\partial \theta_j}}{\frac{\partial F_j(X_j)}{\partial X_j}}.$$

For the network presented in Figure 1, $\mathscr{P}_2 = \{(2,5)\}$, $\mathscr{P}_{2-} = \{(1,3,5),(1,4)\}$. Suppose in one simulation of all arc lengths except for arc 2, we have the realizations of $\{X_i\}_{i\neq 2}$ are given by $x_1 = 9$, $x_3 = 15$, $x_4 = 10$, $x_5 = 7$, then $\|\mathscr{P}_{2-}\| = max(x_1 + x_3 + x_5, x_1 + x_4) = 31$ and $\|\mathscr{P}_2\|^2 = x_5 + 0 = 7$. Therefore, $m_i = max(\|\mathscr{P}_{2-}\| - \|\mathscr{P}_2\|^2, 0) = max(max(x_1 + x_3 + x_5, x_1 + x_4) - (x_5 + 0)), 0) = 24$. When $X_2 = 24$, $\|\mathscr{P}_2\| = \|\mathscr{P}_{2-}\| = \|\mathscr{P}\| = 31$, and when $X_2 > 24$, for example, $X_2 = 25$, $\|\mathscr{P}\| = max(\|\mathscr{P}_2\|, \|\mathscr{P}_{2-}\|) = max(max(x_1 + x_3 + x_5, x_1 + x_4), x_5 + x_2) = 32$, under this realization, $P^*_i = (2,5)$, $P^*_{i-} = (1,3,5)$ and the critical path is $(2,5)$.

3.3.2 Second Derivatives

The IPA estimator for $\frac{\partial^2 C_a(i)}{\partial \theta_i^2}$ is given by:

$$-\frac{\partial^2 F_i(M_i)}{\partial \theta_i^2}.$$

The IPA estimator for $\frac{\partial^2 C_a(i)}{\partial \theta_j^2}$ for $j \neq i$ is not applicable, because $\frac{\partial M_i}{\partial X_j}$ is not a continuous function.

3.3.3 Third Derivatives

The IPA estimator for $\frac{\partial^3 C_a(i)}{\partial \theta_i^3}$ is given by:

$$-\frac{\partial^3 F_i(M_i)}{\partial \theta_i^3}.$$

The IPA estimator for $\frac{\partial^3 C_a(i)}{\partial \theta_j^3}$ for $j \neq i$ is not applicable, because the IPA estimator for the second derivative is not applicable.

3.4 Variance of TAC Estimator

Assume the number of simulation replications is N. Before introducing Lemma 2, we introduce some additional notation: $I_i^{(j)}$ denotes the jth sample obtained from simulation for $I\{\|\mathscr{P}_i\| = \|\mathscr{P}\|\}$ and $\bar{F}_i^{(j)}$ denotes the jth sample obtained from simulation for $\bar{F}_i(M_i)$. Then we have $I_i^{(j)}$ are independent identically distributed as $I\{\|\mathscr{P}_i\| = \|\mathscr{P}\|\}$ for different j, and $\bar{F}_i^{(j)}$ are independent identically distributed as $\bar{F}_i(M_i)$ for different j.

Let $C'_a(i)$ denote the estimator for $C_a(i)$ obtained from (1), and $C''_a(i)$ denotes the estimator for $C_a(i)$ obtained from (8), then for N simulation replications:

$$C'_a(i) = \frac{\sum_{k=1}^{N} I_i^{(k)}}{N}.$$

$$C''_a(i) = \frac{\sum_{k=1}^{N} \bar{F}_i^{(k)}}{N}.$$

Lemma 2 VAR($C''_a(i)$) $\leq$ VAR($C'_a(i)$), $\forall i$, $\forall N$.

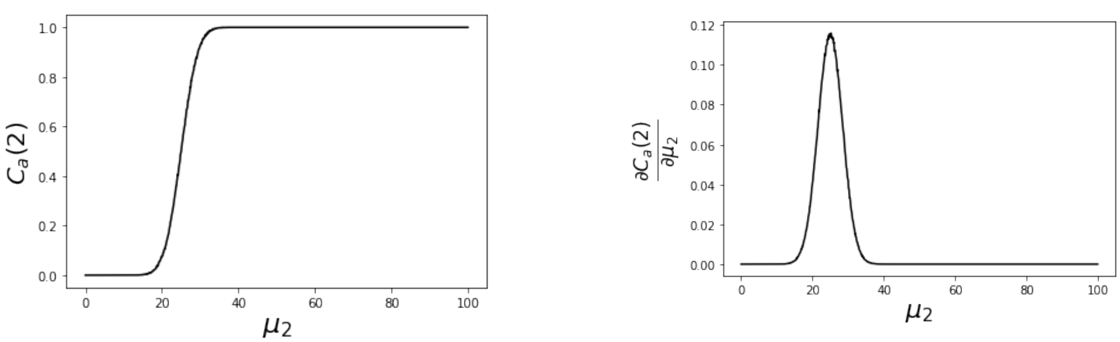
Proof. The proof is similar to Bowman (1995). We have $I_i^{(j)}$ is either 0 or 1 and $0 \leq \bar{F}_i^{(j)} \leq 1$, $\forall i$. By (1) and (8), $\mathbb{E}(I_i^{(j)}) = \mathbb{E}(F_i^{(j)}) = C_a(i)$. Then, for each element in the sample space fixed (a given sample of arc lengths), we have $(F_i^{(j)} - C_a(i))^2 \leq (I_i^{(j)} - C_a(i))^2$, $\forall i$. After taking expectations, we have, VAR($F_i^{(j)}$) $\leq$ VAR($I_i^{(j)}$), $\forall i$. And finally, because different samples are independent identically distributed, we have VAR($C''_a(i)$) $\leq$ VAR($C'_a(i)$), $\forall i$, $\forall N$. □

Lemma 2 shows that both the CAC and TAC arc criticality estimator have lower variances than compared to the IPA estimator of IAC.

4 EXAMPLE AND SIMULATION RESULTS

4.1 Estimators Validations

For the small stochastic network presented in Figure 1, we assume $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, and $\{X_i\}_{i=1}^m$ are independent random variables. The parameter settings are: $\mu_1 = 10$, $\mu_2 = 10$, $\mu_3 = 15$, $\mu_4 = 12$, $\mu_5 = 9$, and $\sigma_i = 2$ for $i = 1,2,3,4,5$. In this example, we are interested in the sensitivity analysis of the criticality of arc 2 and arc 3.

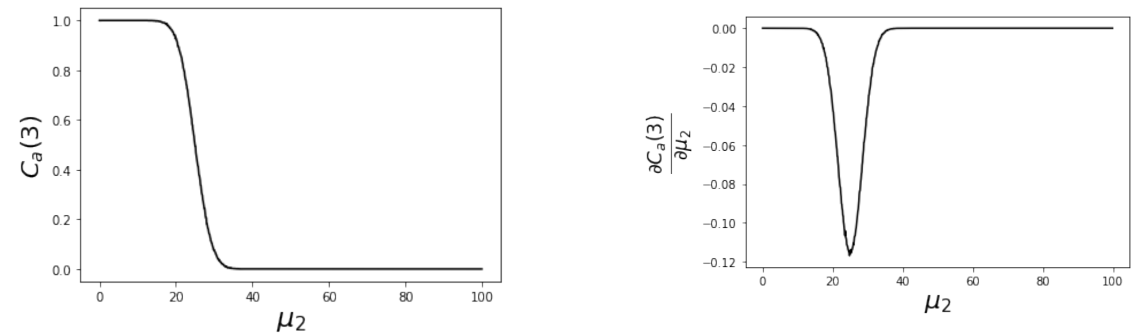


(a) $C_a(2)$ versus μ_2.

(b) $\frac{\partial C_a(2)}{\partial \mu_2}$ versus μ_2.

Figure 3: $C_a(2)$ and IPA derivative for Normal Distribution based on 10,000 independent replications (standard errors <0.01 throughout).

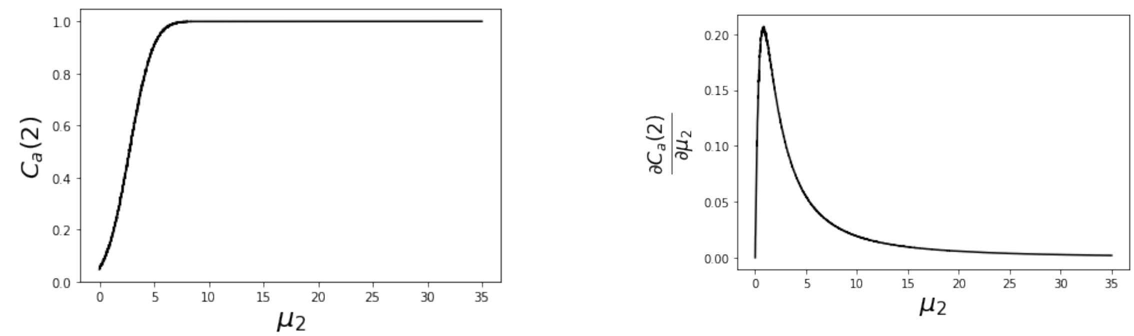
In the simulation, we first simulate the arc criticality using equation (1), where μ_2 ranges from 0.01 to 99.91 with stepsize 0.1, for each given μ_2, we simulate the criticality estimator for 10,000 independent replications and calculate its mean to get an estimator of $C_a(2)$. Here and in the later calculations of arc criticality estimators, we use equation (1) instead of equation (8) to check the validity of IPA estimators before checking the validity of TAC (Threshold Arc Criticality) estimator. We plot $C_a(2)$ versus μ_2 in Figure 3(a). With the same range of μ_2, where μ_2 ranges from 0.01 to 99.91, for each given μ_2, simulate



(a) $C_a(3)$ versus μ_2.

(b) $\frac{\partial C_a(3)}{\partial \mu_2}$ versus μ_2.

Figure 4: $C_a(3)$ and IPA derivative for Normal Distribution based on 10,000 independent replications (standard errors <0.01 throughout).



(a) $C_a(2)$ versus μ_2.

(b) $\frac{\partial C_a(2)}{\partial \mu_2}$ versus μ_2.

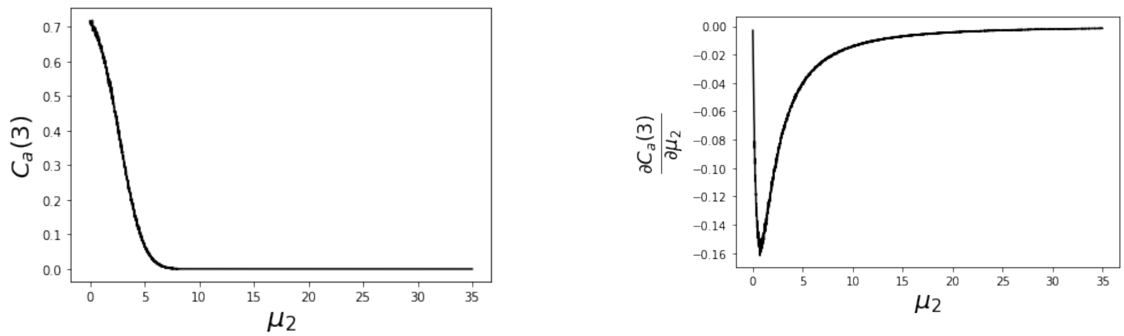
Figure 5: $C_a(2)$ and IPA derivative for Exponential Distribution based on 10,000 independent replications.

the IPA derivative estimator (10) 10,000 independent replications and take its mean as an estimator of $\frac{\partial C_a(2)}{\partial \mu_2}$. We plot the derivative of the function in Figure 3.

Similarly, with the same range of μ_2, we plot $C_a(3)$ and $\frac{\partial C_a(3)}{\partial \mu_2}$ estimated using equation (11) in Figure 4. From the simulation results, it is clear that the IPA estimators are valid stochastic gradient estimators for the Normal distribution example.

Next, we considered $X_i \sim Exp(\mu_i)$, and $\{X_i\}_{i=1}^m$ are independent random variables. The parameter settings are: $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1.5$, $\mu_4 = 1.2$, $\mu_5 = 0.9$. Here, we re-scale the parameters to control the variance of the exponential distributions. In this example, we are interested in the sensitivity analysis of the criticality of arc 2 and arc 3. For each estimation, the number of simulation replications is 10,000, and the range of μ_2 is from 0.005 to 35 with step size 0.01. For estimating $\frac{\partial C_a(3)}{\partial \mu_2}$, μ_2 has the same range from 0.005 to 35 with step size 0.01. The results are presented in Figures 5 and 6. From the simulation results, it is clear that the IPA estimators are valid for the Exponential distribution example.

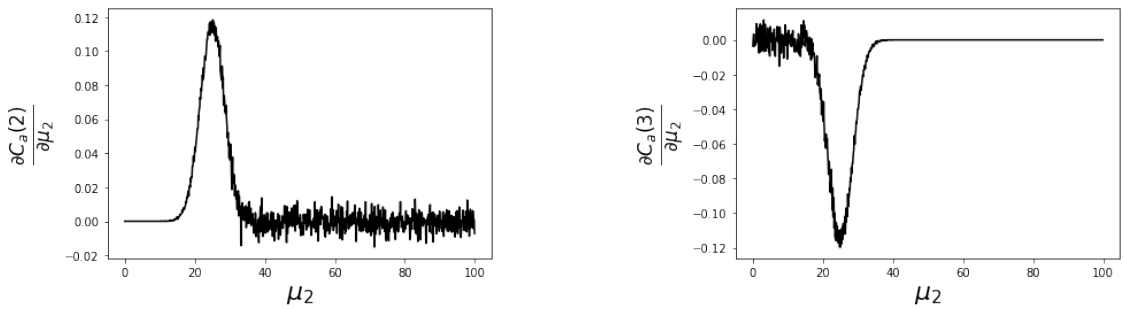
Next, we checked the LR estimator in (9) and compared it with the IPA estimators in (10) and (11) for both Normal distributed and Exponential distributed arc lengths. The number of simulation replications, distribution parameter and range settings are the same as before. The simulation result for the LR estimators are presented in Figures 7 and 8. Compared to the IPA estimator, the simulation results indicate that the LR estimators have larger variance. In Figure 7(a), for Normal distribution, the LR estimator of $\frac{\partial C_a(2)}{\partial \mu_2}$ is



(a) $C_a(3)$ versus μ_2.

(b) $\frac{\partial C_a(3)}{\partial \mu_2}$ versus μ_2.

Figure 6: $C_a(3)$ and IPA derivative for Exponential Distribution based on 10,000 independent replications.



(a) $\frac{\partial C_a(2)}{\partial \mu_2}$ versus μ_2.

(b) $\frac{\partial C_a(3)}{\partial \mu_2}$ versus μ_2.

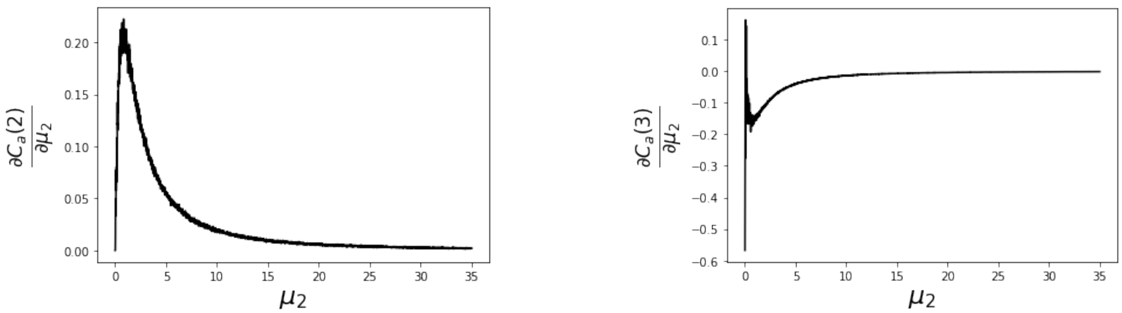
Figure 7: LR derivatives for Normal Distribution based on 10,000 independent replications.

not stable when μ_2 is large. In Figure 7(b), for Normal distribution, the LR estimator of $\frac{\partial C_a(3)}{\partial \mu_2}$ is not stable when μ_2 is small. In Figure 8(b), for Exponential distribution, the LR estimator of $\frac{\partial C_a(3)}{\partial \mu_2}$ is not stable when μ_2 is very close to zero. In conclusion, the LR estimators are unstable due to the large variance at values of μ_2 that have large corresponding criticality values. The IPA estimators in (10) and (11) can overcome this issue, and the overall variance performance of IPA estimators in this example is much better than the LR estimators.

Last step is to check the validation of the estimator in equation (8). For the distributions and parameters setting of Figure 3, the number of simulation replications is 100, and in each simulation, with the same given sample, three estimators (1), (2) and (8) are calculated, and the mean is used to estimate the criticality. The results of the simulation are presented in Figure 9, where (a) corresponds to (1), (b) corresponds to (2), and (c) corresponds to (8). From the simulation results, it is clear that TAC is a valid arc criticality estimator. The result in Figure 9 also indicates that in this example, the variance of CAC is less than the variance of TAC, and both CAC and TAC estimators have lower variance compared to the IAC arc criticality estimate.

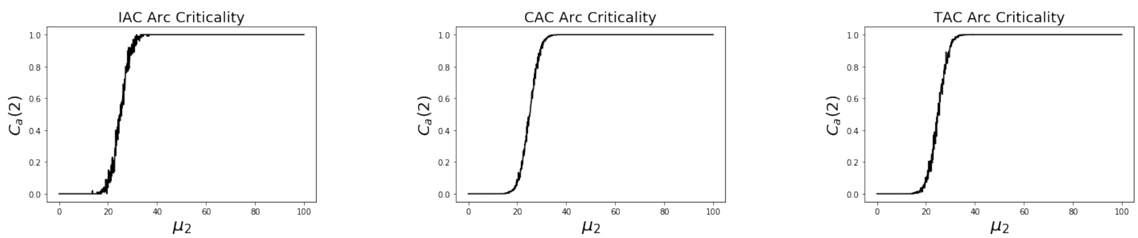
4.2 Sample Standard Deviations

The instability in Figures 7 and 8 is due to the large variance of LR estimators. Numerical results for LR and IPA estimators are presented in the following, where parameter settings for the normal and exponential distributions are the same as in section 4.1.



(a) $\frac{\partial C_a(2)}{\partial \mu_2}$ versus μ_2. (b) $\frac{\partial C_a(3)}{\partial \mu_2}$ versus μ_2.

Figure 8: LR derivatives for Exponential Distribution based on 10,000 independent replications.



(a) Indicator Arc Criticality. (b) Conditional Arc Criticality (CAC). (c) Threshold Arc Criticality (TAC).

Figure 9: Three different arc criticality estimators.

Table 1: Sample standard deviations of derivatives of Normal Distribution with sample size 10,000.

μ_2	0.005	0.1	1	10	20	30	50	90
IPA $\frac{\partial C_a(2)}{\partial \mu_2}$	2.30×10^{-14}	1.83×10^{-13}	1.65×10^{-12}	0.0004	0.059	0.058	1.1×10^{-14}	0
IPA $\frac{\partial C_a(3)}{\partial \mu_2}$	0	2.67×10^{-15}	2.90×10^{-14}	0.0002	0.059	0.059	1.5×10^{-13}	0
LR $\frac{\partial C_a(2)}{\partial \mu_2}$	0	0	0	0	0.183	0.468	0.5	0.5
LR $\frac{\partial C_a(3)}{\partial \mu_2}$	0.497	0.499	0.502	0.5	0.469	0.189	0	0

Table 2 column 1 explains the erratic behavior of the LR estimator in Figure 8(b) for μ_2 very small. And Tables 1 and 2 explain the instability in Figures 7 and 9. In Table 3, $C_a^{'}(2)$ denotes the IAC estimator (1), $C_a^{''}(2)$ denotes the TAC estimator, $C_a^{'''}(2)$ denotes the CAC estimator. Table 3 indicates that the CAC estimator performs the best in sample variance among the three estimators.

5 APPLICATION OF THE DERIVATIVE ESTIMATOR

Arc criticality indicates that arcs with high criticality are more likely to be on the critical path (the longest path), so decreasing the mean of the arc length (activity time) of the arc with highest criticality will decrease the expected longest path length (total project completion time) with high probability. However, the criticality estimation is not sufficient for determining how much to decrease the mean of the arc length of the arc with highest criticality. Decreasing the mean of the length of the arc with high criticality too much may result in it no longer being on the critical path with high probability. Therefore, how to decrease the mean of the an arc while keeping its criticality relatively high at the same time is the key to reducing

Table 2: Sample standard deviations of derivatives of Exponential Distribution with sample size 10,000.

μ_2	0.005	0.1	0.2	2	5	10	30
IPA $\frac{\partial C_a(2)}{\partial \mu_2}$	0.011	0.298	0.303	0.04	0.017	0.009	0.002
IPA $\frac{\partial C_a(3)}{\partial \mu_2}$	0.077	0.300	0.341	0.139	0.054	0.025	0.006
LR $\frac{\partial C_a(2)}{\partial \mu_2}$	0	1.564	1.288	0.396	0.173	0.091	0.032
LR $\frac{\partial C_a(3)}{\partial \mu_2}$	172	8.677	4.183	0.212	0.069	0.322	0.008

Table 3: Sample standard deviations of critical values of Normal Distribution with sample size 10,000.

μ_2	1	10	20	30	90
$C_a^{'}(2)$	0	0	0.263	0.255	0
$C_a^{''}(2)$	1.53×10^{-12}	0.0004	0.159	0.154	0
$C_a^{'''}(2)$	2.53×10^{-14}	0.0003	0.105	0.108	0

the expected total completion time of the project. To solve this issue, one approach is to simulate a large number of times for a large number of points to get a smooth function plot of criticality versus arc mean like Figure 3(a), then try to find the smallest mean value that keeps the critical value above a threshold. Another approach to this issue is to fit a logistic model to the criticality versus arc mean function (Cho and Yum 2004). A third approach is to use the stochastic derivative estimators with Newton's method to find the zero value of a function. In the following, a simple example is presented to explain the third approach, again using the network in Figure 1, with the parameter settings the same as Figure 3. From Figure 3(a), we can see that when μ_2 is 35, arc 2 has a very high criticality. Using the criticality estimator and simulating 100 times, we have an estimation of $C_a(2)$ as 0.9975. By decreasing μ_2, the expected total completion time (the length of the longest path) will also decrease. We are interested in finding the smallest μ_2 that can make the criticality of Arc 2 no less than 0.7. Since criticality of arc 2 as a function of μ_2 is a monotone function, this is the same as finding the x value that makes the y value equals to 0.7 of a function $y = f(x)$.

Define the function in Figure 3 as $G(x)$, where x represents μ_2 and $G(x)$ represents $C_a(2)$. Define a new function $H(x) = G(x) - 0.7$, then apply Newton's method to find the zero point of $H(x)$, with the iterative update:

$$x_{n+1} = x_n - \frac{H(x_n)}{G'(x_n)}.$$

$G'(x_n)$ is estimated by estimating $\frac{\partial C_a(2)}{\partial \mu_2}$ at x_n using equation (10), and $G(x_n)$ is estimated by estimating $C_a(2)$ using equation (2). The starting point is $x_0 = 30$, and for each iteration, 100 simulation replications are used. We stopped when $|G(x_n) - 0.7| < 0.01$, which turns out to be 4 iterations with an estimate of $\mu_2^* = 26.8$, for which the estimate of $C_a(2)$ at μ_2^* using equation (2) based on 10,000 simulation replications turned out to be 0.6999, achieving the target to the level of precision of the stopping rule.

6 CONCLUSIONS AND FUTURE RESEARCH

In this paper, we derived a new Arc Criticality estimator for stochastic networks called the Threshold Arc Criticality (TAC) estimator. The TAC estimator enables IPA derivatives with respect to the arc distribution parameters, which is problematic for previous arc criticality estimators. The variance of the TAC estimator is proved to be theoretically better than IAC estimator. However, the performance of the variance of the TAC estimator is statistically indistinguishable from the variance of the CAC estimator in a small network example. By applying IPA to the TAC estimator, we derived the stochastic derivative estimator of arc criticality with respect to distributional parameters of arc length. Applying the derivative estimator to project management, we can control the amount of decrease of critical arcs so that the expected project completion time can be decreased to the desired level.

For future work, we are interested in applying the Weak Derivative (WD) method and GLR (Peng et al. 2018) estimators for estimating the derivatives in stochastic networks. For Gaussian systems, the WD method outperforms IPA method (Heidergott et al. 2008). For SAN taking the derivative of total completion time with respect to mean of arc distributions, WD method also outperforms IPA and LR in variance performance (Manterola 2011), so it would be interesting to do a similar comparison for arc criticality. Another interesting avenue worth exploring is combining the new estimator with quasi-Monte Carlo methods, which have shown dramatic variance reduction for other SAN estimators (L'Ecuyer and Lemieux 2000). Finally, building on the illustrative application in Section 5, the ultimate goal is to formulate and solve a more realistic optimization problem for project management using the stochastic derivatives of arc criticality.

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