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STOCHASTIC ROOT FINDING VIA BAYES DECISIONS

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ABSTRACT

We consider the root finding problem of a one-dimensional function when the function can be only estimated by noisy responses and a unique root exists between given lower and upper bounds. A new approach, namely the trisection algorithm with Bayes decisions (TAB), is proposed. We investigate the theoretical properties of TAB and empirically compare the proposed algorithm with several existing algorithms.

1 PROBLEM DESCRIPTION

In this study, we consider a one-dimensional root finding problem where the root is unique and located between given lower and upper bounds. Specifically, we assume that there exists a stochastic black-box simulation model (as known as an *oracle*) which returns an observation at each location x. Let $Y_j(x)$ be the jth observation at location x such that $Y_j(x) = g(x) + \varepsilon_j(x)$ where $g(x)$ is a real-valued unknown function and $\varepsilon_j(x)$ is a zero-mean noise component whose distribution relies on the location x. Let a and b be the lower and upper bounds of the root, respectively. Throughout the study, we have the following assumption: For any $j = 1,2,\ldots$ and for any $x \in [a,b]$ and $x' \in [a,b]$ such that $x \neq x'$, (i) $\varepsilon_j(x) \overset{iid}{\sim} N(0,1/\gamma(x))$, and (ii) $\varepsilon_j(x)$ and $\varepsilon_j(x')$ are independent. Normally distributed observations are justified by the Central Limit Theorem when observations are either within-replication averages or batch means (Law and Kelton 2000). Let x^* be the real-valued root such that $g(x^*) = 0$. To simplify the problem, we further assume that a and b are given, $g(x) > 0$ for any $a < x < x^*$, and $g(x) < 0$ for any $b > x > x^*$. The same assumptions are employed in several past studies including Weaber (2013), Rodriguez and Ludkovski (2020a), and Rodriguez and Ludkovski (2020b). See Dunkel and Weber (2010) and Rodriguez and Ludkovski (2020a) for practical examples of this problem.

2 SOLUTION APPROACH

We propose a new approach, namely the trisection algorithm with Bayes decisions (TAB). In TAB, $g(x)$ for any $x \in [a,b]$ is considered as a random variable whose prior and posterior distributions are assumed to follow normal distributions under the Bayesian settings. Let k be the iteration counter of the algorithm and I_k be the promising interval that is supposed to include the root at iteration k. Then, the basic structure of TAB is provided as follows.

(Step 1) Set the iteration counter $k = 1$ and the promising interval $I_1 = [a,b]$.
(Step 2) Select four locations, x_k^1, x_k^2, x_k^3, and x_k^4, where x_k^1 is the leftmost location of I_k, x_k^4 is the rightmost location of I_k, $x_k^2 = \frac{2x_k^1 + x_k^4}{3}$, and $x_k^3 = \frac{x_k^1 + 2x_k^4}{3}$.
(Step 3) Obtain simulation observations among x_k^1, x_k^2, x_k^3, and x_k^4, and update posterior distributions of $g(x_k^i)$ for $i = 1,2,3,4$ if needed.

(Step 4) Update I_{k+1} based on Bayes decisions. If the total simulation budget is consumed, then return the midpoint of I_{k+1} as the best estimate of the root and terminate the algorithm. Otherwise, update $k \leftarrow k+1$ and go to Step 2.

Let $I_k^{s1} = [x_k^1, x_k^3]$, $I_k^{s2} = [x_k^2, x_k^4]$, $I_k^{s3} = [\max\{x_k^1 - 1/2(x_k^4 - x_k^1), a\}, x_k^4]$, and $I_k^{s4} = [x_k^1, \min\{x_k^4 + 1/2(x_k^4 - x_k^1), b\}]$. In Step 4, the algorithm uses Bayes decisions (DeGroot 2005) to set the next promising interval I_{k+1}. For $\ell = 1$ and 2, let d_ℓ be the decision representing that $I_k^{s\ell}$ includes the root and thus the algorithm selects either I_k^{s1} or I_k^{s2} as I_{k+1}. For $\ell = 3$ and 4, let d_ℓ be the decision representing that the root is located in the range $x < x_k^1$ and $x > x_k^4$, respectively, and thus the algorithm selects either I_k^{s3} or I_k^{s4} as I_{k+1}.

Let g_k be a vector of $g(x_k^i)$ for $i = 1,2,3,4$. Then, a combination of the signs of $g(x_k^i)$ determines whether d_ℓ is true or not, and we denote the 0-1 reward function as $R(g_k, d_\ell)$ for $\ell = 1,2,3,4$. The expected reward function corresponding to each d_ℓ at iteration k is defined as $\mathbf{E}[R(g_k, d_\ell)]$, and the Bayes reward function is defined as $\max_{\ell=1,2,3,4} \mathbf{E}[R(g_k, d_\ell)]$. One can find the index of the Bayes decision, denoted by ℓ^*, which is $\arg\max_{\ell=1,2,3,4} \mathbf{E}[R(g_k, d_\ell)]$. Based on the Bayes decision, $I_k^{s\ell^*}$ is selected as I_{k+1}. In this study, we provide theoretical results to calculate or approximate predictive $\mathbf{E}[R(g_k, d_\ell)]$ when a number of additional observations is assigned to each candidate location among x_k^1, x_k^2, x_k^3, and x_k^4 (but not obtained yet). We sample additional observations from a location with the largest improvement in $\mathbf{E}[R(g_k, d_\ell)]$, which leads to efficient simulation budget allocation before updating the posterior distributions of $g(x_k^i)$. The updated posterior distributions are used to select the algorithm's next promising interval.

3 RESULTS AND SUMMARY

We compare the performance of the proposed algorithm, TAB, to the performance of several existing algorithms: Stochastic Approximation (SA) and Polyak-Ruppert (PR) with the settings in Weaber (2013), two versions of the Generalized Probabilistic Bisection Algorithm (G-PBA) in Rodriguez and Ludkovski (2020a), and the spatial G-PBA in Rodriguez and Ludkovski (2020b). To compare the performance of the algorithms, we return the average absolute residual which is defined as the average of $|\hat{x}_T - x^*|$ over 1000 macro replications where x^* is the real root and $\hat{x}_T$ is the best estimate of x^*. Each replication terminates when the total number of observations reaches 100,000.

We first consider a linear $g(x)$ for which the SA-type algorithm is known to perform well. Our algorithm returns the average absolute residuals similar to or slightly higher than those returned by the SA-type algorithms (i.e., SA and PR), but clearly outperforms the G-PBAs. When $g(x)$ is a cubic function known to be more challenging than a linear function, our algorithm achieves a 10 – 40 % reduction in the average absolute residuals compared with all competing algorithms. In addition, we consider a Bermudan put option problem with a discretized Black-Scholes model in Rodriguez and Ludkovski (2020a). In the put option problem, the variance varies at different locations, the random noise may not be normally distributed, and $g(x)$ whose gradient fluctuates is very close to 0 for $x \leq x^*$. The results show that our algorithm achieves a 15 – 96 % reduction in the average absolute residuals compared with the competing algorithms. TAB has two key benefits: (1) the trisection structure enables the algorithm to fix incorrect selections of the promising interval and (2) the Bayes decision leads the algorithm to efficient allocation of simulation observations.

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