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PORTFOLIO RISK MEASUREMENT VIA STOCHASTIC MESH WITH AVERAGE WEIGHT

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ABSTRACT

Nested simulation has been widely used in the risk measurement of derivative portfolio. The convergence rate of the mean squared error (MSE) of the standard nested simulation is $k^{-2/3}$, where k is the simulation budget. To speed the convergence, we propose a stochastic mesh approach with average weight to portfolio risk measurement under the nested setting. We establish the asymptotic properties of the stochastic mesh estimator for portfolio risk, including the bias, variance and then the MSE. In particular, we show that the MSE converges to zero at a rate of k^{-1}, which is the same as that under the non-nested setting. The proposed method also allows for path dependence of financial instruments in the portfolio. Numerical experiments show that the proposed method performs well.

1 INTRODUCTION

A standard nested simulation procedure for portfolio risk measurement proceeds in two levels. In the outer level, scenarios of financial risk factors are simulated over a given risk horizon. Then in the inner level, one simulates a number of samples of security cash flows until the maturity of the securities conditional on a particular scenario of risk factors, and the portfolio value at the risk time horizon is computed. In practice, however, such two-level simulations could be extremely computationally expensive. Typically, the simulation budget can be measured by $k = cn_1n_2$ (see, e.g., Gordy and Juneja 2010; Zhang et al. 2022), where c is a constant depending on the distribution of the loss function and the type of the risk measure, n_1 and n_2 denote the outer- and inner-level sample sizes, respectively. Lee (1998), and Gordy and Juneja (2010) analyzed the nested simulation estimator, and showed that its optimal asymptotic MSE diminishes at rate $k^{-2/3}$ if the underlying scenario space is continuous, while Lee and Glynn (2003) showed that the MSE decays at $\log k/k$ if the underlying scenario space is discrete. Specifically, Gordy and Juneja (2010) demonstrated that the convergence rate of MSE achieves the optimal level when $n_1 = c_1k^{2/3}$ and $n_2 = c_2k^{1/3}$, where c_1 and c_2 are constants, and $c = c_1c_2$. Although the theoretical budget allocation n_1 and n_2 are provided, the constants c_1 and c_2 are typically unknown and very difficult to calculate. Therefore, instead of accelerating the convergence rate of the MSE of the nested simulation estimator, Zhang et al. (2022) proposed a bootstrap-based allocation rule for a given finite simulation budget, so that MSEs could converge at the optimal rate of order $k^{-2/3}$ in practice. Furthermore, Cheng et al. (2022) established the central limit theorem for nested simulation estimators, and then based on it and the budget allocation rule in Zhang et al. (2022), they constructed the unified confidence intervals, which guarantees the MSE and the width of the confidence interval achieve their optimal convergence rates, respectively.

From the convergence rate of MSE, it can be seen that the nested simulation estimator does not achieve the optimal convergence rate of order $1/k$ in Monte Carlo simulation. Therefore, a lot of researchers devoted to establish new algorithms to improve the performance of the nested simulation estimator. Along the same line under the nested setting, Broadie et al. (2011) proposed a sequential allocation of the simulation budget into the inner-level simulation for the estimation of the probability of large losses, and the convergence rate of the MSE is of order $k^{-4/5+\varepsilon}$ for all positive ε. In addition, Sun et al. (2010), Goda (2017) and Cheng and Zhang (2021) investigated the non-nested simulation estimator for the (higher order) central moment of loss function (conditional expectation), in which the essential idea is to construct unbiased estimator for the central moment of loss function, and thus the MSE equals the variance.

Another line of research for portfolio risk measurement under the nested setting is to fit the portfolio loss function on the risk factors by some statistical and machine learning methods. Broadie et al. (2015) proposed the least-squares method (LSM) to fit the loss function, and indicated that the MSE via the LSM converges at a rate k^{-1} to a nonzero level depending on the regression model error. The major drawback of this method is that it is asymptotically biased and the selection of basis functions is difficult. Instead of using the parametric method like LSM, Hong et al. (2017) proposed a kernel smoothing method, which is nonparametric, to fit the loss function, but it could usually suffers from curse of dimensionality if the risk factors are high-dimensional. To overcome this issue, they proposed a decomposition technique to decompose the high-dimensional risk factors in to several low-dimensional ones. Liu and Staum (2010) used a meta-model approach, i.e., stochastic kriging, to estimate expected shortfall. In particular, Zhang et al. (2017) and Zhang et al. (2022) proposed the stochastic mesh or likelihood ratio method to fit the loss function. The estimation is unbiased, and thus the convergence rate of the MSE of the corresponding risk estimator reaches optimal one of order k^{-1}, which is the same as that under a non-nested setting.

In this paper, we study the portfolio risk measurement via stochastic mesh with average weights. This was first proposed by Broadie and Glasserman (1997) (see also Broadie and Glasserman 2004) for pricing of American options. For more work about stochastic mesh approach to American options, see Avramidis and Hyden (1999), Avramidis and Matzinger (2004), Broadie et al. (2000), Liu and Hong (2009), Zhang et al. (2017) and Zhang et al. (2022), and about average weights, see Hesterberg (1988), Veach and Guibas (1995), Veach and Guibas (1995) and Feng and Staum (2017). Compared to the application in pricing of American options, the stochastic mesh method has an advantage in portfolio risk measurement, which has been investigated in Zhang et al. (2017) and Zhang et al. (2022). We construct an estimation of loss function by stochastic mesh with average weights, and then estimate the portfolio risk. Specifically, we show that the rate of convergence of its MSE is of order k^{-1}.

The rest of this paper is organized as follows. The problem is formulated in Section 2. In Section 3, we introduce the stochastic mesh method and the average weight, and then construct the stochastic mesh estimator of portfolio risk. In Section 4, we establish the rate of convergence of the MSE. Preliminary numerical experiments are presented in Section 5, followed by conclusions in Section 6.

2 PROBLEM FORMULATION

Consider a portfolio consisting of a sequence of financial instruments that may include derivative contracts. The value of the portfolio depends on kinds of market variables, such as stock prices, stock indices, exchange rates and other tradable assets. Suppose that the price dynamics of these market variables are governed by a vector valued Markov process $\{S_t \in \mathbb{R}^d, t \geq 0\}$, which is defined on a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathscr{F}$ is a σ-algebra of subsets of Ω and σ-subalgebra $\mathscr{F}_t$ is generated by $\{S_u\}_{0\le u\le t}$, i.e., the set of information available up to t. Hence S_t is adapted to $\{\mathscr{F}_t\}_{t\ge 0}$.

Suppose that the derivative contracts in the portfolio have the same maturity date T. Due to the discrete nature of computer simulation, a stochastic process is often simulated in discrete time points. Therefore, throughout the paper we work with a discretized version of S_t valued at a sequence of time points $0 = t_0 < t_1 < \cdots < t_N = T$. For notational simplicity, we write S_{t_i} as S_i for $i = 0, 1, ..., N$.

The value of the portfolio depends on the market variables. For $i = 0, 1, ..., N$, the value of the portfolio at time t_i can be represented as (see Chapter 6 in Duffie 1996),

$$V_i = \mathbb{E}[Y|\mathscr{F}_i],$$

where Y denotes the summation of all cash flows (weighted with appropriate discounted factors) provided by the instruments in the portfolio and the expectation is taken under a martingale pricing measure.

A risk manager is interested in measuring the risk of the portfolio over a future time horizon. Without loss of generality, we assume that the time horizon up to which the risk is measured is t_τ for an integer τ satisfying $1 \le \tau < N$. The risk of the portfolio is associated with its loss over this time horizon, which is defined as

$$L = V_0 - V_\tau = V_0 - \mathbb{E}[Y|\mathscr{F}_\tau] = \mathbb{E}[V_0 - Y|\mathscr{F}_\tau].$$

Typically, the random variable Y can be viewed as a function of $(S_\tau, S_{\tau+1}, ..., S_N)$, and suppose that $V_0 - Y = h(S_\tau, S_{\tau+1}, ..., S_N)$. Due to the Markov property of S, the loss can be written as

$$L(S_\tau) = \mathbb{E}[h(S_\tau, S_{\tau+1}, ..., S_N)|S_\tau].$$

More generally, if the portfolio consists of path-dependent securities, then the loss at time τ may have the form $L = \mathbb{E}[h(S_1, ..., S_N)|S_1, ..., S_\tau]$, and then portfolio risk is

$$L(x_1, ..., x_\tau) = \mathbb{E}[h(S_1, ..., S_N)|(S_1, ..., S_\tau) = (x_1, ..., x_\tau)].$$

As indicated in Zhang et al. (2017) and Zhang et al. (2022), the results related to $L(x)$ also hold for $L(x_1, ..., x_\tau)$, so we just analyze the case of $L(x)$ for the simplicity of notations.

Here, we consider risk measures formulated as

$$\alpha = \mathbb{E}g(L(S_\tau)), \tag{1}$$

where $g(\cdot)$ is the risk function. In fact, (1) is a nested estimation problem. Obviously, this problem is trivial when $g(\cdot)$ is a linear function. For example, if $g(x) = ax + b$ for some constants a and b, then it could be easily seen that $\alpha = \mathbb{E}(a\mathbb{E}[Y|X] + b) = \mathbb{E}(aY + b)$, and the estimation of α becomes a simple problem. To avoid this, assume that $g(\cdot)$ is nonlinear, and we focus on three kinds of $g(\cdot)$ as follows: when the risk is measured by the *squared tracking error*, $g(\cdot)$ is a quadratic function; when the risk measure is the *expected excess loss*, $g(\cdot)$ is a hockey-stick function, i.e., $g(x) = x^+$; and when the risk measure is the probability of a large loss, $g(\cdot)$ is an indicator function, i.e., $g(x) = 1_{\{x\ge 0\}}$. The latter two functions differ from the first one in smoothness. Specifically, the hockey-stick function has a non-differentiable but continuous point, while the indicator function has a discontinuous point. These two functions are related to two important risk measures, value at risk and conditional value at risk, which both have received increasing attention in recent two decades. In addition, a more general function with a finite number of non-differentiable or discontinuous points could be decomposed into a linear combination of these three types of functions, so the risk measure with this function in (1) has the same convergence rate of MSE as that of risk measures with the three types of functions. For more discussion, see Hong et al. (2017).

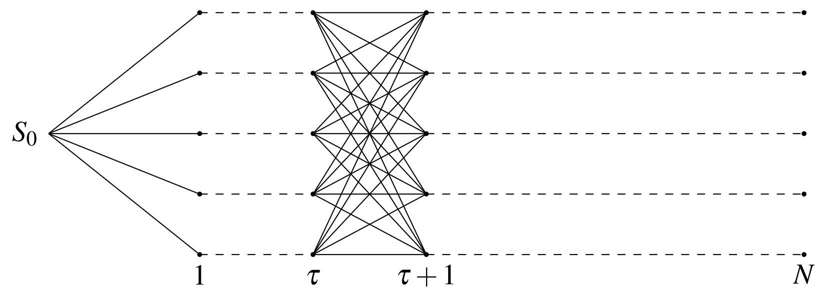

Figure 1: Illustration of stochastic mesh.

3 STOCHASTIC MESH METHOD WITH AVERAGE WEIGHT

In this section, we present the stochastic mesh estimator of $L(x)$, and then deduce the corresponding estimator of α. For more details, see Broadie and Glasserman (2004), Section 8.5 in Glasserman (2004), Zhang et al. (2017), Zhang et al. (2022) and see Liu and Hong (2009) for a new perspective. The construction of stochastic mesh for portfolio risk measurement is illustrated in Figure 1. To begin, we assume that S_0 is a constant and then generate randomly m paths $\{S_k^{(j)}\}_{j=1}^m$, $k=1,...,N$. The mesh points are $S_\tau^{(j)}$ and $S_{\tau+1}^{(j)}$ for $j=1,...,m$. For $k=1,...,N-1$, let $f_k(x,\cdot)$ denote the transition density of S_{k+1} given $S_k=x$, and let $f_k(\cdot)$ denote the probability density from which the points $\{S_k^{(j)}\}_{j=1}^m$ are sampled.

Denote $\mathbf{S}=(S_{\tau+1},...,S_N)$, then we have

$$\begin{aligned} L(x) &= \mathbb{E}[h(S_\tau,\mathbf{S})|S_\tau=x]=\mathbb{E}[h(x,\mathbf{S})|S_\tau=x] \\ &= \int h(x,\mathbf{s})f_\tau(x,s_{\tau+1})f_{\tau+1}(s_{\tau+1},s_{\tau+2})\cdots f_{N-1}(s_{N-1},s_N)d\mathbf{s} \\ &= \int h(x,\mathbf{s})\frac{f_\tau(x,s_{\tau+1})}{f_{\tau+1}(s_{\tau+1})}f_{\tau+1}(s_{\tau+1})f_{\tau+1}(s_{\tau+1},s_{\tau+2})\cdots f_{N-1}(s_{N-1},s_N)d\mathbf{s} \\ &= \mathbb{E}\left[h(x,\mathbf{S})\frac{f_\tau(x,S_{\tau+1})}{f_{\tau+1}(S_{\tau+1})}\right]. \end{aligned} \tag{2}$$

In light of equation (2), the loss function $L(x)$ can be approximated by

$$L_m(x)=\frac{1}{m}\sum_{j=1}^m h(x,\mathbf{S}^{(j)})\frac{f_\tau(x,S_{\tau+1}^{(j)})}{f_{\tau+1}(S_{\tau+1}^{(j)})}. \tag{3}$$

Note that this is a weighted average, and it utilizes the samples in all paths. Furthermore, we generate n samples $\{\tilde{S}_\tau^{(i)}\}_{i=1}^n$, then the estimator of α have been derived:

$$\alpha_{m,n}=\frac{1}{n}\sum_{i=1}^n g(L_m(\tilde{S}_\tau^{(i)})).$$

The asymptotic analyses of $L_m(x)$ and $\alpha_{m,n}$ have been studied in depth in Zhang et al. (2022), including the convergence rate of bias, variance, MSE, the central limit theorem and the construction of the confidence interval.

In this paper, we investigate an alternative weight in the approximation (3), called average weight in Broadie and Glasserman (2004), which is denoted by

$$\bar{w}_{n,ij} \triangleq \frac{f_\tau(\tilde{S}_\tau^{(i)},S_{\tau+1}^{(j)})}{\frac{1}{n-1}\sum_{l\neq i}f_\tau(\tilde{S}_\tau^{(l)},S_{\tau+1}^{(j)})}$$

With this average weight, the stochastic mesh estimation of the loss at a given scenario $\tilde{S}_\tau^{(i)}$ is

$$\bar{L}_{mn}(\tilde{S}_\tau^{(i)}) = \frac{1}{m}\sum_{j=1}^{m} h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})\bar{w}_{n,ij}, \tag{4}$$

and the corresponding risk estimator is

$$\bar{\alpha}_{m,n} = \frac{1}{n}\sum_{i=1}^{n} g(\bar{L}_{mn}(\tilde{S}_\tau^{(i)})).$$

In the next section, we will analyze the asymptotic properties of $\bar{L}_m(\tilde{S}_\tau^{(i)})$ and $\bar{\alpha}_{m,n}$.

It is worth mentioning that if the portfolio consists of path-dependent securities, such as Asian options, barrier options and look-back options, the stochastic mesh with average weight could be also applied. Actually, in this case, the loss at time τ may have the form $L = \mathbb{E}[h(S_1,...,S_N)|S_1,...,S_\tau]$, and then portfolio risk is

$$L(x_1,...,x_\tau) = \mathbb{E}[h(S_1,...,S_N)|(S_1,...,S_\tau) = (x_1,...,x_\tau)].$$

Similar to (2) and (3), $L(x_1,...,x_\tau)$ can be approximated by

$$L_m(x_1,...,x_\tau) = \frac{1}{m}\sum_{j=1}^{m} h(x_1,...,x_\tau,\mathbf{S}^{(j)})\frac{f_\tau(x_\tau, S_{\tau+1}^{(j)})}{f_{\tau+1}(S_{\tau+1}^{(j)})},$$

and then (4) can be rewritten as

$$\bar{L}_{mn}(\tilde{S}_1^{(i)},...,\tilde{S}_\tau^{(i)}) = \frac{1}{m}\sum_{j=1}^{m} h(\tilde{S}_1^{(i)},...,\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})\bar{w}_{n,ij}.$$

Therefore, the analyses for $\bar{L}_{mn}(\tilde{S}_\tau^{(i)})$ also hold for $\bar{L}_{mn}(\tilde{S}_1^{(i)},...,\tilde{S}_\tau^{(i)})$, and it would be aesthetic to extend scope of stochastic mesh method with average weight to path-dependent assets.

4 ANALYSIS

In this section, we provide asymptotic analysis of the loss estimator $\bar{L}_{mn}(\tilde{S}_\tau^{(i)})$ and the corresponding risk estimator $\bar{\alpha}_{m,n}$. Specifically, in subsection 4.1, we show the convergence rate of $\bar{L}_{mn}(\tilde{S}_\tau^{(i)})$ towards $L(\tilde{S}_\tau^{(i)})$ under some mild assumptions. According to these results, in the following subsections, we continue to investigate the MSE of $\bar{\alpha}_{m,n}$ for the three types of functions $g(\cdot)$: a smooth function, a hockey-stick function and an indicator function.

4.1 Analysis for $\bar{L}_m(\tilde{S}_\tau^{(i)})$

In the present subsection, we investigate asymptotic properties of $L_m(\tilde{S}_\tau^{(i)})$. For the simplicity of notations, denote

$$w_{ij} \triangleq \frac{f_\tau(\tilde{S}_\tau^{(i)}, S_{\tau+1}^{(j)})}{f_{\tau+1}(S_{\tau+1}^{(j)})} \quad \text{and} \quad w \triangleq \frac{f_\tau(\tilde{S}_\tau, S_{\tau+1})}{f_{\tau+1}(S_{\tau+1})},$$

and note that

$$\frac{w_{ij}}{\overline{w}_{n,ij}} = \frac{\frac{1}{n-1}\sum_{l\neq i} f_\tau(\tilde{S}_\tau^{(l)}, S_{\tau+1}^{(j)})}{f_{\tau+1}(S_{\tau+1}^{(j)})} = \frac{1}{n-1}\sum_{l\neq i} w_{lj} \triangleq \widetilde{w}_{n,ij}.$$

Before proceeding, we introduce the following lemma.

Lemma 1 (Theorem 3.5 in Zhang et al. 2022) If $\mathbb{E}\left[|h(\widetilde{S}_\tau,\mathbf{S})w|^{2p}\right]<\infty$ for some positive integer p, then

$$\mathbb{E}\left[\left(L_m(\tilde{S}_\tau^{(i)})-L(\tilde{S}_\tau^{(i)})\right)^{2p}\right]=\mathscr{O}\left(\frac{1}{m^p}\right) \text{ as } m\to\infty.$$

Lemma 1 demonstrates that the $2p$-moment of $L_m(\widetilde{S}_\tau)$ is of order $1/m^p$ provided a mild moment condition on $h(\widetilde{S}_\tau,\mathbf{S})w$. To obtain the convergence of $\bar{L}_{mn}(\tilde{S}_\tau^{(i)})$ to $L(\tilde{S}_\tau^{(i)})$, we just have to analyze the convergence of $\bar{L}_{mn}(\tilde{S}_\tau^{(i)})-L_m(\tilde{S}_\tau^{(i)})$, which is provided in the following lemma.

Lemma 2 For $i=1,\ldots,n$ and $l\neq i$, the following inequalities hold.

(i) If g is differentiable, $\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})w_{ij}\right|^2\right]<\infty$ and $\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})w_{ij}\widetilde{w}_{n,ij}^{-3}\right|^2\right]<\infty$, then

$$\left|\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)})-L_m(\tilde{S}_\tau^{(i)})\right)\right]\right|\leq\mathscr{O}\left(\frac{1}{n}\right). \tag{5}$$

(ii) If $\mathbb{E}\left[h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})^4\bar{w}_{n,ij}^4\right]<\infty$ and $\mathbb{E}\left[w_{lj}^4\right]<\infty$, then

$$\mathbb{E}\left[\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)})-L_m(\tilde{S}_\tau^{(i)})\right)^2\right]\leq\mathscr{O}\left(\frac{1}{n}\right).$$

(iii) If $\mathbb{E}\left[h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})^8\overline{w}_{n,ij}^8\right]<\infty$ and $\mathbb{E}\left[w_{lj}^4\right]<\infty$, then

$$\mathbb{E}\left[\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)})-L_m(\tilde{S}_\tau^{(i)})\right)^4\right]\leq\mathscr{O}\left(\frac{1}{n^2}\right).$$

(iv) Denote $\overline{w}_{n,ij,x}\triangleq\frac{f_\tau(x,S_{\tau+1}^{(j)})}{\frac{1}{n-1}\sum_{l\neq i}f_\tau(\tilde{S}_\tau^{(l)},S_{\tau+1}^{(j)})}$. If for any x, $\mathbb{E}\left[h(x,Y_j)^4\overline{w}_{n,ij,x}^4\right]<\infty$ and $\mathbb{E}\left[w_{lj}^4\right]<\infty$, then

$$\mathbb{E}\left[(\bar{L}_{mn}(x)-L_m(x))^2\right]\leq\mathscr{O}\left(\frac{1}{n}\right).$$

In the Appendix, we only provide the proof of (5) for the limited space. Combining Lemmas 1 and 2, we can obtain the convergence rate of $\bar{L}_{mn}(\tilde{S}_\tau^{(i)})$ in $\mathscr{L}^2$ and $\mathscr{L}^4$, and we will use these results to prove the convergence rate of MSE for the three kinds of functions $g(\cdot)$ in the following subsections.

4.2 Analysis for $\bar{\alpha}_{m,n}$

We establish the convergence rate of the MSE of the risk estimator $\bar{\alpha}_{m,n}$ here, and do not provide its proof for the limited space.

Theorem 1 If Assumption 1 shown in the Appendix holds, $\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})w_{ij}\widetilde{w}_{n,ij}^{-3}\right|^2\right]$ and $\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})w_{ij}\right|^2\right]$ are finite, and one of the following sets of assumptions hold:

1. The risk function $g(\cdot)$ is twice differentiable with a bounded second derivative, $\mathbb{E}\left[h(\tilde{S}_\tau^{(i)},\mathbf{S}^{(j)})^4\bar{w}_{n,ij}^4\right]<\infty$, $\mathbb{E}\left[w_{lj}^4\right]<\infty$,

2. The risk function $g(\cdot)$ is a hockey-stick function, and Assumption 2 holds,
3. The risk function $g(\cdot)$ is an indicator function, and Assumption 2 holds,

then for any m and n,

$$\text{MSE}(\bar{\alpha}_{m,n}) \leq \mathscr{O}(m^{-1}) + \mathscr{O}(n^{-1}).$$

In particular, let $m = n$, we have $\text{MSE}(\bar{\alpha}_{m,n}) \leq O(n^{-1})$.

The result in Theorem 1 is consistent with our intuition that the MSE of the estimator $\bar{\alpha}_{m,n}$ decays at the rate $\max\{m^{-1}, n^{-1}\}$. That is, the convergence rate of MSE of the stochastic mesh estimator is k^{-1} if $m = n$, which is the same as the convergence rate in the case of non-nested Monte Carlo simulation, whereas the rate given by nested simulation is $k^{-2/3}$. In practice, to accelerate the convergence rate of the estimator, we just have to increase both of m and n. This is easier to be controlled than some parametric or nonparametric methods, where the convergence rate of MSE may depend on the choices of basis functions, kernel functions and bandwidth, respectively. In particular, when the risk function $g(\cdot)$ is a hockey-stick function, the MSE derived by the LSM (see Broadie et al. 2015) decays at the rate k^{-1} until the MSE reaches a bias level determined by the regression model error. When the risk function $g(\cdot)$ is an indicator function, the convergence rate of the MSE derived by the nested sequential simulation (see Broadie et al. 2011) is of order $k^{-4/5+\delta}$ for all positive δ. They both decay slower than those of $\bar{\alpha}_{m,n}$.

From Theorem 1, it easily follows that the stochastic mesh estimator $\bar{\alpha}_{m,n}$ converges in probability to the true portfolio risk α, i.e., $\bar{\alpha}_{m,n} \xrightarrow{\mathbb{P}} \alpha$, as $\min\{m,n\} \to \infty$.

5 NUMERICAL EXPERIMENTS

In this section, we use a simple example to examine the performances of the proposed method. In this example, we consider a portfolio consisting of three European options, which are based on underlying assets following geometric Brownian motion process,

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where B_t is a standard Brownian motion process, μ is the return of stock under the real-world probability measure. Then,

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\},$$

and the transition density function is

$$f_\tau(x,y) = \frac{1}{\sigma\sqrt{\Delta t}y}\phi\left(\frac{\log(y/x) - (\mu - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right),$$

where ϕ is the standard normal density, and Δt is the time interval from x to y.

We assume that options in the portfolio have the maturity T. We want to measure the portfolio risk at a future time τ ($\tau < T$). In the simulation we first simulate S_τ under the real-world probability measure and then simulate $\mathbf{S}$ under the risk neutral probability measure. Note that the payoff of the portfolio at time T is $V_T(S_T)$. Let a constant V_0 denote the value of the portfolio at time 0, which can be calculated by the Black-Scholes formula. At time τ, portfolio loss is $L = \mathbb{E}[V_0 - V_T(S_T)|S_\tau]$.

We want to measure the portfolio risk that is represented by

$$\alpha = \mathbb{E}g(L),$$

where we consider three cases: a quadratic function $g(x) = x^2$, a hockey-stick function $g(x) = (x - y_0)^+$, and an indicator function $g(x) = 1_{\{x > y_0\}}$, respectively. Here y_0 is a pre-specified threshold. For the underlying asset, assume that initial price $S_0 = 100$, the real-world drift $\mu = 8\%$, risk-free interest rate $r = 5\%$ and volatility $\sigma = 20\%$. For the portfolio, the mature time is $T = 1$ year, and the risk horizon is $\tau = 1/12$ year, i.e., 1 month. The strikes of the three European options is $K = 90$, 100 and 110, respectively. The loss threshold is set to $y_0 = 5.8235$, the 90th percentile of L.

To measure its performance, we need the true value of α as a benchmark. Note that in this example, the analytical expression of $L(\tilde{S}_\tau)$ can be derived using the Black-Scholes formula, and thus we can generate a large amount (10^9) of samples of L and use sample mean of $g(L)$ to accurately approximate α. We then use this accurate estimate as a benchmark to measure the performance of our estimator. Specifically, we treat the benchmark as the theoretical value and consider the bias, variance, MSE and relative root MSE (RRMSE) of our estimators of portfolio risk, where RRMSE is defined as the percentage of the root MSE to the benchmark. All results reported are estimated based on 1000 independent replications.

In Figure 2, we plot the estimated absolute biases, the standard deviations, and the square roots of the MSEs relative to the stochastic mesh estimator of portfolio risk with respect to different sample sizes. Note that the y-axes of the two plots are in the same scale but different levels. From the plot, on one hand, we see that the MSEs of the estimators decrease as the sample size increases, and the bias is small and stable when the sample size is greater than 4,000. On the other hand, the MSE is affected mainly by the variance instead of bias.

In Figure 3, we plot the logarithm of the estimated MSE with respect to the logarithm of different sample sizes. It can be seen that the convergence rate of MSE of the stochastic mesh estimators is k^{-1}, which is consistent with the theoretical results. From the Figures 2 and 3, we see that the stochastic mesh estimators have the desired properties.

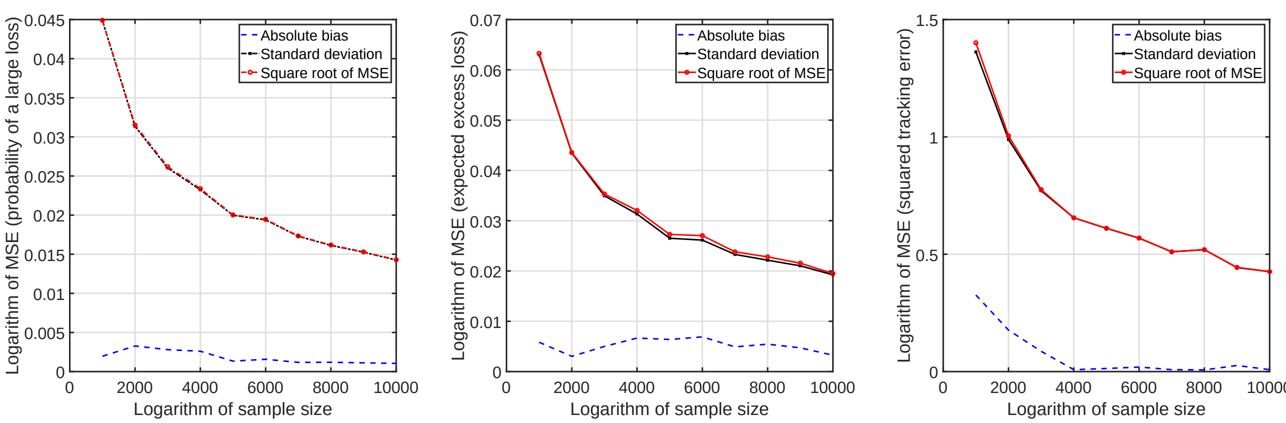

Figure 2: Estimated absolute bias, square root variance, and square root MSE of the stochastic mesh estimators.

6 CONCLUSIONS

In this paper, we have studied the stochastic mesh method with average weight for portfolio risk measurement under the nested setting. We have analyzed the asymptotic MSE of the stochastic mesh estimator of portfolio risk. Its convergence rate is k^{-1}, where k denotes the total computational budget, which is the same as that by the non-nested simulation. Numerical results show that the stochastic mesh method illustrated consistency with the theoretical results.

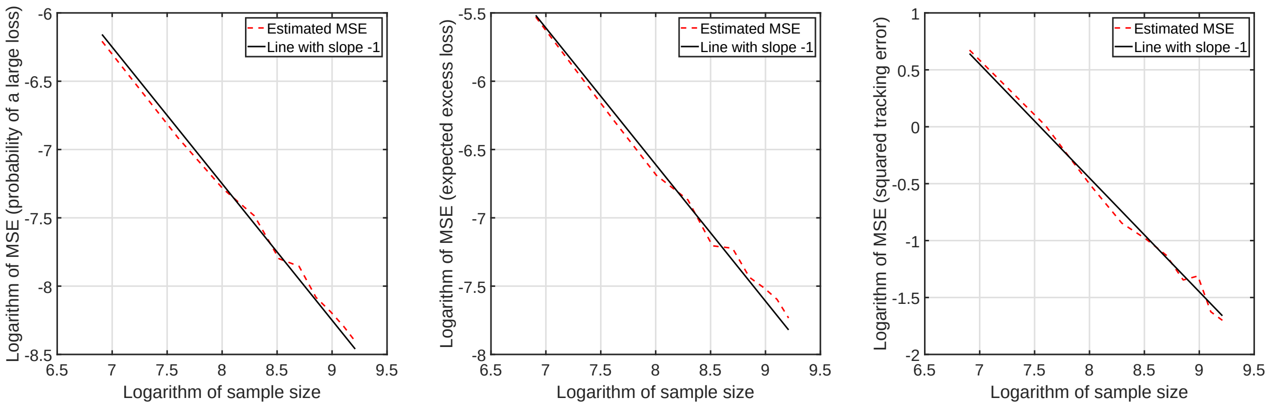

Figure 3: Illustration of convergence rate of MSE of the stochastic mesh estimators.

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A APPENDICES

A.1 Assumptions

In the following, we first introduce two assumptions which are similar to Assumption 1 in Gordy and Juneja (2010), where there is a variable $\widetilde{Z}_m$ with a nontrivial limiting distribution as $m \to \infty$ such that $L_m(S_\tau) - L(S_\tau) = \widetilde{Z}_m/\sqrt{m}$.

Assumption 1 **(i)** The joint density $p_m(\cdot,\cdot)$ of $L(S_\tau)$ and $\widetilde{Z}_m$ and its partial derivatives

$$\frac{\partial}{\partial y} p_m(y,z) \quad \textit{and} \quad \frac{\partial^2}{\partial y^2} p_m(y,z)$$

exist for each m and for all (y,z).

(ii) For $m \geq 1$, there exist nonnegative functions $\bar{p}_{0,m}(\cdot)$ and $\bar{p}_{1,m}(\cdot)$ such that

$$\begin{aligned} p_m(y,z) &\leq \bar{p}_{0,m}(z), \\ \left| \frac{\partial}{\partial y} p_m(y,z) \right| &\leq \bar{p}_{1,m}(z) \end{aligned}$$

for all y,z. In addition,

$$\sup_m \int_{-\infty}^{\infty} |z|^r \bar{p}_{i,m}(z) dz < \infty$$

for $i = 0,1$, and $0 \leq r \leq 3$.

Assumption 2 1. For any $i,k \in \{1,\ldots,n\}$ and $i \neq k$, the joint density $q_m(\ell_1,\ell_2,z_1,z_2)$ of $(L(\tilde{S}_\tau^{(i)}), L(\tilde{S}_\tau^{(k)}), \widetilde{Z}_m(X_i), \widetilde{Z}_m(X_k))$ and its partial derivatives $\frac{\partial}{\partial \ell_u} q_m(\ell_1,\ell_2,z_1,z_2)$ $(u = 1,2)$ exist for every m and for all (ℓ_1,ℓ_2,z_1,z_2).

2. For every $m \geq 1$, there exist nonnegative functions $\bar{q}_{v,m}(z_1,z_2), (v = 0,1)$ such that for $u = 1,2$,

$$q_m(\ell_1,\ell_2,z_1,z_2) \leq \bar{q}_{0,m}(z_1,z_2) \text{ and } \left| \frac{\partial}{\partial \ell_u} q_m(\ell_1,\ell_2,z_1,z_2) \right| \leq \bar{q}_{1,m}(z_1,z_2), \quad \forall (\ell_1,\ell_2,z_1,z_2).$$

3. For $v = 0, 1$ and any $r_1, r_2 \geq 0$ with $r_1 + r_2 \leq 3$,

$$\sup_m \int_{\mathbb{R}} |z_1|^{r_1} |z_2|^{r_2} \bar{q}_{v,m}(z_1, z_2) dz_1 dz_2 < \infty.$$

A.2 Proof of Lemma 2 (i)

To prove Lemma 2, we first introduce a result in Zhang et al. (2022).

Lemma 3 Consider identically distributed random variables $\{R_j\}_{j=1}^m$ such that $\mathbb{E}[R_1^{2p}] < \infty$ for some positive integer p. In addition, conditional on an arbitrary σ-field $\mathscr{G}$, $\{R_j\}_{j=1}^m$ are mutually independent and $\mathbb{E}[R_j|\mathscr{G}] = 0$ for all $1 \leq j \leq m$. Then,

$$\mathbb{E}\left[\left(\frac{1}{m}\sum_{j=1}^m R_j\right)^{2p}\right] = \frac{c_1}{m^p}\mathbb{E}\left[R_1^{2p}\right] + \mathscr{O}\left(\frac{1}{m^{p+1}}\right) = \mathscr{O}(m^{-p}), \text{ as } m \to \infty,$$

where $c_1 = \binom{2p}{2}\binom{2p-2}{2}\cdots\binom{2}{2}/p!$. In particular, for $p = 1$,

$$\mathbb{E}\left[\left(\frac{1}{m}\sum_{j=1}^m R_j\right)^{2}\right] = \frac{\mathbb{E}\left[R_1^2\right]}{m}.$$

In the following, we prove Lemma 2 (i). Note that

$$\begin{aligned}
&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)}) - L_m(\tilde{S}_\tau^{(i)})\right)\right] = \mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\frac{1}{m}\sum_{j=1}^m \left(h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})\bar{w}_{n,ij} - h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\right)\right] \\
=&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})\bar{w}_{n,ij} - h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\right)\right] = \mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})\left(\overline{w}_{n,ij} - w_{ij}\right)\right] \\
=&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\left(\widetilde{w}_{n,ij}^{-1} - 1\right)\right].
\end{aligned}$$

By the Taylor expansion of x^{-1},

$$\widetilde{w}_{n,ij}^{-1} - 1 = -(\widetilde{w}_{n,ij} - 1) + \breve{w}^{-3}(\widetilde{w}_{n,ij} - 1)^2,$$

where $\breve{w}$ is a random variable between $\widetilde{w}_{n,ij}$ and 1. Then it follows that

$$\begin{aligned}
&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)}) - L_m(\tilde{S}_\tau^{(i)})\right)\right] \\
=&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\left(-(\widetilde{w}_{n,ij} - 1) + \breve{w}^{-3}(\widetilde{w}_{n,ij} - 1)^2\right)\right] \\
=&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\left(\mathbb{E}\left[-(\widetilde{w}_{n,ij} - 1)\middle|X_i, Y_j\right] + \breve{w}^{-3}(\widetilde{w}_{n,ij} - 1)^2\right)\right] \\
=&\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\breve{w}^{-3}(\widetilde{w}_{n,ij} - 1)^2\right],
\end{aligned}$$

where the third equality is due to $\mathbb{E}\left[\widetilde{w}_{n,ij} - 1\middle|X_i, Y_j\right] = 0$. By Cauchy-Schwartz inequality,

$$\begin{aligned}
\left|\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)}) - L_m(\tilde{S}_\tau^{(i)})\right)\right]\right| \leq& \mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\breve{w}^{-3}(\widetilde{w}_{n,ij} - 1)^2\right|\right] \\
\leq& \left(\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\breve{w}^{-3}\right|^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\widetilde{w}_{n,ij} - 1)^4\right]\right)^{1/2}.
\end{aligned}$$

Note that applying Lemma 3 with $\mathscr{G} = \sigma(X_i, Y_j)$ and $p = 2$,

$$\mathbb{E}\left[(\widetilde{w}_{n,ij} - 1)^4\right] = \frac{3}{(n-1)^2}\mathbb{E}\left[(w_{kj} - 1)^4\right] + \mathscr{O}\left(\frac{1}{(n-1)^3}\right),\ k \neq i,$$

where the equality is due to the definition of $\widetilde{w}_{n,ij}$ and $\mathbb{E}\left[w_{kj} - 1\middle| X_i, Y_j\right] = 0$ for $k \neq i$.

Therefore,

$$\begin{aligned}
&\left|\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)}) - L_m(\tilde{S}_\tau^{(i)})\right)\right]\right| \\
\leq &\left(\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\breve{w}^{-3}\right|^2\right]\right)^{1/2}\left(\frac{3}{(n-1)^2}\mathbb{E}\left[(w_{kj} - 1)^4\right] + \mathscr{O}\left(\frac{1}{(n-1)^3}\right)\right)^{1/2} \\
= &\frac{\sqrt{3}}{n-1}\left(\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}\breve{w}^{-3}\right|^2\right]\right)^{1/2}\left(\mathbb{E}\left[(w_{kj} - 1)^4\right]\right)^{1/2} + o(1/n),\ k \neq i.
\end{aligned}$$

Because $\breve{w}$ is between $\widetilde{w}_{n,ij}$ and 1, its inverse $\breve{w}^{-1}$ is between 1 and $\widetilde{w}_{n,ij}^{-1}$, and so $\breve{w}^{-3} \leq \max\{1, \widetilde{w}_{n,ij}^{-3}\} \leq 1 + \widetilde{w}_{n,ij}^{-3}$. Therefore,

$$\begin{aligned}
&\left|\mathbb{E}\left[g'(L(\tilde{S}_\tau^{(i)}))\left(\bar{L}_{mn}(\tilde{S}_\tau^{(i)}) - L_m(\tilde{S}_\tau^{(i)})\right)\right]\right| \\
\leq &\frac{\sqrt{3}}{n-1}\left(\mathbb{E}\left[\left|g'(L(\tilde{S}_\tau^{(i)}))h(\tilde{S}_\tau^{(i)}, \mathbf{S}^{(j)})w_{ij}(1 + \widetilde{w}_{n,ij}^{-3})\right|^2\right]\mathbb{E}\left[(w_{kj} - 1)^4\right]\right)^{1/2} + o(1/n),\ k \neq i.
\end{aligned}$$

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