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IMPORTANCE SAMPLING FOR COVAR ESTIMATION

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ABSTRACT

Measuring systemic risk has been an important problem in financial risk management. The CoVaR, as one of the commonly used systemic risk measures, could capture the tail dependency of the losses between financial institutions and financial systems. CoVaR is estimated via several statistical methods like quantile regression. In this paper, considering the complexity of the constituent securities in the financial institution and financial systems, we propose a simulation approach to estimate the CoVaR. We investigate the use of importance sampling to reduce the variance of the CoVaR estimator, and propose an efficient importance sampling distribution based on large deviation principles. We also illustrate the effectiveness of our approach via numerical experiments.

1 INTRODUCTION

Since the 2008 financial crisis, measuring systemic risk has aroused people's concern and been a critical research problem in financial risk management. Different from the market risk measures of financial institutions, e.g., Value-at-Risk (VaR) or expected shortfall (ES), which evaluate the tail behavior of the loss of financial institutions in isolation (Jorion 2006), the system risk measure could capture the tail dependency and co-movements between individual financial institutions and financial systems. Previous research has established several systemic risk measures from different perspectives, such as Acharya et al. (2017) and Brownlees and Engle (2017), and the CoVaR proposed by Tobias and Brunnermeier (2016) and Girardi and Ergün (2013) is one of the most commonly used systemic risk measures.

The CoVaR could be defined as the VaR of one financial institution (financial system) conditional on another institution (financial system) being under distress relative to its median state, which is usually represented by the institution suffering a large loss, i.e., a large quantile. Specifically, we adopt the definition of CoVaR from Girardi and Ergün (2013) in this paper. Let X and Y be the losses of two financial institutions or financial systems, and VaR_α^X be the α-VaR of X, i.e.,

$$\Pr\{X \geq \mathrm{VaR}_\alpha^X\} = \alpha. \tag{1}$$

Then, the CoVaR of Y at a confidence level $1-\alpha$ (or α-CoVaR), denoted by $\mathrm{CoVaR}_\alpha^{Y|X}$, is given by

$$\Pr\left\{Y \geq \mathrm{CoVaR}_\alpha^{Y|X} \middle| X \geq \mathrm{VaR}_\alpha^X\right\} = \alpha. \tag{2}$$

Accurate assessment of CoVaR could help the regulator grasp the picture of systemic risk and propose policies to mitigate the risk. Some statistical methods to estimate CoVaR have been proposed. For example, Tobias and Brunnermeier (2016) provided the quantile regression as the primary estimation method; Nolde et al. (2022) proposed a semi-parametric model under multivariate extreme value theory to estimate CoVaR;

Sun et al. (2020) built copula-based GARCH models to estimate CoVaR and time-varying CoVaR. Note that these methods regard all the constituent securities in a financial institution as a whole asset, and use the overall capitalization data of the financial institution to estimate the CoVaR. However, the constituent securities in a financial institution may have specific properties and need to be treated separately. For example, if a financial institution consists of thousands of derivative securities (e.g., options and swaps, etc.), then we need to evaluate these securities according to their specific payoffs and term structures over a large number of underlying assets, and cannot regard them as a whole asset. Therefore, we need to propose new methods for estimating CoVaR in consideration of the constituent securities in the financial institution.

Inspired by the methods for estimating market risk measures like VaR or ES, we could use Monte Carlo simulation to evaluate the constituent securities as well as estimate the CoVaR of the financial institutions. Estimating VaR or ES via Monte Carlo simulation has been studied by a number of scholars, e.g., Jamshidian and Zhu (1997), Picoult (1999), Gordy and Juneja (2010), Broadie et al. (2015), and Hong et al. (2017), and refers to Hong et al. (2014) for a comprehensive review. To capture extreme risk events, extreme VaR (extreme quantile) or ES, i.e., the confidence level α is very small, needs to be estimated, so a large number of simulation samples may be necessary to achieve the required precision. To remedy this issue, importance sampling (IS) is usually applied to reduce the variance of the estimator and improve the estimation efficiency (see Glynn 1996; Glasserman et al. 2000; He et al. 2022). However, as pointed out by He et al. (2022), when using IS in estimating quantile, we may be in a dilemma: selecting a good IS distribution requires the knowledge of the quantile at hand, which is the goal to begin with and thus forms a circular challenge. More precisely, if we want to estimate the quantile q_α such that $\Pr\{Y \geq q_\alpha\} = \alpha$ for some small α and model output Y, we need to determine a good or optimal IS distribution of Y for evaluating the indicator function $\mathbf{1}\{Y \geq q_\alpha\}$, which could highly depend on q_α. To untie this circularity, a large deviations principle is commonly used to first obtain an estimate of the quantile based on a tail approximation, and then choose a good IS distribution based on this quantile estimate, see Glynn (1996) and Glasserman et al. (2000). Another approach is to use adaptive IS, which reaches the optimal IS distribution and the true quantile simultaneously, see Bardou et al. (2009) and Pan et al. (2020). Recently, He et al. (2022) proposed a new adaptive IS regime with stochastic approximation and sample average approximation for both quantile estimation and general stochastic root-finding problem that suffer the same circular dilemma, and established strong consistency and asymptotic normality of the resulting estimators.

This paper proposes an efficient IS approach for estimating the CoVaR based on large deviations principles. We first use the IS approach from Glynn (1996) to obtain an estimate of the VaR of X. Then, we establish a tail approximation of the joint probability of X and Y. By embedding the VaR estimate of X, we can approximate the CoVaR of Y via this tail approximation. Next, based on both the VaR estimate and the CoVaR estimate, we derive an efficient IS distribution to generate simulation samples and estimate the CoVaR. Finally, we conduct numerical experiments to demonstrate the effectiveness of our IS approach.

The rest of this paper is as follows. Section 2 formulates the CoVaR estimation as a quantile estimation problem and proposes an exponential twisting IS regime. Section 3 presents an efficient IS distribution for CoVaR estimation via large deviations principles. Section 4 conducts two numerical examples to show the effect of our IS approach. Section 5 concludes our paper.

2 PROBLEM FORMULATION

The CoVaR is defined by the VaR of the financial system (or financial institution) conditional on an institution being under distress, which can capture the cross-sectional tail-dependence between the financial system (or financial institution) and one specific financial institution. Suppose that the portfolio loss of a financial institution is denoted by X, and the portfolio loss of financial system (or another financial institution) is denoted by Y. As long as we can generate simulation samples of X and Y, we can use Monte Carlo (MC) simulation to estimate $\mathrm{CoVaR}_\alpha^{Y|X}$.

2.1 Monte Carlo Simulation Approach

Let $\{(X_1,Y_1),(X_2,Y_2),\dots,(X_n,Y_n)\}$ be the n pairs of simulation samples generated from a joint probability density function $f(x,y)$ (with c.d.f $F(x,y)$). The MC estimator of $\text{CoVaR}_\alpha^{Y|X}$ can be derived from a two-stage estimation.

The first stage is to estimate the VaR of X. Let $F_{X,n}(x)$ denote the empirical distribution of the portfolio loss X based on n simulation samples,

$$F_{X,n}(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{X_i \le x\}.$$

The empirical quantile estimator for VaR_α^X is given by

$$\widehat{\text{VaR}}_\alpha^X = F_{X,n}^{-1}(1-\alpha).$$

Note that, $\widehat{\text{VaR}}_\alpha^X$ could be regarded as the root of the equation $F_{X,n}(x) = 1-\alpha$, then it can be represented by

$$\widehat{\text{VaR}}_\alpha^X = \inf\{x : F_{X,n}(x) \ge 1-\alpha\}.$$

For the second stage, we screen out the simulation sample pairs of $(X_i,Y_i), i = 1,2,\dots,n$ that X_i is greater than or equal to $\widehat{\text{VaR}}_\alpha^X$, and then estimate the $(1-\alpha)$-quantile with respect to Y. Specifically, let $F_{Y|X,n}$ be the empirical distribution of portfolio loss Y conditional on the event $\{X \ge \widehat{\text{VaR}}_\alpha^X\}$, and there are $\lfloor n\alpha \rfloor$ (usually n should satisfy $\lfloor n\alpha \rfloor \ge 1$) pairs of simulation replications satisfying

$$F_{Y|X,n}(y) = \frac{1}{\lfloor n\alpha \rfloor}\sum_{i=1}^{n} 1\{Y_i \le y, X_i \ge \widehat{\text{VaR}}_\alpha^X\}.$$

Then, the empirical quantile estimator of Y conditional on $\{X \ge \widehat{\text{VaR}}_\alpha^X\}$, i.e., the CoVaR estimator $\text{CoVaR}_\alpha^{Y|X}$ is given by

$$\widehat{\text{CoVaR}}_\alpha^{Y|X} = F_{Y|X,n}^{-1}(1-\alpha).$$

Similar to VaR estimator, the $\text{CoVaR}_\alpha^{Y|X}$ can also be written as a root by

$$\widehat{\text{CoVaR}}_\alpha^{Y|X} = \inf\left\{y : F_{Y|X,n}(y) \ge 1-\alpha\right\}.$$

Under appropriate conditions, if we assume that the probability density function of portfolio loss X, denoted as f_X, is strictly positive in a small neighborhood of VaR_α^X, the central limit theorem of $\widehat{\text{VaR}}_\alpha^X$ shows the following result,

$$\sqrt{n}\left(\widehat{\text{VaR}}_\alpha^X - \text{VaR}_\alpha^X\right) \xrightarrow{d} \frac{\sqrt{\alpha(1-\alpha)}}{f_X\left(\text{VaR}_\alpha^X\right)}\mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is a standard normal distribution. Similar to the VaR estimator, we also has the central limit theorem for the CoVaR estimator, and the proof can be found in Jiang and Yun (2022).

Proposition 1 Let $f_{Y|X}$ be the probability density function of the portfolio loss Y conditional on event $\{X \ge \text{VaR}_\alpha^X\}$, i.e.,

$$f_{Y|X}(y) = \frac{1}{\alpha}\int_{\text{VaR}_\alpha^X}^{\infty} f(x,y)dx.$$

Suppose that $f_{Y|X}(y)$ is strictly positive in a small neighborhood of $\mathrm{CoVaR}_\alpha^{Y|X}$. Then

$$\sqrt{\lfloor n\alpha \rfloor}\left(\widehat{\mathrm{CoVaR}}_\alpha^{Y|X} - \mathrm{CoVaR}_\alpha^{Y|X}\right) \xrightarrow{d} \frac{\alpha\sqrt{(1-\alpha)}}{f_{Y|X}\left(\mathrm{CoVaR}_\alpha^{Y|X}\right)}\mathscr{N}(0,1). \tag{3}$$

Note that the MC estimator $\widehat{\mathrm{CoVaR}}_\alpha^{Y|X}$ may not be efficient. As seen in (3), the effective number of simulation samples for estimating the CoVaR is only $\lfloor n\alpha \rfloor$. If α is small, i.e., we are interested in extreme events, we have to use a very large number of simulation samples to obtain a relatively accurate estimate of CoVaR. In addition, both the variances of $\widehat{\mathrm{VaR}}_\alpha^{X}$ and $\widehat{\mathrm{CoVaR}}_\alpha^{Y|X}$ are potentially very large because $f_X(\mathrm{VaR}_\alpha^X)$ and $f_{Y|X}(\mathrm{CoVaR}_\alpha^{Y|X})$ are evaluated at the right tail and likely to be close to zero. Therefore, we need to consider variance reduction techniques to improve the estimation efficiency.

2.2 Importance Sampling Approach

In this paper, we propose an exponential twisting approach to construct the IS distributions. Let $\mathbf{Z} = (X,Y)^\top$ be a random vector consisting of X and Y. Let $\mathbf{z} = (x,y)^\top$ and $\boldsymbol{\theta} = (\theta_x, \theta_y)^\top$. Define the logarithmic moment generating function of random vector $\mathbf{Z}$ as

$$\Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = \log \mathbb{E}\left[e^{\boldsymbol{\theta}^\top \mathbf{Z}}\right] = \log \int_{\mathbb{R}^2} e^{\boldsymbol{\theta}^T \mathbf{z}} f(\mathbf{z}) d\mathbf{z},$$

where $f(\mathbf{z}) \triangleq f(x,y)$ be the joint probability density function of $\mathbf{Z} = (X,Y)^\top$.

Let $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\boldsymbol{\Theta} = \{\boldsymbol{\theta} : \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) < \infty\}$, the exponential twisting changes the original distribution $f(\mathbf{z})$ to one of the exponential family $f(\mathbf{z};\boldsymbol{\theta})$ as

$$f(\mathbf{z};\boldsymbol{\theta}) = e^{\boldsymbol{\theta}^T \mathbf{z} - \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})} f(\mathbf{z}).$$

We will show how to choose the twisting parameter $\boldsymbol{\theta}$ (also called the IS parameter) in the next section. Now suppose that a good IS parameter $\boldsymbol{\theta}^* = (\theta_x^*, \theta_y^*)^\top$ is given to us, and we can generate simulation samples under the IS distribution $f(\mathbf{z};\boldsymbol{\theta}^*)$. Let $\{(X_1^*,Y_1^*),(X_2^*,Y_2^*),\ldots,(X_n^*,Y_n^*)\}$ be the new sample pairs, and denote $\Lambda_X(\theta_x)$ as the logarithmic moment generating function of random variable X. Similar to the MC estimator, we need a two-stage estimation. In the first stage, we estimate the VaR of X under the IS distribution. The empirical distribution function of X under the IS distribution (denoted by X^*) is given by

$$\tilde{F}_{X^*,n}(x;\theta_x^*) = \frac{1}{n}\sum_{i=1}^{n} 1\{X_i^* \le x\}\ell_X(X_i^*, \theta_x^*),$$

where $\ell_X(X_i^*, \theta_x^*) \triangleq \exp(\theta_x^* X_i^* - \Lambda_X(\theta_x^*))$ is the likelihood ratio. According to Glynn (1996), when α is small, we may set the empirical quantile estimator with IS as

$$\widetilde{\mathrm{VaR}}_\alpha^X = \inf\left\{x : \frac{1}{n}\sum_{i=1}^{n} 1\{X_i^* \ge x\}\ell_X(X_i^*, \theta_x^*) \le \alpha\right\},$$

which has a better asymptotic variance.

In the second stage, instead of considering the conditional distribution of $Y|X$, we consider the joint distribution of (X,Y). By (2), we have

$$P\left\{Y \ge \mathrm{CoVaR}_\alpha^{Y|X} \middle| X \ge \mathrm{VaR}_\alpha^X\right\} = \frac{P\left\{Y \ge \mathrm{CoVaR}_\alpha^{Y|X}, X \ge \mathrm{VaR}_\alpha^X\right\}}{P\left\{X \ge \mathrm{VaR}_\alpha^X\right\}} = \alpha.$$

Then combining with (1), we have

$$P\left\{Y \geq \mathrm{CoVaR}_\alpha^{Y|X}, X \geq \mathrm{VaR}_\alpha^X\right\} = \alpha^2. \tag{4}$$

Under the IS distribution, if $\mathrm{CoVaR}_\alpha^{Y|X}$ and VaR_α^X are given, then we can approximate left hand side of Equation (4) by

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Y_i^* \geq \mathrm{CoVaR}_\alpha^{Y|X}\right\} \mathbf{1}\left\{X_i^* \geq \mathrm{VaR}_\alpha^X\right\} \ell(X_i^*, Y_i^*, \boldsymbol{\theta}^*),$$

where $\ell(X_i^*, Y_i^*, \boldsymbol{\theta}^*) \triangleq \exp(\mathbf{Z}_i^{*\top}\boldsymbol{\theta}^* - \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}^*)) = \exp((X_i^*, Y_i^*)\boldsymbol{\theta}^* - \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}^*))$ is the likelihood ratio. Next, replace VaR_α^X by $\widetilde{\mathrm{VaR}}_\alpha^X$, $\mathrm{CoVaR}_\alpha^{Y|X}$ can be regarded as the root of the following equation (the variable is y)

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Y_i^* \geq y\right\} \mathbf{1}\left\{X_i^* \geq \widetilde{\mathrm{VaR}}_\alpha^X\right\} \ell(X_i^*, Y_i^*, \boldsymbol{\theta}^*) = \alpha^2.$$

Therefore, we can set the estimator of CoVaR with IS as

$$\widetilde{\mathrm{CoVaR}}_\alpha^{Y|X} = \inf\left\{y : \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Y_i^* \geq y\right\} \mathbf{1}\left\{X_i^* \geq \widetilde{\mathrm{VaR}}_\alpha^X\right\} \ell(X_i^*, Y_i^*, \boldsymbol{\theta}^*) \leq \alpha^2\right\}.$$

3 CHOICE OF IMPORTANCE SAMPLING PARAMETER

In this section, we first recall the choice of efficient IS parameter of VaR in Glynn (1996), and then propose our method to choose efficient IS parameters for CoVaR. At the end of this section, we present the whole algorithm for estimating CoVaR with the efficient IS parameter.

3.1 IS Parameters for VaR

As we have mentioned in Section 1, to determine an efficient IS parameter, we first need to obtain an estimate of the VaR. Such an estimate can be derived via large deviations principles. Recall that $\Lambda_X(\theta) = \log \mathbb{E}[\exp(\theta X)]$ is the logarithmic moment generating function of X. In Glynn (1996), the tail probability of X can be approximated via

$$P(X > x) \approx \exp(-x\tilde{\theta} + \Lambda_X(\tilde{\theta})), \tag{5}$$

with $x \gg \mathbb{E}[X]$, where $\tilde{\theta}$ is the root of the equation $\Lambda_X'(\tilde{\theta}) = x$, and $\Lambda_X'(\tilde{\theta})$ is the derivative of $\Lambda_X(\tilde{\theta})$ on $\tilde{\theta}$. Then we can derive a quantile approximation via (5). Specifically, let $\tilde{\theta}^*$ be the root of the following equation

$$-\tilde{\theta}^*\Lambda_X'(\tilde{\theta}^*) + \Lambda_X(\tilde{\theta}^*) = \log\alpha. \tag{6}$$

Then according to (5), we have

$$P\left(X > \Lambda_X'(\tilde{\theta}^*)\right) \approx \alpha.$$

That is, the VaR of X can be approximated via large deviations principle, which is given by

$$\overline{VaR}_\alpha^X \triangleq \Lambda_X'(\tilde{\theta}^*). \tag{7}$$

Then by exponential twisting, the IS distribution is given by

$$f(x;\tilde{\theta}^*) = e^{\tilde{\theta}^* x - \Lambda_X(\tilde{\theta}^*)} f(x),$$

Note that the mean of this IS distribution is

$$\int_{\Omega} x e^{\tilde{\theta}^* x - \Lambda_X(\tilde{\theta}^*)} f(x)dx = \frac{\int_{\Omega} x e^{\tilde{\theta}^* x} f(x)dx}{\int_{\Omega} e^{\tilde{\theta}^* x} f(x)dx} = \frac{d}{d\tilde{\theta}^*} \log \int_{\Omega} e^{\tilde{\theta}^* x} f(x)dx = \Lambda'_X(\tilde{\theta}^*).$$

That is, this IS distribution is equivalent to changing the mean of the original distribution to $\Lambda'_X(\tilde{\theta}^*)$, which is the approximation of the quantile via the large deviations principle.

Example 1 Consider a normal random variable $X \sim N(\mu, \sigma^2)$ with density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

then the IS distribution is a normal distribution with mean $\Lambda'_X(\tilde{\theta}^*) = \mu + \sigma^2 \tilde{\theta}^*$, i.e., the density function of the IS distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(x-\mu-\sigma^2\theta^*\right)^2}{2\sigma^2}}.$$

In addition, if $\mu = 1$, $\sigma^2 = 2.25$, and $\alpha = 0.05$, then $\theta^* = 2.9670$ and $\overline{VaR}_{\alpha}^{X} = 8.6758$. The true α-VaR of X is 7.8359. Although the estimate of VaR via the large deviations principle is a rough approximation, it is good enough to derive an efficient IS parameter (see Section 4).

3.2 IS Parameters for CoVaR

In order to obtain an efficient IS parameter, we first need to obtain an estimate of CoVaR. Similar to Glynn (1996), we can approximate the probability (4) via a tail approximation. Let $E = (x, \infty) \times (y, \infty) \subset \Re^2$, where $x \gg \mathbb{E}[X]$ and $y \gg \mathbb{E}[Y]$. Recall that $\Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = \log \mathbb{E}[\exp(\boldsymbol{\theta}^\top \mathbf{Z})]$ is the logarithmic moment generating function of $\mathbf{Z}$. Then we have the tail approximation (Theorem 2.2.30, Dembo and Zeitouni (1998))

$$P(X > x, Y > y) = P(\mathbf{Z} \in E) \approx \exp(-\Lambda^*_{\mathbf{Z}}(\mathbf{z}))$$

where

$$\Lambda^*_{\mathbf{Z}}(\mathbf{z}) = \sup_{\boldsymbol{\theta} \in \Re^2} \left\{ \boldsymbol{\theta}^\top \mathbf{z} - \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) \right\}. \tag{8}$$

In addition, the logarithmic moment generating function $\Lambda_{\mathbf{Z}}(\boldsymbol{\theta})$ is convex, i.e., the Hessian matrix $\nabla^2 \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})$ is positive definite. Then, we can obtain $\Lambda^*_{\mathbf{Z}}(\mathbf{z})$ by solving the optimization problem (8). That is, for given $\mathbf{z}$, the optimal solution that maximizes (8) is the root of the equation

$$\nabla \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = \mathbf{z}. \tag{9}$$

Plug (9) into (8), we have

$$\Lambda^*_{\mathbf{Z}}(\mathbf{z}) = \boldsymbol{\theta}^\top \nabla \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) - \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}).$$

Go back to (4), if α is small, then we can solve the following equation to obtain the estimate of CoVaR via the large deviations theory.

$$-\boldsymbol{\theta}^\top \nabla \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) + \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = 2 \log \alpha. \tag{10}$$

Let

$$\nabla \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = \left(\frac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_x}, \frac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_y} \right)^\top,$$

and (10) can rewritten as

$$-\theta_x \frac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_x} - \theta_y \frac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_y} + \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = 2 \log \alpha, \tag{11}$$

Note that there are two variables θ_x and θ_y in above equation. To uniquely determine them, we need another equation.

Note that in Section 3.1, VaR_α^X can be approximated by $\overline{\mathrm{VaR}}_\alpha^X$, then by (9), we have another equation that

$$\frac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_x} = \overline{\mathrm{VaR}}_\alpha^X. \tag{12}$$

That is, in Equation (9), x equals $\overline{\mathrm{VaR}}_\alpha^X$. Based on the distribution of $\mathbf{z}$, the logarithmic moment generating function can be calculated. Then, (11) and (12) formulate the system of equations with two equations and two unknown variables, which can be solved by mathematical software such as MATHEMATICA and MATLAB. Therefore, by solving (11) and (12), we can obtain the root $\boldsymbol{\theta}^*$. Then the importance sampling distribution is given by

$$f(\mathbf{z}, \boldsymbol{\theta}^*) = \exp(\mathbf{z}^\top \boldsymbol{\theta}^* - \Lambda(\boldsymbol{\theta}^*)) f(\mathbf{z}). \tag{13}$$

Next, we provide the algorithm for estimating the CoVaR with IS in Algorithm 1. Firstly, we solve Equation (6) to obtain $\tilde{\theta}^*$ and calculate the approximated VaR by (7). Secondly, we solve Equations (11) and (12) to obtain $\boldsymbol{\theta}^*$. Thirdly, we choose the IS distribution as (13) to generate simulation samples and estimate the VaR and CoVaR as introduction in Section 2.2.

Algorithm 1 CoVaR with importance sampling

Initialization: Original joint distribution $f(\mathbf{z})$ and its logarithmic moment generating function $\Lambda_{\mathbf{Z}}(\boldsymbol{\theta})$, original marginal distribution $f_X(x)$ and its logarithmic moment generating function $\Lambda_X(\theta)$, the quantile level α, the total number of simulation samples n.

Step 1. Solve the equation

$$-\tilde{\theta}^* \Lambda_X'(\tilde{\theta}^*) + \Lambda_X(\tilde{\theta}^*) = \log(\alpha)$$

to obtain $\tilde{\theta}^*$ and calculate $\overline{\mathrm{VaR}}_\alpha^X = \Lambda_X'(\tilde{\theta}^*)$;

Step 2. Solve the equations

$$\begin{cases} -\theta_x \dfrac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_x} - \theta_y \dfrac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_y} + \Lambda_{\mathbf{Z}}(\boldsymbol{\theta}) = 2\log\alpha \\ \dfrac{\partial \Lambda_{\mathbf{Z}}(\boldsymbol{\theta})}{\partial \theta_x} = \overline{\mathrm{VaR}}_\alpha^X \end{cases}$$

to obtain $\boldsymbol{\theta}^* = (\theta_x^*, \theta_y^*)^\top$;

Step 3. Choose the IS distribution as

$$f(\mathbf{z}, \boldsymbol{\theta}^*) = \exp(\mathbf{z}^\top \boldsymbol{\theta}^* - \Lambda(\boldsymbol{\theta}^*)) f(\mathbf{z}),$$

and generate simulation samples $\mathbf{Z}_i^* = (X_i^*, Y_i^*)^\top, i = 1, 2, \ldots, n$, under the IS distribution;

Step 4. Estimate VaR_α^X via

$$\widetilde{\mathrm{VaR}}_\alpha^X = \inf\left\{ x : \frac{1}{n} \sum_{i=1}^n 1\{X_i^* \geq x\} \ell_X(X_i^*, \theta_x^*) \leq \alpha \right\},$$

where $\ell_X(X_i^*, \theta^*) = f_X(X_i^*)/f_X(X_i^*; \theta^*)$ is the likelihood of X_i^*, $i = 1, 2, \ldots, n$;

Step 5. Estimate $CoVaR_\alpha^{Y|X}$ via

$$\widetilde{\mathrm{CoVaR}}_\alpha^{Y|X} = \inf\left\{ y : \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{Y_i^* \geq y\} \mathbf{1}\left\{ X_i^* \geq \widetilde{\mathrm{VaR}}_\alpha^X \right\} \ell(X_i^*, Y_i^*, \boldsymbol{\theta}^*) \leq \alpha^2 \right\}, \tag{14}$$

where $\ell(X_i^*, Y_i^*, \boldsymbol{\theta}^*) = f(X_i^*, Y_i^*)/f(X_i^*, Y_i^*; \theta^*)$ is the likelihood of $\mathbf{Z}_i^*$, $i = 1, 2, \ldots, n$.

Output: CoVaR estimator is $\widetilde{CoVaR}_\alpha^{Y|X}$.

4 NUMERICAL EXAMPLE

In this section, we consider two numerical examples. In the first example, we consider X and Y to be correlated normal random variables. In the second example, we assume that X and Y are financial options whose underlying assets are correlated.

4.1 Normal Distribution

We first consider $\mathbf{Z} = (X,Y)^\top \sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$ to be a multivariate normal distribution, where $\boldsymbol{\mu} = (\mu_X,\mu_Y)^\top$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2, & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y, & \sigma_Y^2 \end{pmatrix}.$$

Specifically, let $\mu_X = 1$, $\mu_Y = 2$, $\sigma_X = 1.5$, $\sigma_Y = 2$, and $\rho = 0.8$, and we change the simulation sample size n from 500 to 128000, and set $\alpha = 0.05$, 0.005, and 0.0005, respectively. All experiments are replicated 1000 times. Then we obtain Tables 1-3.

Table 1: Means, variances, and variance reduction ratios for 0.05-VaR and 0.05-CoVaR in the normal distribution example; the true CoVaR and VaR is 7.5457 and 3.4673, respectively.

n	CoVaR					VaR				
	IS		MC		Ratio	IS		MC		Ratio
	mean	variance	mean	variance		mean	variance	mean	variance	
500	7.535	5.87E-03	7.366	3.61E-01	61.5	3.466	2.69E-03	3.450	1.97E-02	7.3
1000	7.545	2.81E-03	7.528	1.84E-01	65.5	3.467	1.24E-03	3.462	1.02E-02	8.2
2000	7.542	1.39E-03	7.461	8.63E-02	62.2	3.464	6.70E-04	3.461	5.22E-03	7.8
4000	7.544	7.05E-04	7.515	4.84E-02	68.7	3.467	3.23E-04	3.469	2.47E-03	7.7
8000	7.546	3.61E-04	7.520	2.39E-02	66.3	3.467	1.53E-04	3.468	1.33E-03	8.7
16000	7.545	1.82E-04	7.535	1.28E-02	70.2	3.468	7.91E-05	3.468	6.50E-04	8.2
32000	7.546	8.89E-05	7.541	6.65E-03	74.8	3.467	4.16E-05	3.468	3.05E-04	7.3
64000	7.546	4.62E-05	7.543	3.08E-03	66.7	3.467	1.82E-05	3.467	1.55E-04	8.5
128000	7.546	2.41E-05	7.547	1.55E-03	64.3	3.467	1.00E-05	3.467	7.84E-05	7.8

Table 2: Means, variances, and variance reduction ratios for 0.005-VaR and 0.005-CoVaR in the normal distribution example; the true CoVaR and VaR is 10.0667 and 4.8637, respectively.

n	CoVaR					VaR				
	IS		MC		Ratio	IS		MC		Ratio
	mean	variance	mean	variance		mean	variance	mean	variance	
500	10.057	4.89E-03	7.550	1.19E+00	243.9	4.861	1.78E-03	4.862	1.06E-01	59.6
1000	10.063	2.17E-03	8.129	7.63E-01	352.1	4.863	8.70E-04	4.813	4.83E-02	55.5
2000	10.064	1.26E-03	8.678	6.68E-01	529.4	4.863	4.53E-04	4.838	2.40E-02	53.0
4000	10.067	6.21E-04	9.083	5.11E-01	823.4	4.863	2.10E-04	4.849	1.21E-02	57.6
8000	10.066	2.96E-04	9.496	4.82E-01	1624.8	4.863	1.09E-04	4.857	6.78E-03	62.2
16000	10.066	1.51E-04	9.867	3.99E-01	2651.1	4.864	5.71E-05	4.858	3.26E-03	57.1
32000	10.067	7.64E-05	10.201	3.70E-01	4843.7	4.864	2.93E-05	4.862	1.75E-03	59.7
64000	10.067	3.92E-05	10.090	1.50E-01	3833.8	4.864	1.33E-05	4.864	8.90E-04	67.2
128000	10.067	1.75E-05	10.019	7.09E-02	4045.9	4.864	6.97E-06	4.864	4.11E-04	59.0

Table 3: Means, variances, and variance reduction ratios for 0.0005-VaR and 0.0005-CoVaR in the normal distribution example; the true CoVaR and VaR is 12.0208 and 5.9358, respectively.

n	CoVaR					VaR				
	IS		MC		Ratio	IS		MC		Ratio
	mean	variance	mean	variance		mean	variance	mean	variance	
500	12.011	4.69E-03	6.817	2.00E+00	426.4	5.933	1.44E-03	5.562	3.27E-01	226.8
1000	12.017	1.95E-03	7.206	1.69E+00	864.2	5.936	7.13E-04	5.855	2.81E-01	394.7
2000	12.019	1.15E-03	8.045	1.16E+00	1007.1	5.935	3.74E-04	5.764	1.25E-01	334.6
4000	12.021	5.92E-04	8.536	9.98E-01	1685.8	5.936	1.72E-04	5.836	7.02E-02	408.1
8000	12.020	2.64E-04	9.043	7.95E-01	3007.8	5.936	8.57E-05	5.878	4.05E-02	472.3
16000	12.021	1.42E-04	9.519	6.06E-01	4256.0	5.936	4.35E-05	5.914	2.07E-02	475.8
32000	12.021	6.56E-05	9.941	5.29E-01	8064.5	5.936	2.30E-05	5.919	1.16E-02	505.2
64000	12.021	3.61E-05	10.327	4.34E-01	12019.1	5.936	1.09E-05	5.931	5.49E-03	502.9
128000	12.021	1.76E-05	10.735	3.91E-01	22236.8	5.936	5.79E-06	5.933	2.68E-03	463.7

In the tables, "IS" means the estimates with IS from Algorithm 1, and "MC" means the naive MC method. We present both the estimates for CoVaR and VaR, and observe the following: (i) The IS distributions derived from large deviations principles are efficient, and they can achieve good variance reduction effects. As α becomes smaller, the variance reduction ratios increase significantly. For example, in Tables 1-3 where α takes 0.05, 0.005, and 0.0005, respectively, fixing the sample size as 128000, the variance reduction ratios for IS are 64.3, 4045.9, and 22236.8, respectively. (ii) As the CoVaR involves rarer event than the VaR (the VaR corresponds to the probability α, whereas the CoVaR corresponds to the probability α^2 as seen in (4)), the variance reduction ratios of CoVaRs are much larger than those of VaRs. For example, in Table 2, the variance reduction ratios for CoVaR and VaR are 4045.9 and 59.0, respectively. (iii) In Table 3, when the number of simulation samples n is small (e.g., from 500 to 32000), the naive MC method is invalid since almost all the indicator functions in (14) are zero. Whereas, with small n, say $n = 500$, we still have an accurate estimate for CoVaR when using IS.

4.2 Option Portfolio

As we have mentioned in Section 1, the constituent securities in financial institutions and systems may have special structure. In this subsection, we consider the CoVaR of two financial institutions, and assume that one consists of an European call option and the other consists of an European put option. Set the initial prices of these two underlying stocks $S_1(0) = S_2(0)) = 100$, the strike prices $K_1 = K_2 = 100$, i.e., they are both at-the-money options, and the maturities $T_1 = T_2 = 0.5$. Let the risk-free interest rate $r = 0.05$. In addition, we write the two stock prices as a vector $\mathbf{S}(t) = (S_1(t), S_2(t))^\top$, and the values of these two options as $\mathbf{V} = [V_1(S_1(t),t), V_2(S_2(t),t)]^\top$. The loss L_i of option i over time interval Δt is given by $L_i = V_i(S_i(t),t) - V_i(S_i(t+\Delta t), t+\Delta t)$. Similar to the setting in Glasserman et al. (2000), we assume 250 trading days in a year and set $\Delta t = 0.04$. Let $\Delta\mathbf{S}/\mathbf{S} \sim N(0, \mathbf{\Sigma})$, where

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_X^2, & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y, & \sigma_Y^2 \end{pmatrix}.$$

The volatilities of two stocks are $\sigma_X = 0.3$ and $\sigma_y = 0.4$, respectively, and the correlation $\rho = 0.5$.

When determining the IS parameters $\tilde{\theta}^*$ and $\boldsymbol{\theta}^*$, we consider a delta-approximation of the option price, i.e., let $L_i \approx \tilde{L}_i = a_i + b_i\Delta_i S_i$, where $a_i = -\Theta_i \Delta t$ with $\Theta_i = \partial V_i/\partial t$ and $b_i = -\delta_i$ with $\delta_i = \partial V_i/\partial S_i$. Note that the $(\tilde{L}_1, \tilde{L}_2)$ is multivariate normal distribution, then its logarithmic moment generating function is known, and we can apply Algorithm 1 to estimate the CoVaR of $\tilde{L}_2$. Similar to the setting in the first numerical example, we change the simulation sample size n from 500 to 128000, and set $\alpha = 0.05$, 0.005, and 0.0005, respectively. Still, all experiments are replicated 1000 times. Then we obtain Tables 4-6.

Table 4: Means, variances, and variance reduction ratios for 0.05-VaR and 0.05-CoVaR in the option portfolio example.

n	CoVaR					VaR				
	IS		MC		Ratio	IS		MC		Ratio
	mean	variance	mean	variance		mean	variance	mean	variance	
500	1.756	4.64E-02	1.385	9.97E-01	21.5	5.163	8.97E-03	5.192	4.53E-02	5.0
1000	1.755	2.28E-02	1.629	6.16E-01	27.0	5.166	4.22E-03	5.209	2.16E-02	5.1
2000	1.760	1.10E-02	1.558	2.65E-01	24.1	5.165	2.21E-03	5.226	1.16E-02	5.3
4000	1.758	5.26E-03	1.591	1.38E-01	26.2	5.170	1.05E-03	5.226	6.31E-03	6.0
8000	1.759	2.48E-03	1.638	7.89E-02	31.8	5.169	5.29E-04	5.227	3.01E-03	5.7
16000	1.760	1.35E-03	1.635	3.82E-02	28.3	5.170	2.67E-04	5.229	1.38E-03	5.2
32000	1.760	6.75E-04	1.654	1.83E-02	27.1	5.171	1.25E-04	5.230	6.88E-04	5.5
64000	1.760	3.41E-04	1.657	9.38E-03	27.5	5.170	6.72E-05	5.230	3.68E-04	5.5
128000	1.760	1.79E-04	1.658	4.57E-03	25.5	5.170	3.62E-05	5.229	1.87E-04	5.2

Table 5: Means, variances, and variance reduction ratios for 0.005-VaR and 0.005-CoVaR in the option portfolio example.

n	CoVaR					VaR				
	IS		MC		Ratio	IS		MC		Ratio
	mean	variance	mean	variance		mean	variance	mean	variance	
500	2.804	4.55E-02	-2.137	6.71E+00	147.4	7.006	3.17E-03	7.022	1.26E-01	39.7
1000	2.822	2.41E-02	-0.966	3.75E+00	155.8	7.009	1.45E-03	6.993	6.12E-02	42.1
2000	2.822	1.07E-02	0.126	2.86E+00	266.4	7.010	7.57E-04	7.018	3.10E-02	40.9
4000	2.822	5.76E-03	0.952	1.94E+00	336.3	7.012	3.69E-04	7.033	1.50E-02	40.8
8000	2.823	2.68E-03	1.797	1.47E+00	548.8	7.011	1.75E-04	7.045	7.82E-03	44.7
16000	2.826	1.52E-03	2.330	9.39E-01	617.7	7.011	9.60E-05	7.051	3.97E-03	41.4
32000	2.826	6.61E-04	2.915	7.59E-01	1148.7	7.012	4.38E-05	7.049	2.01E-03	45.8
64000	2.825	3.57E-04	2.760	3.54E-01	991.3	7.011	2.16E-05	7.052	9.96E-04	46.1
128000	2.825	1.72E-04	2.629	1.74E-01	1007.7	7.012	1.16E-05	7.053	4.91E-04	42.3

Table 6: Means, variances, and variance reduction ratios for 0.0005-VaR and 0.0005-CoVaR in the option portfolio example.

n	CoVaR					VaR				
	IS		MC		Ratio	IS		MC		Ratio
	mean	variance	mean	variance		mean	variance	mean	variance	
500	3.498	4.77E-02	-6.168	1.76E+01	368.2	8.012	1.37E-03	7.701	2.23E-01	162.0
1000	3.512	2.38E-02	-6.704	1.93E+01	808.6	8.015	6.07E-04	7.972	1.42E-01	234.3
2000	3.523	1.18E-02	-4.475	1.07E+01	904.3	8.013	3.22E-04	7.879	7.28E-02	226.3
4000	3.522	6.08E-03	-3.645	7.69E+00	1263.5	8.015	1.56E-04	7.954	4.32E-02	276.9
8000	3.526	2.83E-03	-2.522	5.42E+00	1913.5	8.014	7.05E-05	7.997	2.30E-02	326.4
16000	3.527	1.43E-03	-1.424	3.89E+00	2715.8	8.015	4.06E-05	8.022	1.21E-02	298.9
32000	3.528	7.29E-04	-0.479	2.59E+00	3556.7	8.015	1.91E-05	8.031	6.58E-03	343.9
64000	3.528	3.77E-04	0.359	1.87E+00	4963.9	8.015	9.20E-06	8.035	3.41E-03	371.2
128000	3.528	1.80E-04	1.175	1.38E+00	7688.4	8.015	4.69E-06	8.042	1.73E-03	369.2

We observe the following: (i) Instead of considering the distribution of L_1 and L_2 themselves, we use their delta-approximations ($\tilde{L}_1$ and $\tilde{L}_2$) to determine the IS parameters $\tilde{\theta}^*$ and $\boldsymbol{\theta}^*$. These parameters we choose may not be optimal but still make the IS distribution very efficient. The good variance reduction effect can be achieved. As α becomes smaller, the variance reduction ratios increase significantly. For example, in Tables 4-6 where α takes 0.05, 0.005, and 0.0005, respectively, fixing the sample size as 128000, the variance reduction ratios for IS are 25.5, 1007.7, and 7688.4, respectively. For the problem without moment generating function or with complicated moment generating function, the approximated IS distribution shown in this example could be a good way to reduce variance. (ii) Similar to the first example, the variance reduction ratios of CoVaRs are much larger than those of VaRs. For example, in Table 6, the variance reduction ratios for CoVaR and VaR are 7688.4 and 369.2, respectively. (iii) In Table 6, the naive MC of CoVaR is invalid even for large sample size ($n = 128000$) since almost all the indicator functions in (14) are zero. In Table 5, it is still invalid until $n = 32000$. The naive MC of VaR is relatively acceptable for all cases.

5 CONCLUSION

In this paper, we propose a simulation approach to estimate the CoVaR, which is an important systemic risk measure to capture the tail dependency of two financial institutes or systems. With the importance sampling technique based on large deviations principles, algorithm is designed to reduce the variance of the CoVaR estimator. The importance sampling parameters are chosen carefully by solving two equations. And the good variance reduction effect are shown in both two numerical examples. Specifically, IS parameter is usually insensitive, so that the simplification of moment generating function can still achieve desirable results.

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