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COMBINING RETROSPECTIVE APPROXIMATION WITH IMPORTANCE SAMPLING FOR OPTIMISING CONDITIONAL VALUE AT RISK

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ABSTRACT

This paper investigates the use of retrospective approximation solution paradigm in solving risk-averse optimization problems effectively via importance sampling (IS). While IS serves as a prominent means for tackling the large sample requirements in estimating tail risk measures such as Conditional Value at Risk (CVaR), its use in optimization problems driven by CVaR is complicated by the need to tailor the IS change of measure differently to different optimization iterates and the circularity which arises as a consequence. The proposed algorithm overcomes these challenges by employing a univariate IS transformation offering uniform variance reduction in a retrospective approximation procedure well-suited for tuning the IS parameter choice. The resulting simulation based approximation scheme enjoys both the computational efficiency bestowed by retrospective approximation and logarithmically efficient variance reduction offered by importance sampling.

1 INTRODUCTION

Conditional value at risk (CVaR) serves as a widely used risk measure towards assessing tail risks in quantitative risk management and operations research (see McNeil, Frey, and Embrechts 2015; Rockafellar and Uryasev 2000). For a loss $\ell(\boldsymbol{X},\boldsymbol{\theta})$ associated with a decision choice $\boldsymbol{\theta}$ under a random realization $\boldsymbol{X}$, let $v_\beta(\boldsymbol{\theta})$ denote the $(1-\beta)$-th quantile of $\ell(\boldsymbol{X},\boldsymbol{\theta})$. Then its CVaR at the tail-level $\beta \in (0,1)$ is given by,

$$C_\beta(\boldsymbol{\theta}) = E\left[\ell(\boldsymbol{X},\boldsymbol{\theta}) \mid \ell(\boldsymbol{X},\boldsymbol{\theta}) \geq v_\beta(\boldsymbol{\theta})\right],$$

which measures the average loss over the worst β-fraction of the realizations. Under the assumption that $\ell(\boldsymbol{X},\cdot)$ is convex, $C_\beta(\boldsymbol{\theta})$ has been shown to possess favourable properties such as convexity, subadditivity and coherence (see Acerbi and Tasche 2002). Minimizing CVaR $C_\beta(\boldsymbol{\theta})$ over a compact set Θ enjoys the following variational representation (see Rockafellar and Uryasev 2000),

$$c_\beta = \inf_{u\in\mathbb{R},\boldsymbol{\theta}\in\Theta}\left[u+\beta^{-1}E\left(\ell(\boldsymbol{X},\boldsymbol{\theta})-u\right)^+\right] = \inf_{u\in\mathbb{R},\boldsymbol{\theta}\in\Theta} f(u,\boldsymbol{\theta}), \tag{1}$$

which is a convenient starting point for solving optimization problems. Consequently, CVaR has served as one of the primary vehicle for introducing risk aversion in a variety of planning and resource allocation problems in operations research and quantitative finance.

Since (1) is rarely solvable in closed form, Monte Carlo methods are often deployed to obtain an approximate solution. Given N i.i.d. samples of data $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_N$ from the distribution of $\boldsymbol{X}$, define the sample averaged objective,

$$\hat{f}_n(u,\boldsymbol{\theta}) = \left[u+\frac{1}{N\beta}\sum_{i=1}^{N}(\ell(\boldsymbol{X}_i,\boldsymbol{\theta})-u)^+\right]$$

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