Proceedings of the 2020 Winter Simulation Conference
K.-H. Bae, B. Feng, S. Kim, S. Lazarova-Molnar, Z. Zheng, T. Roeder, and R. Thiesing, eds.

SENSITIVITY ANALYSIS OF ARC CRITICALITIES IN STOCHASTIC ACTIVITY NETWORKS

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ABSTRACT

Using Monte Carlo simulation, this paper proposes a new algorithm for estimating the arc criticalities of stochastic activity networks. The algorithm is based on the following result: given the length of all arcs in a network except for the one arc of interest, which is on the critical path (longest path) if and only if its length is greater than a threshold. Therefore, the new algorithm is named Threshold Arc Criticality (TAC). By applying Infinitesimal Perturbation Analysis (IPA) to TAC, an unbiased estimator of the stochastic derivative of the arc criticalities with respect to parameters of arc length distributions can be derived.

1 INTRODUCTION

This manuscript is a short extended abstract of the WSC 2020 submission con269. A stochastic activity network (SAN) is a directed acyclic graph with arcs representing activities and the direction of arcs representing the precedence relationships between different activities. For example, all arcs whose tail node is i have to be completed in order for the directed arc with head node i to start. The lengths of arcs represent the length of the time for completing the activities. And the lengths of the activities completion times are all random variables with known distributions and parameters. Stochastic activity networks have been widely used in project evaluation and review techniques (PERT).

In SANs, a path is a route starting from the source node and ending at the sink node. The length of a path is the sum of the lengths of all arcs on the path. The path with the longest length is called the critical path. The length of the critical path represents the project completion time, which is of interest to project managers and is to be minimized.

The criticality of a given arc is the probability that the arc is on the critical path. Previous works mostly focus on minimizing the expected project completion time, which requires estimating the stochastic derivative. For project managers, in order to minimize the project completion time, knowledge about the criticality of each arcs is important, since reducing the length of arcs with high criticality can reduce the completion time with high probability.

The sensitivity analysis of arc criticality with respect to the distribution parameters of each arcs is of interest. And both the sensitivity analysis of expected completion time and arc criticalities require techniques for estimating stochastic derivatives.

2 NOTATION

$\bar{F}_i(x) = 1 - F_i(x)$, $F_i(x)$ is the distribution function of arc i,
$C_a(i) = \Pr\{\text{arc } i \text{ is on the critical path}\}$ = criticality of arc i, $\mathscr{P}$ = set of all paths,
$\mathscr{P}_i$ = set of paths containing arc i, $\mathscr{P}_{i-}$ = set of paths not containing arc i,
P_i^* = set of arcs on the longest path corresponding to path set $\mathscr{P}_i$,
P_{i-}^* = set of arcs on the longest path corresponding to path set $\mathscr{P}_{i-}$,

$\|\cdot\|$ is an operator that calculates the length of the longest path for a given set of paths, $\|\cdot\|^i$ is an operator that calculates the length of the longest path for a given set of paths under the condition that arc i has length 0

3 THRESHOLD ARC CRITICALITY ESTIMATOR

Lemma 1 $C_a(i|X_j = x_j, 1 \le j \le m, j \ne i) = \Pr(X_i \ge m_i)$, where $m_i = max(\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i, 0)\Big|_{X_j = x_j, 1 \le j \le m, j \ne i}$ and it is a continuous function of $\{x_j\}_{1 \le j \le m, j \ne i}$.

From Lemma 1, a new expression for arc criticality is given by:

$$C_a(i) = \mathbb{E}_{M_i}(\bar{F}_i(M_i))$$

And the TAC estimator for $C_a(i)$ is given by:

$$1 - F_i(M_i)$$

where

$$M_i = max(\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i, 0).$$

4 DERIVATIVE ESTIMATOR OF ARC CRITICALITIES

The following IPA estimators are derived from Threshold Arc Criticality (TAC):
The IPA estimator for $\frac{\partial C_a(i)}{\partial \theta_i}$ is given by:

$$-\frac{\partial F_i(M_i)}{\partial \theta_i}. \tag{1}$$

The IPA estimator for $\frac{\partial C_a(i)}{\partial \theta_j}$ when $j \ne i$ is given by:

$$-\frac{\partial F_i(M_i)}{\partial \theta_j} = \frac{dF_i(M_i)}{dM_i} \times I\{\|\mathscr{P}_{i-}\| - \|\mathscr{P}_i\|^i \ge 0\} \times (I\{j \in P^*_{i-}\} - I\{j \in P^*_i\}) \times \left(\frac{\frac{\partial F_j(X_j)}{\partial \theta_j}}{\frac{\partial F_j(X_j)}{\partial X_j}}\right) \tag{2}$$

5 EXAMPLE

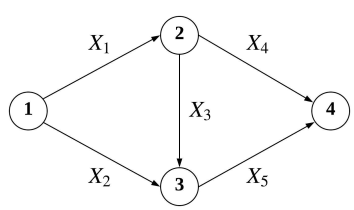

Figure 1: A small stochastic network

For the network presented in Figure 1, $\mathscr{P}_2 = \{(2,5)\}$, $\mathscr{P}_{2-} = \{(1,3,5),(1,4)\}$. Suppose in one simulation of all arc lengths except for arc 2, we have the realizations of $\{X_i\}_{i \ne 2}$ are given by $x_1 = 9$, $x_3 = 15, x_4 = 10, x_5 = 7$, then $\|\mathscr{P}_{2-}\| = max(x_1 + x_3 + x_5, x_1 + x_4) = 31$ and $\|\mathscr{P}_2\|^2 = x_5 + 0 = 7$. Therefore, $m_i = max(\|\mathscr{P}_{2-}\| - \|\mathscr{P}_2\|^2, 0) = max(max(x_1 + x_3 + x_5, x_1 + x_4) - (x_5 + 0)), 0) = 24$. When $X_2 = 24$, $\|\mathscr{P}_2\| = \|\mathscr{P}_{2-}\| = \|\mathscr{P}\| = 31$, and when $X_2 > 24$, for example, $X_2 = 25$, $\|\mathscr{P}\| = max(\|\mathscr{P}_2\|, \|\mathscr{P}_{2-}\|) = max(max(x_1 + x_3 + x_5, x_1 + x_4), x_5 + x_2) = 32$, under this realization, $P^*_2 = (2,5)$, $P^*_{2-} = (1,3,5)$ and the critical path is $(2,5)$.