Proceedings of the 2023 Winter Simulation Conference
C. G. Corlu, S. R. Hunter, H. Lam, B. S. Onggo, J. Shortle, and B. Biller, eds.

MEAN-VARIANCE PORTFOLIO OPTIMIZATION WITH NONLINEAR DERIVATIVE SECURITIES

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ABSTRACT

In this paper, we propose a simulation approach to mean-variance optimization for portfolios comprised of derivative securities. The key of the proposed method is on the development of an unbiased and consistent estimator of the covariance matrix of asset returns which do not admit closed-form formulas but require Monte Carlo estimation, leading to a sample-based optimization problem that is easy to solve. We characterize the asymptotic properties of the proposed covariance estimator, and the solution to and the objective value of the sample-based optimization problem. Performance of the proposed approach is demonstrated via numerical experiments.

1 INTRODUCTION

Portfolio optimization is one of the central problems in financial engineering, which aims to allocate the wealth of a decision-maker among different assets, such as stocks, bonds, and derivatives, to earn the highest possible future wealth and control the risk at the same time. The classical mean-variance (MV) model developed by Markowitz (1952) has been serving as an important tool for this problem. In the MV model, the risk is measured by the variance of the portfolio return, providing an efficient means for investors to balance the trade-off between the risk and the expected return of the portfolio.

One of the limitations of the classical MV model in the existing literature is that it considers only portfolios of primary assets (Jewell et al. 2013), while optimization of mixed portfolios with primary and derivative instruments has received increasing attention in recent years. Rockafellar and Uryasev (2000) introduced a technique for optimizing the CVaR (Conditional Value-at-Risk) of a portfolio that includes both primary and derivative assets. Alexander et al. (2006) observed that CVaR minimization for a portfolio of derivative securities is ill-posed. Furthermore, it has shown that this predicament can be overcome by including transaction costs. Carr and Madan (2001) analyzed the optimal investment and equilibrium pricing of primary and derivative instruments. Haugh and Lo (2001) considered portfolio optimization with non-standard asset classes and showed how to approximate dynamic positions in options

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by minimizing the mean-squared error. Dert and Oldenkamp (2000) proposed a model that maximizes the expected return of a portfolio consisting of a single index stock and several European options while guaranteeing a maximum loss. Jewell et al. (2013) considered the MV optimization for a portfolio with derivative assets using a quadratic program based on the delta-gamma approximation of portfolio losses. Zymler et al. (2013) developed two conservative approximations for the Value-at-Risk (VaR) of a derivative portfolio by evaluating the worst-case VaR, based on convex pircewise linear delta-gamma approximation of the derivative returns.

The return of a portfolio with derivative securities is often represented as $L(X) \triangleq \mathrm{E}[Y|X]$, where the random vector X denotes risk factors up to a given risk horizon, and Y is discounted portfolio return at maturities dates of the derivative securities. To solve the MV optimization problem for such a portfolio, the most challenging part stems from the fact that the functional form of $L(\cdot)$ is usually unknown, thus making the estimation of the covariance matrix of $L(X)$ a difficult task. In this paper, with a given simulation model of (X,Y), we propose a simulation-based estimator of the covariance matrix, which is distribution-free, and requires only two independent simulation outputs of Y for a given scenario of risk factors X. We show the unbiasedness and consistency of the estimator, and study the incorporation of this estimator into the sample-based MV portfolio optimization problem. We also characterze the asymptotic properties of the solution to and the objective value of the sample-based optimization problem. Compared to existing methods that rely on approximations of the unknown conditional expectation $\mathrm{E}[Y|X]$ and estimate the covariance of the approximation, our method provides an unbiased and efficient estimator without resorting to approximations.

The rest of the paper is organized as follows. We formulate the problem in Section 2. The estimator of the covariance matrix is proposed in Section 3, along with a theoretical analysis on the estimator and the resulting sample-based optimization problem. We introduce a scaling method to transform the estimated covariance matrix into a positive semi-definite matrix in Section 4. Numerical experiments are presented in Section 5, followed by conclusions in Section 6. Lengthy proofs are provided in the appendix.

2 PROBLEM FORMULATION

Consider a portfolio comprised of a risk-free asset with known return r_f and K risky assets that may include derivative securities. Let r_t^k, $k=1,\cdots,K$ be the anticipated return at time $0 \le t \le T$ per dollar invested in the kth risky asset, where T denotes the maximum maturity date of all the derivative securities. The return of each risky asset depends on its value at time t that relies on a collection of financial risk factors, such as stock prices, stock indices, exchange rates and other tradable assets. Denote by X_t all relevant risk factors up to time t that is sufficient to determine the values of all risky assets in the portfolio at time t. Then, the return of the kth asset at time t is a function of X_t, and can be represented as

$$r_k(X_t) = \frac{\mathrm{E}[C_k|X_t] - V_0^k}{V_0^k} = \mathrm{E}\left[\left. \frac{C_k - V_0^k}{V_0^k} \right| X_t \right] = \mathrm{E}\left[Y_k | X_t \right],$$

where C_k denotes the cumulative cash flow for asset k from time zero to its maturity (weighted with appropriate discounted factors), V_0^k is the initial price (the price at the current time) of asset k and is a known constant, and $Y_k \triangleq (C_k - V_0^k)/V_0^k$. Here, the conditional expectation $\mathrm{E}[C_k|X_t]$ is taken under a martingale pricing measure and represents the price of asset k at time t (see Duffie 2010, Chapter 6). Following the convention in the related literature, it is assumed that such a martingale pricing measure exists throughout the paper.

Denoting vectors by bold letter, we let $\mathbf{r}(X_t) = (r_1(X_1), \ldots, r_K(X_t))^\top$. Allocating a fraction $\mathbf{z} \in \mathbb{R}^K$ of wealth to risky assets and the remainder $(1 - \mathbf{z}^\top \mathbf{1}_K)$ to the risk-free asset, the return of the portfolio at time t is written as

$$R(X_t) = \mathbf{z}^\top \mathbf{e}(X_t) + r_f,$$

where

$$\mathbf{e}(X_t) = \mathbf{r}(X_t) - r_f \mathbf{1}_K = (r_1(X_t) - r_f, \cdots, r_K(X_t) - r_f)^\top$$

denotes the vector of excess returns, and $\mathbf{1}_K$ denotes a size-K column of ones.

The investor is seeking the best allocation of wealth among the portfolio (i.e., making a decision on $\mathbf{z}$) to earn the highest possible wealth up to a pre-specified time horizon τ, $0 < \tau \leq T$. Under the framework of the MV portfolio optimization model by Markowitz (1952), her objective is to determine an optimal tradeoff between the expected return defined as $\mathrm{E}[R(X_\tau)]$ and the risk that is measured by the variance of the portfolio return denoted as $\mathrm{Var}(R(X_\tau))$ at time τ. The optimal portfolio can thus be determined by solving the following MV optimization problem:

$$\max_{\mathbf{z}\in\mathscr{Z}} U(\mathbf{z}) = \max_{\mathbf{z}\in\mathscr{Z}} \mathrm{E}[R(X_\tau)] - \frac{\gamma}{2} \cdot \mathrm{Var}(R(X_\tau)), \tag{1}$$

where the parameter $\gamma \geq 0$ measures the level of relative risk aversion of the investor. The objective function $U(\mathbf{z})$ is referred to as the expected utility function. If short sale is not allowed, the set $\mathscr{Z}$ can be set as $\mathscr{Z} = \{\mathbf{z} \in \mathbb{R}^K : \mathbf{z} \geq 0, \mathbf{z}^\top \mathbf{1}_K \leq 1\}$.

We further express the expected return $\mathrm{E}[R(X_\tau)]$ in the form of

$$\mathrm{E}[R(X_\tau)] = \mathbf{z}^\top \mathrm{E}\left[\mathbf{e}(X_\tau)\right] + r_f = \mathbf{z}^\top \mu_\tau + r_f,$$

while writing the variance of the portfolio return $\mathrm{Var}(R(X_\tau))$ as follows:

$$\mathrm{Var}(R(X_\tau)) = \mathbf{z}^\top \mathrm{Cov}\left(\mathbf{e}(X_\tau)\right) \mathbf{z} = \mathbf{z}^\top \Sigma_\tau \mathbf{z},$$

where Σ_τ denotes the covariance matrix of assets' returns at time τ, i.e.,

$$\begin{aligned}
\Sigma_\tau &= \mathrm{Cov}\left(\mathbf{r}(X_\tau)\right) \\
&\triangleq \begin{pmatrix}
\mathrm{Var}\left(r_1(X_\tau)\right) & \mathrm{Cov}\left(r_1(X_\tau), r_2(X_\tau)\right) & \ldots & \mathrm{Cov}\left(r_1(X_\tau), r_K(X_\tau)\right) \\
\mathrm{Cov}\left(r_2(X_\tau), r_1(X_\tau)\right) & \mathrm{Var}\left(r_2(X_\tau)\right) & \ldots & \mathrm{Cov}\left(r_2(X_\tau), r_K(X_\tau)\right) \\
\ldots & \ldots & \ldots & \ldots \\
\mathrm{Cov}\left(r_K(X_\tau), r_1(X_\tau)\right) & \ldots & \ldots & \mathrm{Var}\left(r_K(X_\tau)\right)
\end{pmatrix}.
\end{aligned}$$

Hence, the MV optimization problem (1) can be rewritten as

$$\max_{\mathbf{z}\in\mathscr{Z}} U(\mathbf{z}) = \max_{\mathbf{z}\in\mathscr{Z}} \mathbf{z}^\top \mu_\tau + r_f - \frac{\gamma}{2} \cdot \mathbf{z}^\top \Sigma_\tau \mathbf{z}. \tag{2}$$

Note that in the case when short sale is allowed without any constraint, i.e., $\mathscr{Z} = \mathbb{R}^K$, the optimal portfolio weights of the problem in (2) has an explicit form as

$$\mathbf{z}^* = \frac{1}{\gamma} \Sigma_\tau^{-1} \mu_\tau,$$

and the optimal value is given by

$$u^* = \frac{1}{\gamma} \mu_\tau^\top \Sigma_\tau^{-1} \mu_\tau + r_f.$$

More generally, solving the portfolio optimization problem (2) requires to approximate two crucial quantities, i.e., the mean vector μ_τ and the covariance matrix Σ_τ. The challenge in approximating μ_τ and Σ_τ stems from the fact that the closed-form expression of $r_k(X_\tau)$ may not be available for many commonly used pricing models for derivative securities. In many financial applications, a portfolio may include complex derivative securities, the valuation of which cannot be carried out analytically but may require Monte Carlo simulation. In what follows, we propose a simulation method to solve the MV portfolio optimization model by incorporating a new covariance estimator for the returns.

3 A SIMULATION METHOD

3.1 Unbiased Estimators of Covariance Matrix and Mean Return

For any $k,l=1,\dots,K$, consider the estimation of each element $(\Sigma_\tau)_{kl}=\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)$ of the covariance matrix. Let Y_k' denote a random variable that follows the same distribution as Y_k and is independent of Y_k conditional on X_τ, where in practical applications observations of Y_k and Y_k' can be obtained from real data sets or simulated from a calibrated financial model. Inspired by Goda (2017), $\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)$ can be simply rewritten as follows:

$$\begin{aligned}\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right) &= \mathrm{Cov}\left(\mathrm{E}[Y_k|X_\tau],\mathrm{E}[Y_l|X_\tau]\right)=\mathrm{E}\left[\mathrm{E}[Y_k|X_\tau]\cdot\mathrm{E}[Y_l|X_\tau]\right]-\mathrm{E}[Y_k]\cdot\mathrm{E}[Y_l]\\ &= \mathrm{E}\left[\mathrm{E}[Y_k|X_\tau]\cdot\mathrm{E}[Y_l'|X_\tau]\right]-\mathrm{E}[Y_k]\cdot\mathrm{E}[Y_l]=\mathrm{E}\left[\mathrm{E}[Y_k\cdot Y_l'|X_\tau]\right]-\mathrm{E}[Y_k]\cdot\mathrm{E}[Y_l]\\ &= \mathrm{E}[Y_k\cdot Y_l']-\mathrm{E}[Y_k]\cdot\mathrm{E}[Y_l],\end{aligned}$$

where Y_k and Y_l' are conditionally independent given X_τ. Notice that $\mathrm{Cov}\left(r_k(X_\tau),r_k(X_\tau)\right)=\mathrm{Var}\left(r_k(X_\tau)\right)$.

Therefore, an estimator for the element $(\Sigma_\tau)_{kl}$ can be constructed as follows:

$$\hat{V}_{kl}\triangleq\frac{1}{n}\sum_{i=1}^{n}Y_k^{(i)}\cdot Y_l'^{(i)}-\frac{1}{n}\sum_{i=1}^{n}Y_k^{(i)}\cdot\frac{1}{n}\sum_{i=1}^{n}Y_l'^{(i)},$$

where for each i, $Y_k^{(i)}$ and $Y_l'^{(i)}$ are sampled independently from the conditional distribution of Y_k and Y_l given $X_\tau^{(i)}$, respectively. The estimators $\hat{V}_{kl}$ is biased in general, and a bias-corrected version is given by

$$\widetilde{V}_{kl}\triangleq\frac{n}{n-1}\hat{V}_{kl}=\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_k^{(i)}-\bar{Y}_k\right)\cdot\left(Y_l'^{(i)}-\bar{Y}_l'\right).$$

Unbiasedness of the proposed estimator $\widetilde{V}_{kl}$ is summarized in the following theorem, whose proof is provided in Section A.1 of the appendix.

Theorem 1 For any $k,l=1,\dots,K$, we have

$$\mathrm{E}[\widetilde{V}_{kl}]=\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right).$$

Moreover, with the simulated samples, the mean of the return μ_τ can be estimated in a straightforward manner and an estimator is given by

$$\hat{\mu}_\tau\triangleq\left(\frac{1}{2n}\sum_{i=1}^{n}(Y_1^{(i)}+Y_1'^{(i)}),\cdots,\frac{1}{2n}\sum_{i=1}^{n}(Y_K^{(i)}+Y_K'^{(i)})\right)^\top-r_f\mathbf{1}_K.$$

Denote the unbiased estimator of the covariance matrix by $\widetilde{\Sigma}_\tau$ with $\widetilde{\Sigma}_\tau=(\widetilde{V}_{kl})_{K\times K}$ for any $k,l=1,\dots,K$. Then, we propose to solve the following sample-based MV optimization problem:

$$\max_{\mathbf{z}\in\mathscr{Z}}\hat{U}_n(\mathbf{z})=\max_{\mathbf{z}\in\mathscr{Z}}\mathbf{z}^\top\hat{\mu}_\tau+r_f-\frac{\gamma}{2}\cdot\mathbf{z}^\top\widetilde{\Sigma}_\tau\mathbf{z}. \tag{3}$$

3.2 Asymptotic Analysis

Throughout the analysis, it is assumed that Y_k (and Y_k') has finite fourth moment for any $k=1,\dots,K$. The following proposition presents the consistency and central limit theorem for the covariance estimator $\widetilde{\Sigma}_\tau$, whose proof is provided in Section A.2 of the appendix.

Proposition 1 For each element of $\widetilde{\Sigma}_\tau$, as $n\to\infty$,

(a) $\widetilde{V}_{kl} \longrightarrow \mathrm{Cov}\left(r_k(X_\tau), r_l(X_\tau)\right)$ w.p.1, and

(b) $\sqrt{n}\left(\widetilde{V}_{kl} - \mathrm{Cov}\left(r_k(X_\tau), r_l(X_\tau)\right)\right) \xrightarrow{d} N(0, \sigma_{kl}^2)$, where "$\xrightarrow{d}$" denotes convergence in distribution, and

$$\sigma_{kl}^2 \triangleq \mathrm{Var}\left((Y_k - \mathrm{E}[Y_k]) \cdot \left(Y_l' - \mathrm{E}[Y_l]\right)\right).$$

To facilitate the analysis for the sample-based optimization problem in (3), we make the following regularity assumption.

Assumption 1 The feasible set $\mathscr{X}$ is compact.

Since the decision set $\mathscr{X}$ is compact, the original MV optimization problem in (2) and the approximating problem in (3) have nonempty sets of optimal solutions, denoted by $\mathscr{X}^*$ and $\hat{\mathscr{X}}_n$, respectively. Further let u^* and $\hat{u}_n$ denote their corresponding optimal values.

Denote the semi-approximate value as $U(\hat{\mathbf{z}})$, where $\hat{\mathbf{z}} \in \hat{\mathscr{X}}_n$. The following proposition characterizes the asymptotic properties of the optimization problem in (3), whose proof is provided in Section A.3 of the appendix.

Proposition 2 Under Assumption 1,

(a) $\hat{u}_n \to u^*$ w.p.1 as $n \to \infty$;

(b) The event $\{\hat{\mathscr{X}}_n \subset \mathscr{X}^*\}$ occurs w.p.1 for sufficiently large n; and

(c) $\inf\{U(\mathbf{z}) : \mathbf{z} \in \hat{\mathscr{X}}_n\} \to u^*$ w.p.1 as $n \to \infty$.

Proposition 2(a) states that the optimal value of the sample-based problem in (3) converges to the true optimal value with probability 1 as sample size increases. Proposition 2(b) guarantees the convergence of the approximated solution set, and it follows that if $\mathscr{X}^* = \{\mathbf{z}^*\}$ is a singleton, then $\hat{\mathscr{X}}_n = \{\mathbf{z}^*\}$ w.p.1 for sufficiently large n. In particular, if the original MV optimization problem (2) has a unique optimal solution $\mathbf{z}^*$, then w.p.1 for sufficiently large n the approximating problem (3) has a unique optimal solution $\hat{\mathbf{z}}_n$ and $\hat{\mathbf{z}}_n = \mathbf{z}^*$. To measure the quality of the approximated solution set $\hat{\mathscr{X}}_n$, we compare the difference between $\inf\{U(\mathbf{z}) : \mathbf{z} \in \hat{\mathscr{X}}_n\}$ and u^*. Proposition 2(c) shows that the difference converges to zero as n goes to infinity. Furthermore, if the sample-based problem in (3) has a unique optimal solution $\hat{\mathbf{z}}_n$, the corresponding utility value $U(\hat{\mathbf{z}}_n)$ converges to the true optimal value w.p.1.

4 IMPLEMENTATION ISSUES OF COVARIANCE ESTIMATOR

Our proposed covariance estimator does not guarantee the symmetry of $\widetilde{\Sigma}_\tau$. Thus an additional step is needed to transform the estimator into a symmetric matrix. Notice that $\widetilde{V}_{kl}$ and $\widetilde{V}_{lk}$, $k \neq l$ are both unbiased and consistent estimators of the covariance $\mathrm{Cov}\left(r_k(X_\tau), r_l(X_\tau)\right)$, so is $(\widetilde{V}_{kl} + \widetilde{V}_{lk})/2$. In this way, we can easily transform $\widetilde{\Sigma}_\tau$ into a symmetric matrix by taking

$$\bar{\Sigma}_\tau = \frac{1}{2}\left(\widetilde{\Sigma}_\tau + \widetilde{\Sigma}_\tau^\top\right).$$

We further transform $\bar{\Sigma}_\tau$ into a positive semi-definite (PSD) matrix by using a scaling method; see, e.g., Rousseeuw and Molenberghs (1993). To do so, we first compute the corresponding correlation matrix $\bar{R}$ given by

$$\bar{\mathbf{R}} \triangleq \mathbf{D}\bar{\Sigma}_\tau\mathbf{D},$$

where $\mathbf{D}$ is the diagonal matrix with diagonal elements $1/\sqrt{(\bar{\Sigma}_\tau)_{kk}}$, $k = 1, \ldots, K$.

During practical implementation, the diagonal values of $(\bar{\Sigma}_\tau)_{kk}$, $k = 1, \ldots, K$ may be non-positive, especially when the sample size n is small. In this case, we propose a truncation by using $\max\{\varepsilon, (\bar{\Sigma}_\tau)_{kk}\}$ instead of $(\bar{\Sigma}_\tau)_{kk}$, where ε is chosen to be a relatively small positive constant.

Then, we transform the correlation matrix $\bar{\mathbf{R}}$ into a PSD pseudo-correlation matrix $\check{\mathbf{R}}$ via a scaling method. By definition, a K-by-K matrix $\mathbf{R}$ will be called a pseudo-correlation matrix if $\mathbf{R}$ is symmetric, $R_{kk} = 1$, and $|R_{kl}| \leq 1$. Essentially, the scaling method solves the following optimization problem:

$$\begin{aligned} \min_{\check{\mathbf{R}} \in \mathscr{S}_+^K} \quad & \|\check{\mathbf{R}} - \bar{\mathbf{R}}\|^2 \\ s.t. \quad & \check{\mathbf{R}} \text{ is a pseudo-correlation matrix,} \end{aligned} \tag{4}$$

where $\mathscr{S}_+^K \in \mathbb{R}^K$ denotes the PSD set and $\|A\|$ denotes its Euclidean norm, i.e., $\|A\| = [\mathrm{tr}(AA^\top)]^{1/2}$.

Slightly abusing the notation, we let $\check{\mathbf{R}}$ denote the solution to the above optimization problem. In this way, $\check{\mathbf{R}}$ is the closest approximation to $\bar{\mathbf{R}}$ in the set of PSD pseudo-correlation matrices. Because the PSD set is a compact and convex subset of $\mathbb{R}^{K \times K}$, it follows that $\check{\mathbf{R}}$ always exists and is unique, and that the transformation $\bar{\mathbf{R}} \to \check{\mathbf{R}}$ is a continuous mapping.

Obviously, the optimization problem in (4) is a semi-definite program, which can be exactly solved by commonly used solvers. Then, we obtain the corresponding PSD covariance estimator $\check{\Sigma}_\tau$ by computing

$$\check{\Sigma}_\tau \triangleq \mathbf{D}^{-1} \check{\mathbf{R}} \mathbf{D}^{-1}.$$

Therefore, during implementation, we propose to solve the following approximation optimization problem:

$$\max_{\mathbf{z} \in \mathscr{Z}} \check{U}_n(\mathbf{z}) = \max_{\mathbf{z} \in \mathscr{Z}} \mathbf{z}^\top \hat{\mu}_\tau + r_f - \frac{\gamma}{2} \cdot \mathbf{z}^\top \check{\Sigma}_\tau \mathbf{z}. \tag{5}$$

5 NUMERICAL EXPERIMENTS

Consider a portfolio that comprised of ten derivative securities written on five underlying assets. We assume that the dynamics of the asset prices are governed by the following multidimensional geometric Brownian motion (GBM):

$$\mathrm{d}S_i(t) = \mu_i S_i(t)\mathrm{d}t + \sigma_i S_i(t)\mathrm{d}W_i(t), \qquad i = 1, \ldots, 5$$

where $S_i(t)$ represents the price of ith asset at time t, $i = 1, \ldots, 5$. Each W_i is a standard one-dimensional Brownian motion (BM), with $W_i(t)$ and $W_j(t)$ having correlation ρ_{ij}.

The maturities of derivatives are the same, denoted as T. Detailed configuration of derivative securities are summarized as follows.

- Derivatives 1, 2 are European vanilla call options written on the 1st asset, with payoffs $(S_1(T) - K_1)^+$ and $(S_1(T) - K_2)^+$ respectively, where K_1 and K_2 are strike prices.
- Derivatives 3, 4 are binary call options written on the 2nd asset, with payoffs $\mathbb{1}\{S_2(T) > K_1\}$ and $\mathbb{1}\{S_2(T) > K_2\}$ respectively, where the indicator function $\mathbb{1}\{A\}$ takes value 1 if A occurs and 0 otherwise.
- Derivatives 5, 6 are up-and-out call options written on the 3rd asset, with payoffs $(S_3(T) - K_1)^+ \mathbb{1}\{\max_{0 \leq t \leq T} S_3(t) \leq U\}$ and $(S_3(T) - K_2)^+ \mathbb{1}\{\max_{0 \leq t \leq T} S_3(t) \leq U\}$ respectively, where U is barrier level.
- Derivatives 7, 8 are down-and-out call options written on the 4th asset, with payoffs $(S_4(T) - K_1)^+ \mathbb{1}\{\min_{0 \leq t \leq T} S_4(t) \geq H\}$ and $(S_4(T) - K_2)^+ \mathbb{1}\{\min_{0 \leq t \leq T} S_4(t) \geq H\}$ respectively, where H is barrier level.
- Derivatives 9, 10 are geometric Asian call options written on the 5th asset, with payoffs $\left((\Pi_{k=1}^p S_5(t_k))^{1/p} - K_1\right)^+$ and $\left((\Pi_{k=1}^p S_5(t_k))^{1/p} - K_2\right)^+$ respectively, where p is the number of observations in forming the geometric average.

Market parameters of the portfolio are specified as follows:

- Set initial asset prices to be $S_i(0) = 100$, and returns μ_i and volatilities σ_i are set to be 8% and 10% respectively for $i = 1, \ldots, 5$. Correlation ρ_{ij} between ith and jth asset is set to be 0.5 if $i \neq j$.
- Set strike prices to be $K_1 = 90$ and $K_2 = 100$, and interest rate as $r = 5\%$, maturity as $T = 1$ and time horizon $\tau = 2/24$. For geometric Asian options, $p = 24$ and t_k's is evenly spaced in $[0, T]$. For barrier options, we set $U = 120$ and $H = 85$. When simulating $\max_{0 \le t \le T} S_3(t)$ and $\min_{0 \le t \le T} S_4(t)$, 24 time steps are used and Brownian bridge approximation is applied for any two adjacent time points.
- The level of relative risk aversion is set as $\gamma = 1/100$, with $r_f = 0.5\%$. When truncating the diagonal element of the estimated covariance matrix to ensure positivity, ε is set as 0.01.

Our goal is to allocate a fraction $\mathbf{z} \in \mathbb{R}^{10}$ of wealth to the ten derivative securities and the remainder to the risk-free asset so as to maximize the mean-variance utility function at a future time τ. In this example, the risk factors are given by

$$X_\tau = \left[S_i(\tau), \max_{0 \le t \le \tau} S_3(t), \min_{0 \le t \le \tau} S_4(t), \prod_{t_k \le \tau} S_5(t_k) \right], \quad i = 1, \ldots, 5.$$

We examine the proposed method for the MV optimization problems with two different constraints:

Problem a. The feasible set is defined as $\mathscr{Z} = \{\mathbf{z} \in \mathbb{R}^K : \mathbf{z} \ge 0, \mathbf{z}^\top \mathbf{1}_K \le 1\}$ to exclude short sale.

Problem b. The feasible set is defined as $\mathscr{Z} = \{\mathbf{z} \in \mathbb{R}^K : -1 \le \mathbf{z} \le 1, -1 \le 1 - \mathbf{z}^\top \mathbf{1}_K \le 1\}$ to allow short sale to certain extent.

During implementation, we vary the sample size n within the range $\{10^3, 10^{3.25}, 10^{3.5}, \ldots, 10^6\}$ and the number of replications is set to be $1{,}000$ for each sample size. Due to unavailability of closed-form solutions, we use simulation with a very large sample size (i.e., 10^8) to approximate the true values of the optimal solutions and optimal objective values, which are then used as benchmarks in evaluating the proposed method.

In this example, mean-variance portfolio optimization in (2) with both constraints specified in Problems (a) and (b) has unique optimal solutions, denoted by $\mathbf{z}_a^*$ and $\mathbf{z}_b^*$, respectively. The approximating problem in (5) also produces unique solutions with different sample size n, denoted by $\check{\mathbf{z}}_n^a$ and $\check{\mathbf{z}}_n^b$, and the approximated utility values are denoted by $\check{U}_n(\check{\mathbf{z}}_n^a)$ and $\check{U}_n(\check{\mathbf{z}}_n^b)$. In addition, the semi-approximate utility values are given by $U(\check{\mathbf{z}}_n^a)$ and $U(\check{\mathbf{z}}_n^b)$, respectively.

In Tables 1 and 2, approximated solutions $\check{\mathbf{z}}_n^a$ and $\check{\mathbf{z}}_n^b$ with different sample sizes n are compared to optimal solutions $\mathbf{z}_a^*$ and $\mathbf{z}_b^*$, where $\check{\mathbf{z}}_n^a$ and $\check{\mathbf{z}}_n^b$ are randomly drawn from 1000 replications. These tables show that the approximated solution $\check{\mathbf{z}}_n$ is the same as optimal one $\mathbf{z}^*$ when sample size n is sufficiently large, which is consistent with our asymptotic analysis.

Table 1: Holding ratio $\check{\mathbf{z}}_n^a$ for Problem (a).

sample size	derivative index									
	1	2	3	4	5	6	7	8	9	10
10^3	0	1	0	0	0	0	0	0	0	0
10^4	0	0	0	0	0	0	0	0	0	1
10^5	0	0	0	0	0	0	0	0	0	1
10^6	0	0	0	0	0	0	0	0	0	1
optimal	0	0	0	0	0	0	0	0	0	1

Table 2: Holding ratio $\check{\mathbf{z}}_n^b$ for Problem (b).

sample size	derivative index									
	1	2	3	4	5	6	7	8	9	10
10^3	1	1	-1	1	-1	-1	1	1	-1	1
10^4	1	1	-1	-1	-1	-1	1	1	1	1
10^5	1	1	-1	-1	-1	-1	1	1	1	1
10^6	1	1	-1	-1	-1	-1	1	1	1	1
optimal	1	1	-1	-1	-1	-1	1	1	1	1

In Figure 1, we depict the bias squared, variance and mean squared error (MSE) of the estimated values $\check{U}_n(\check{\mathbf{z}}_n^a)$ and $\check{U}_n(\check{\mathbf{z}}_n^b)$. As shown in Figure 1, as n increases, all the error measures decrease. Moreover, the rate of convergence of the MSE is plotted in Figure 2, which decays approximately at a rate of n^{-1} for both Problems (a) and (b). To measure the quality of the semi-approximated utility values, we estimate bias squared, variance and MSE of $U(\check{\mathbf{z}}_n^a)$ and $U(\check{\mathbf{z}}_n^b)$ in Figure 3, showing that the MSE decreases and converges to zero as the sample size increases, and a main part of the MSE comes from its bias while its variance is relatively small.

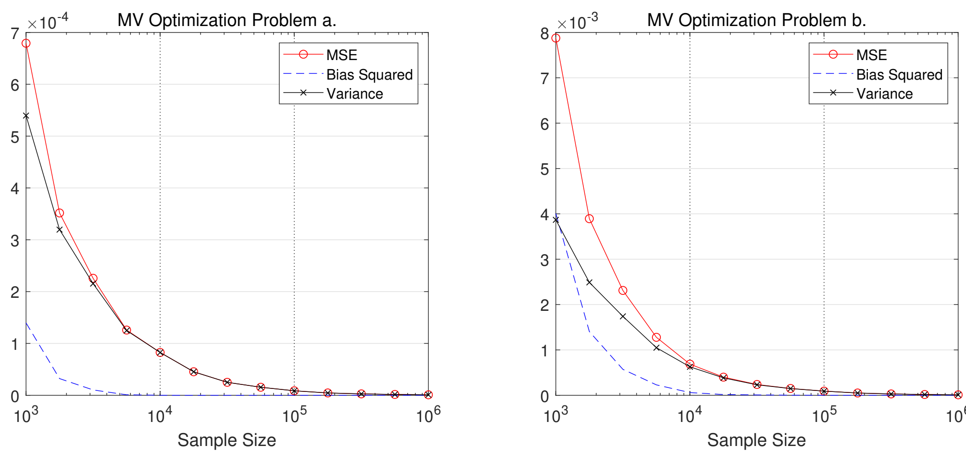

Figure 1: Bias squared, variance, and MSE of $\check{U}_n(\check{\mathbf{z}}_n^a)$ and $\check{U}_n(\check{\mathbf{z}}_n^b)$ for Problems (a) and (b).

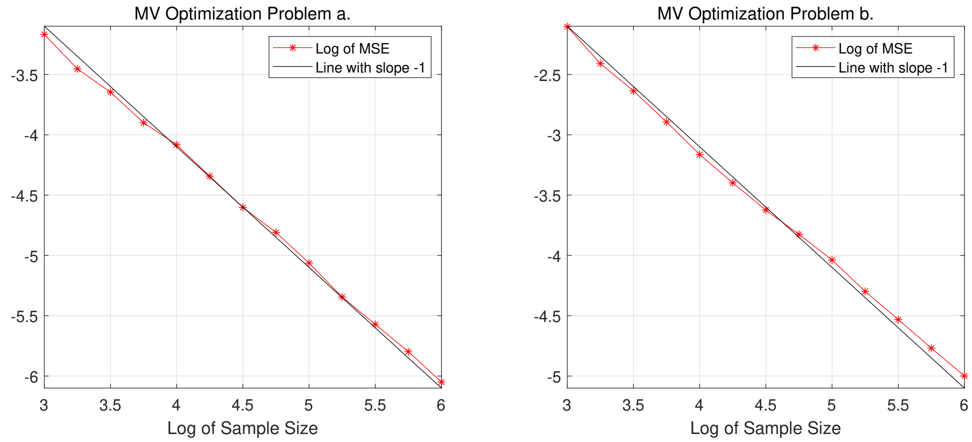

Figure 2: Illustration of convergence rates of MSEs of $\check{U}_n(\check{\mathbf{z}}_n^a)$ and $\check{U}_n(\check{\mathbf{z}}_n^b)$.

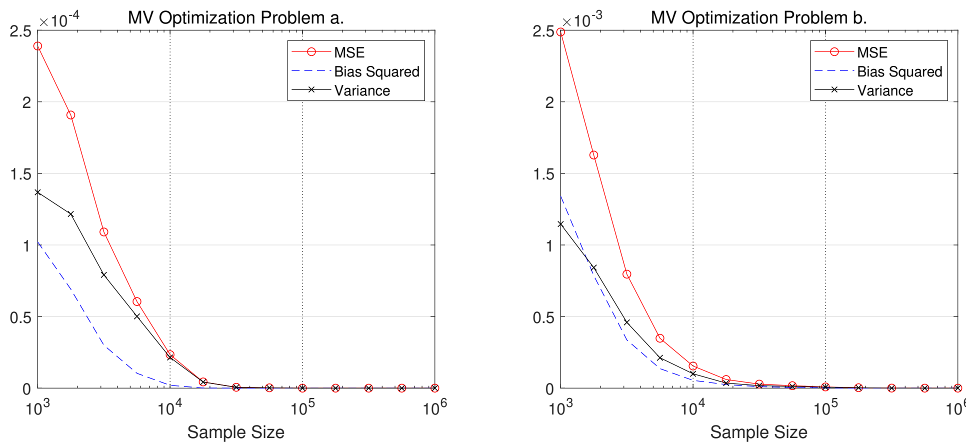

Figure 3: Bias squared, variance, and MSE of $U(\check{\mathbf{z}}_n^a)$ and $U(\check{\mathbf{z}}_n^b)$ for Problems (a) and (b).

6 CONCLUSIONS

We have proposed a simulation method for solving the mean-variance optimization for portfolios that may include derivative securities, at the heart of which is the development of an unbiased and consistent estimator of the covariance matrix of the asset returns. The proposed method leads naturally to a sample-based mean-variance optimizaton problem which is easy to solve. We have analyzed the asymptotic properties of the solution to and the objective value of the sample-based mean-variance optimization problem, and demonstrated its performances via a numerical example.

ACKNOWLEDGMENTS

This research is partially supported by the Research Grants Council (RGC) of Hong Kong via General Research Fund under project 11508620 and via NSFC/RGC Joint Research Scheme under project N_CityU105/21, and InnoHK initiative, The Government of the HKSAR, and Laboratory for AI-Powered Financial Technologies.

A APPENDIX

A.1 Proof of Theorem 1

Note that

$$\mathrm{E}\left[\hat{V}_{kl}\right]=\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(i)}\right]-\frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^{n}Y_k^{(i)}\cdot\sum_{i=1}^{n}Y_l'^{(i)}\right].$$

By conditional independence of $Y_k^{(i)}$ and $Y_l'^{(i)}$ given $X_\tau^{(i)}$, we have

$$\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(i)}\right]=\mathrm{E}\left[\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(i)}\middle|X_\tau^{(i)}\right]\right]=\mathrm{E}\left[\mathrm{E}\left[Y_k^{(i)}\middle|X_\tau^{(i)}\right]\cdot\mathrm{E}\left[Y_l'^{(i)}\middle|X_\tau^{(i)}\right]\right]=\mathrm{E}\left[\mathrm{E}\left[Y_k|X_\tau\right]\cdot\mathrm{E}\left[Y_l|X_\tau\right]\right].$$

Moreover, this conditional independence also implies

$$\begin{aligned}\frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^{n}Y_k^{(i)}\cdot\sum_{i=1}^{n}Y_l'^{(i)}\right]&=\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(i)}\right]+\frac{1}{n^2}\sum_{i\neq j}\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(j)}\right]\\&=\frac{1}{n}\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(i)}\right]+\frac{n-1}{n}\mathrm{E}\left[Y_k^{(i)}\right]\cdot\mathrm{E}\left[Y_l'^{(j)}\right]=\frac{1}{n}\mathrm{E}\left[\mathrm{E}\left[Y_k|X_\tau\right]\cdot\mathrm{E}\left[Y_l|X_\tau\right]\right]+\frac{n-1}{n}\mathrm{E}\left[Y_k\right]\cdot\mathrm{E}\left[Y_l\right].\end{aligned}$$

Thus, we have

$$\mathrm{E}\left[\hat{V}_{kl}\right]=\frac{n-1}{n}\left(\mathrm{E}\left[\mathrm{E}\left[Y_k|X_\tau\right]\cdot\mathrm{E}\left[Y_l|X_\tau\right]\right]-\mathrm{E}\left[Y_k\right]\cdot\mathrm{E}\left[Y_l\right]\right)=\frac{n-1}{n}\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right),$$

and hence $\mathrm{E}\left[\widetilde{V}_{kl}\right]=\frac{n}{n-1}\mathrm{E}\left[\hat{V}_{kl}\right]=\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)$.

A.2 Proof of Proposition 1

(a) Decompose $\hat{V}_{kl}$ into two parts,

$$\begin{aligned}\hat{V}_{kl}&=\frac{1}{n}\sum_{i=1}^{n}\left(Y_k^{(i)}-\frac{1}{n}\sum_{i=1}^{n}Y_k^{(i)}\right)\cdot\left(Y_l'^{(i)}-\frac{1}{n}\sum_{i=1}^{n}Y_l'^{(i)}\right)\\&=\frac{1}{n}\sum_{i=1}^{n}\left(Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(Y_l'^{(i)}-\mathrm{E}[Y_l]\right)-\left(\frac{1}{n}\sum_{i=1}^{n}Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n}Y_l'^{(i)}-\mathrm{E}[Y_l]\right)\triangleq V_{1n}-V_{2n}.\end{aligned}$$

It follows from conditional independence of $Y_k^{(i)}$ and $Y_l'^{(i)}$ that

$$\begin{aligned}\mathrm{E}[V_{1n}]&=\mathrm{E}\left[\left(Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(Y_l'^{(i)}-\mathrm{E}[Y_l]\right)\right]\\&=\mathrm{E}\left[Y_k^{(i)}\cdot Y_l'^{(i)}\right]-\mathrm{E}\left[Y_k^{(i)}\right]\cdot\mathrm{E}[Y_l]-\mathrm{E}[Y_k]\cdot\mathrm{E}\left[Y_l'^{(i)}\right]+\mathrm{E}[Y_k]\cdot\mathrm{E}[Y_l]\\&=\mathrm{Cov}\left(\mathrm{E}[Y_k|X_\tau],\mathrm{E}[Y_l|X_\tau]\right)=\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right).\end{aligned}$$

Then, it follows from the strong law of large numbers (SLLN) that w.p.1, $V_{1,n}\to\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)$ as $n\to\infty$. It can also be easily verified that $V_{2,n}\to\ 0$ w.p.1 as $n\to\infty$. Combing this with the fact that $\widetilde{V}_{kl}=\frac{n}{n-1}\hat{V}_{kl}$ leads to the conclusion of part (a).

(b) Note that

$$\begin{aligned}&\sqrt{n}\left(\hat{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right)\\&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(Y_l'^{(i)}-\mathrm{E}[Y_l]\right)-\mathrm{Cov}\left(r^k(X_\tau),r^l(X_\tau)\right)\right]\\&\quad-\sqrt{n}\cdot\left(\frac{1}{n}\sum_{i=1}^{n}Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n}Y_l'^{(i)}-\mathrm{E}[Y_l]\right)\triangleq V_{3n}-V_{4n}.\end{aligned}$$

It follows from the central limit theorem and the fact $\mathrm{E}\left[\left(Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(Y_l^{\prime(i)}-\mathrm{E}[Y_l]\right)\right]=\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)$ that

$$V_{3n}=\frac{1}{\sqrt{n}}\sum_{i=1}^n\left[\left(Y_k^{(i)}-\mathrm{E}[Y_k]\right)\cdot\left(Y_l^{\prime(i)}-\mathrm{E}[Y_l]\right)-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right]\xrightarrow{d}N(0,\sigma_{kl}^2),$$

where $\sigma_{kl}^2=\mathrm{Var}\left((Y_k-\mathrm{E}[Y_k])\cdot\left(Y_l'-\mathrm{E}[Y_l]\right)\right)$.

Moreover, as $n\to\infty$,

$$V_{4n}=\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(Y_k^{(i)}-\mathrm{E}[Y_k]\right)\right)\cdot\left(\frac{1}{n}\sum_{i=1}^n\left(Y_l^{\prime(i)}-\mathrm{E}[Y_l]\right)\right)\triangleq S_{1n}\cdot S_{2n}\xrightarrow{p}0,$$

because $S_{1n}\xrightarrow{d}N\left(0,\mathrm{Var}(Y_k)\right)$ and $S_{2n}\xrightarrow{p}0$ by the central limit theorem and SLLN, respectively.

Therefore, $\sqrt{n}\left(\hat{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right)\xrightarrow{d}N(0,\sigma_{kl}^2)$, and hence

$$\sqrt{n}\left(\widetilde{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right)=\frac{n}{n-1}\cdot\sqrt{n}\left(\hat{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right)+\frac{\sqrt{n}}{n-1}\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\xrightarrow{d}N(0,\sigma_{kl}^2).$$

A.3 Proof of Proposition 2

Note that w.p.1, as $n\to\infty$,

$$\begin{aligned}
&\sup_{\mathbf{z}\in\mathscr{Z}}|\hat{U}_n(\mathbf{z})-U(\mathbf{z})|=\sup_{\mathbf{z}\in\mathscr{Z}}\left|\mathbf{z}^\top\left(\hat{\mu}_\tau-\mu_\tau\right)-\frac{\gamma}{2}\cdot\mathbf{z}^\top\left(\widetilde{\Sigma}_\tau-\Sigma_\tau\right)\mathbf{z}\right|\\
&\le\sup_{\mathbf{z}\in\mathscr{Z}}\left|\mathbf{z}^\top\left(\hat{\mu}_\tau-\mu_\tau\right)\right|+\frac{\gamma}{2}\cdot\sup_{\mathbf{z}\in\mathscr{Z}}\left|\mathbf{z}^\top\left(\widetilde{\Sigma}_\tau-\Sigma_\tau\right)\mathbf{z}\right|\\
&\le\sup_{\mathbf{z}\in\mathscr{Z}}\left|\sum_{k=1}^K z_k\left(\frac{1}{n}\sum_{i=1}^n Y_k^{(i)}-\mathrm{E}[Y_k]\right)\right|+\frac{\gamma}{2}\cdot\sup_{\mathbf{z}\in\mathscr{Z}}\left|\sum_{k=1}^K\sum_{l=1}^K z_kz_l\left(\widetilde{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right)\right|\\
&\le\sum_{k=1}^K\sup_{\mathbf{z}\in\mathscr{Z}}\left|z_k\left(\frac{1}{n}\sum_{i=1}^n Y_k^{(i)}-\mathrm{E}[Y_k]\right)\right|+\frac{\gamma}{2}\cdot\sum_{k=1}^K\sum_{l=1}^K\sup_{\mathbf{z}\in\mathscr{Z}}\left|z_kz_l\left(\widetilde{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right)\right|\\
&=\sum_{k=1}^K\left|\frac{1}{n}\sum_{i=1}^n Y_k^{(i)}-\mathrm{E}[Y_k]\right|\sup_{\mathbf{z}\in\mathscr{Z}}|z_k|+\frac{\gamma}{2}\cdot\sum_{k=1}^K\sum_{l=1}^K\left|\widetilde{V}_{kl}-\mathrm{Cov}\left(r_k(X_\tau),r_l(X_\tau)\right)\right|\sup_{\mathbf{z}\in\mathscr{Z}}|z_kz_l|\longrightarrow 0,
\end{aligned}$$

where the convergence is due to Proposition 1(a) and SLLN.

Define $\delta_n\triangleq\sup_{\mathbf{z}\in\mathscr{Z}}|\hat{U}_n(\mathbf{z})-U(\mathbf{z})|$. Then, $U(\mathbf{z})-\delta_n\le\hat{U}_n(\mathbf{z})\le U(\mathbf{z})+\delta_n$, and hence by definition, for $\hat{\mathbf{z}}_n\in\hat{\mathscr{Z}}_n$ and $\mathbf{z}^*\in\mathscr{Z}^*$,

$$\hat{U}_n(\hat{\mathbf{z}}_n)\geqslant\hat{U}_n(\mathbf{z}^*)\geqslant U(\mathbf{z}^*)-\delta_n,\quad\text{and}\quad\hat{U}_n(\hat{\mathbf{z}}_n)\leqslant U(\hat{\mathbf{z}}_n)+\delta_n\leqslant U(\mathbf{z}^*)+\delta_n,$$

implying that

$$|\hat{u}_n-u^*|\le\delta_n\to 0$$

w.p.1, as $n\to\infty$, which leads to the conclusion of part (a), and

$$u^*-\inf\{U(\mathbf{z}):\mathbf{z}\in\hat{\mathscr{Z}}_n\}\le 2\delta_n\to 0$$

w.p.1, as $n\to\infty$, which implies the conclusion of part (c).

Define $\rho\triangleq u^*-\max_{\mathbf{z}\in\mathscr{Z}\setminus\mathscr{Z}^*}U(\mathbf{z})$. Because for any $\mathbf{z}\in\mathscr{Z}\setminus\mathscr{Z}^*$, $u^*>U(\mathbf{z})$ and $\mathscr{Z}$ is compact, we have $\rho>0$. Choose sufficiently large n such that $\delta_n<\rho/2$. Then, $|u^*-\hat{u}_n|\le\delta_n<\rho/2$, implying $u^*-\rho/2<\hat{u}_n$. Furthermore, because $|\hat{U}_n(\mathbf{z})-U(\mathbf{z})|\le\delta_n<\rho/2$ and thus $\hat{U}_n(\mathbf{z})<U(\mathbf{z})+\rho/2$, and for any $\mathbf{z}\in\mathscr{Z}\setminus\mathscr{Z}^*$, $U(\mathbf{z})<\max_{\mathbf{z}\in\mathscr{Z}\setminus\mathscr{Z}^*}U(\mathbf{z})=u^*-\rho$, it holds that $\hat{U}_n(\mathbf{z})<u^*-\rho/2$ for any $\mathbf{z}\in\mathscr{Z}\setminus\mathscr{Z}^*$. Therefore, it can be seen that if $\mathbf{z}\in\mathscr{Z}\setminus\mathscr{Z}^*$, $\hat{U}_n(\mathbf{z})<\hat{u}_n$ and hence $z\notin\hat{\mathscr{Z}}_n$, which completes the proof of part (b).

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