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A NEW APPROACH TO SENSITIVITY ANALYSIS BASED ON DIRAC DELTA FAMILY METHODS

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ABSTRACT

In this paper, we propose a new approach to sensitivity analysis by utilizing the Dirac Delta family method. In a novel way, we combine it with the classical infinitesimal perturbation analysis (IPA) estimator, and propose a new class of Dirac-Delta based sensitivity estimators, which we name as the Delta-Family IPA estimators. We establish an explicitly computable error bound for the Delta-Family IPA estimators, which bypasses the usual technical assumption of interchangeability of limit and differentiation as in the literature of IPA stochastic derivatives estimators. Numerical examples of Greeks computations in the case of European call options and Asian digital options illustrate the improved efficiency of the proposed method as compared to the IPA method.

1 INTRODUCTION

Stochastic derivatives estimation is an important research area with far-reaching connections to contextual areas in operations research. For example, it plays a crucial role in gradient-based optimization methods, which are popular in machine learning applications, e.g., training of artificial neural network, (Peng et al. 2022), in statistics, e.g., estimating score function in maximum likelihood estimation, (Peng et al. 2020), and in financial engineering, e.g., options Greeks estimation, (Chen and Liu 2014), etc.

In the literature, there are two prominent classes of methods for stochastic derivatives estimation: the infinitesimal perturbation analysis (IPA), and the likelihood ratio method (LRM). There is a vast literature on the IPA and LRM and their comparison, see for example (L'Ecuyer 1990; Fu 2006; Liu and Hong 2011; Cui et al. 2020) and the references therein. There have also been various variants and extensions of these two methods in the literature. However, to clearly illustrate the idea and advantages of the proposed method, in this paper, we shall focus on comparing with the original IPA method. In particular, we utilize the Dirac Delta family method to design a new stochastic derivative estimator, the Delta-Family IPA estimator. We shall name it DF-IPA estimator in subsequent discussions. The possible combination of the Dirac Delta family method with variants of IPA estimators shall be delegated to future research. In numerical experiments, we found that our method almost always outperforms the traditional IPA and LRM. Even under extreme parameter conditions, our method is not worse than IPA and LRM. By adjusting ε, our method can achieve a smaller root mean square error.

A summary of the existing representative methods together with our proposed method is provided in Table 1. It can be seen that our method allows the presence of both discontinuous sample performance function and the structural parameter. On the other hand, the limitation of our method should be acknowledged. Our method relies on the choice of ε. If ε is too small, then the results may be inaccurate. Conversely, if ε is too large, the computation can become time-consuming and the results may be unstable. The determination of the optimal choice of ε is left as future research.

Table 1: Comparison of key methods.

Methods	discontinuity of sample performance	structural parameter
IPA	Not allow	Allow
LRM	Allow	Not allow
DF-IPA	Allow	Allow

The contributions of the paper are as follows:

1. We propose a new class of unbiased stochastic derivatives estimator based on the Dirac Delta family method, which combines the Dirac Delta family representation with the IPA method.
2. We establish explicitly computable theoretical error bound for the DF-IPA estimator, bypassing the need for the hard-to-verify interchangeability conditions as in the literature. This broadens the scope of applications in which our method can be employed. It is of independent theoretical interest.
3. Numerical experiments demonstrate the improved efficiency of the proposed method against the IPA and LRM, as measured by smaller root mean squared errors (RMSEs)

The remainder of the paper is organized as follows: Section 2 presents the main results on the construction of the Delta-Family IPA estimator and analyzes its theoretical properties, including the convergence and the explicitly computable error bound. Section 3 illustrates the new estimator in two applications: computing Greeks of European options under the Variance Gamma process, and Greeks for Asian digital options. Section 4 concludes the paper with a discussion of future research directions.

2 MAIN RESULTS

Recall that a Dirac delta function, denoted by δ, is a generalized function on the real line that is zero everywhere except at zero, with an integral of one over the entire real line. To elucidate our method, we first introduce the following definition of (Dirac) Delta family. For an open subset $\Omega \subset \mathbb{R}$, let $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions on Ω with compact support. A function f belongs to $L^\infty(\Omega)$ if there exists a real number M such that the absolute value of f is bounded by M almost everywhere in Ω.

Definition 1 A family of functions $\delta_\varepsilon \in L^\infty(\Omega)$ is said to be a Delta family on Ω if for each $\phi \in C_c^\infty(\Omega)$ and $x \in \Omega$, the following holds:

$$\lim_{\varepsilon \to 0} \int_\Omega \delta_\varepsilon(x-y)\phi(y)dy = \phi(x). \tag{1}$$

One can see that the Delta family essentially converges weakly to the Dirac Delta function as $\varepsilon \to 0$. Recall the sifting property of the Dirac Delta function, i.e.,

$$\int_\Omega \delta(x-y)\phi(y)dy = \phi(x). \tag{2}$$

Then by (1) and (2), we can replace the Dirac Delta function by its equivalent distributional representations using the tools of Delta family method, i.e., through Delta sequences. As for the Delta sequences, there are many possible choices as documented in (Walter and Blum 1979), see Table 2 below.

For simplicity, throughout this paper, we utilize the Delta sequence based on the Normal density function:

$$\delta(x-a) = \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{(x-a)^2}{4\varepsilon}},$$

where the convergence is in the weak sense and accordingly, the indicator function can be represented as

$$\mathbb{1}_{\{x \geq a\}} = \int_{-\infty}^{x-a} \delta(u)du = \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{x-a} e^{-\frac{u^2}{4\varepsilon}} du.$$

Table 2: Different types of Delta sequences.

Types	Delta sequences $\delta(x-y)$
Normal density	$\frac{1}{2\sqrt{\pi\varepsilon}}e^{-\frac{(x-y)^2}{4\varepsilon}}$
Lorentzian type	$\frac{1}{\pi}\frac{\varepsilon}{(x-y)^2+\varepsilon^2}$
Fourier integral	$\frac{1}{2\pi}\int_{-1/\varepsilon}^{1/\varepsilon} e^{i(x-y)t}dt$
Trigonometric function	$\frac{1}{2\pi}\frac{\sin\left[\left(\frac{1}{\varepsilon}+\frac{1}{2}\right)(x-y)\right]}{\sin\left(\frac{1}{2}(x-y)\right)}$

2.1 Delta Family Method Combined with IPA Estimator

Let the random variable S depend on a parameter θ and have a probability density f. For the differentiable functions g and h, we consider the payoff of the following form $G(S) = g(S)\mathbb{1}_{\{h(S)\geq 0\}}$, and denote $p(\theta) := \mathbb{E}[G(S)]$. The goal is to calculate $p'(\theta)$, the sensitivity of $p(\theta)$ with respect to θ.

From Theorem 1 of (Liu and Hong 2011), under some technical conditions for g and h, $p'(\theta)$ can be expressed as follows:

$$p'(\theta) = \mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\right] - \partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0}. \tag{3}$$

Alternatively, by using the Dirac Delta family method, first we have the following representation:

$$p(\theta) = \mathbb{E}\left[g(S)\mathbb{1}_{\{h(S)\geq 0\}}\right] = \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right], \tag{4}$$

such that one can differentiate w.r.t. θ and obtain the following representation:

$$p'(\theta) = \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi\varepsilon}}\left(\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right] + \mathbb{E}\left[g(S)e^{-\frac{(h(S))^2}{4\varepsilon}}\partial_\theta h(S)\right]\right). \tag{5}$$

Note that in order to obtain the above expression (5), it requires interchanging the limit and the differentiation. We will make (5) rigorous in Proposition 3 by not only showing that (5) holds but also providing an explicitly computable error bound between $p'(\theta)$ and the right hand side in (5) for any sufficiently small ε.

From the above, it can be observed that our approach is to first apply the Dirac Delta family method to represent the probability density function, and then carry out the differentiation. However, in the previous literature, the differentiation is usually carried out first, and then followed by the smoothing. More specifically, (Liu and Hong 2009) apply Gaussian kernel smoothing to the expression (3), which is obtained *after* the differentiation procedure. Our approach is instead to first represent $p(\theta)$ as in (4), and then apply differentiation after the smoothing procedure. This explains the main distinction of our method from the literature.

Next, as promised, we shall provide an explicit error bound, and more specifically, we would like to upper bound the difference between (3) and

$$\frac{1}{2\sqrt{\pi\varepsilon}}\left(\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right] + \mathbb{E}\left[g(S)e^{-\frac{(h(S))^2}{4\varepsilon}}\partial_\theta h(S)\right]\right). \tag{6}$$

We first introduce the following lemma that provides an error bound on the difference between the first terms in (3) and (6).

Lemma 1 For any $\delta > 0$, we have

$$\begin{aligned}&\left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\right] - \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right]\right| \\ &\leq 2\left(\mathbb{E}|\partial_\theta g(S)|\right)^{1/2}\left(\mathbb{P}(-\delta\leq h(S)\leq\delta)\right)^{1/2} + 2e^{-\frac{\delta^2}{4\varepsilon}}\mathbb{E}\left[|\partial_\theta g(S)|\right].\end{aligned}$$

In particular, for any $0 < \varepsilon < 1$, by taking $\delta = \sqrt{4\varepsilon \log(1/\varepsilon)}$, we have

$$\begin{aligned}
&\left| \mathbb{E}\left[\partial_\theta g(S) \mathbb{1}_{\{h(S)\geq 0\}}\right] - \frac{1}{2\sqrt{\pi\varepsilon}} \mathbb{E}\left[\partial_\theta g(S) \int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}} du\right] \right| \\
&\leq 2\left(\mathbb{E}|\partial_\theta g(S)|\right)^{1/2} \left(\mathbb{P}(-\sqrt{4\varepsilon\log(1/\varepsilon)} \leq h(S) \leq \sqrt{4\varepsilon\log(1/\varepsilon)})\right)^{1/2} + 2\varepsilon \mathbb{E}\left[|\partial_\theta g(S)|\right].
\end{aligned}$$

Next, we have the following lemma that provides an error bound on the difference between the second terms in (3) and (6).

Lemma 2 Assume that h is monotonically increasing and s_* is the unique value such that $h(s_*) = 0$ and $h'(s_*) \in (0, \infty)$. Moreover, $g(s_*)\partial_\theta h(s_*) f(s_*) \neq 0$, where f is the probability density function of S. Then, there exists some $C_0 > 0$ which can be computed out explicitly, such that for any sufficiently small ε,

$$\left| \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{\infty} g(s) e^{-\frac{(h(s))^2}{4\varepsilon}} \partial_\theta h(s) f(s) ds + \partial_y \mathbb{E}\left[g(S)\partial_\theta h(S) \mathbb{1}_{\{h(S)\geq y\}}\right] \Big|_{y=0} \right| \leq C_0 \varepsilon.$$

By applying the above two technical lemmas to (3) and (6), we obtain the following proposition in a straightforward way, and thus its proof is omitted.

Proposition 3 Assume that h is monotonically increasing and s_* is the unique value such that $h(s_*) = 0$ and $h'(s_*) \in (0, \infty)$. Moreover, $g(s_*)\partial_\theta h(s_*) f(s_*) \neq 0$, where f is the probability density function of S. Then, there exists some $C_0 > 0$ which can be computed out explicitly, such that for any sufficiently small ε,

$$\begin{aligned}
&\left| p'(\theta) - \frac{1}{2\sqrt{\pi\varepsilon}} \left(\mathbb{E}\left[\partial_\theta g(S) \int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}} du\right] + \mathbb{E}\left[g(S) e^{-\frac{(h(S))^2}{4\varepsilon}} \partial_\theta h(S)\right] \right) \right| \\
&\leq C_0\varepsilon + 2\left(\mathbb{E}|\partial_\theta g(S)|\right)^{1/2} \left(\mathbb{P}(-\sqrt{4\varepsilon\log(1/\varepsilon)} \leq h(S) \leq \sqrt{4\varepsilon\log(1/\varepsilon)})\right)^{1/2} + 2\varepsilon \mathbb{E}\left[|\partial_\theta g(S)|\right].
\end{aligned}$$

Note that Proposition 3 essentially states that it is possible to replace $p'(\theta)$ by the expression in (6) and the error is controllable by an explicitly computable upper bound given in Proposition 3. In proving the result, we bypass the traditional hard-to-verify interchangeability conditions in the IPA literature. Thus in practice, we can conveniently use (5) to compute $p'(\theta)$, as we not only know the explicit form of the maximum error, but also know that this maximum error is converging to zero as ε is taken arbitrarily small.

Note that if we further suppose that h and g are independent of θ, which implies that h and g themselves do not contain θ. Then $\partial_\theta h(S)$ can be interpreted as $h'(S)\frac{\partial S}{\partial \theta}$, and $\partial_\theta h(s)$ is given by:

$$\partial_\theta h(s) = h'(S) \frac{\partial S}{\partial \theta}\Big|_{S=s}. \tag{7}$$

Similarly, $\partial_\theta g(S) = g'(S)\frac{\partial S}{\partial \theta}$, and $\partial_\theta g(s) = g'(S)\frac{\partial S}{\partial \theta}|_{S=s}$.

Remark 4 The assumptions of Proposition 3 are satisfied in practice, and they are also met in subsequent numerical examples. For most financial options, h is a linear function that increases monotonically. $g(s_*)\partial_\theta h(s_*) f(s_*) \neq 0$ means $g(s_*)$, $\partial_\theta h(s_*)$ and $f(s_*)$ are all not equal to zero. By (7), $\partial_\theta h(s_*) = h'(S)\frac{\partial S}{\partial \theta}\Big|_{S=s_*}$. When $h'(s_*) \in (0, \infty)$, we only need $\frac{\partial S}{\partial \theta}\Big|_{S=s_*} \neq 0$.

Therefore, by Proposition 3, we can represent the Delta-family IPA estimator (denoted as DF-IPA) equivalently as

$$\begin{aligned}
\text{DF-IPA} &= \lim_{\varepsilon\to 0} \frac{1}{2\sqrt{\pi\varepsilon}} \left\{ \partial_\theta g(S) \int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}} du + g(S) e^{-\frac{(h(S))^2}{4\varepsilon}} \partial_\theta h(S) \right\} \\
&= \lim_{\varepsilon\to 0} \frac{1}{2\sqrt{\pi\varepsilon}} \left\{ \partial_\theta S \left(g'(S) \int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}} du + g(S) e^{-\frac{(h(S))^2}{4\varepsilon}} h'(S) \right) \right\}.
\end{aligned} \tag{8}$$

2.2 Applications to Greeks of European Options

Let us take a simple example of the European call option payoff at maturity, i.e. $G(S) = (S-K)^+$ where $g(S) = h(S) = S-K$. Then we have

$$\begin{aligned}\mathbb{E}[(S_T-K)^+] &= \mathbb{E}\left[(S_T-K)\mathbb{1}_{\{S_T\geq K\}}\right] = \mathbb{E}\left[(S_T-K)\int_{-\infty}^{S_T-K}\delta(u)du\right] \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[(S_T-K)\int_{-\infty}^{S_T-K}e^{-\frac{u^2}{4\varepsilon}}du\right].\end{aligned}$$

To summarize, we have

$$\mathbb{E}[(S_T-K)^+] = \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}[a_{\text{DF}-\text{IPA}}(S_T-K)], \tag{9}$$

where

$$a_{\text{DF}-\text{IPA}}(x) := x\int_{-\infty}^{x}e^{-\frac{s^2}{4\varepsilon}}ds.$$

For the DF-IPA approach, by taking the derivative of (9) w.r.t. θ and by chain rule, we obtain:

$$\frac{\partial}{\partial\theta}\mathbb{E}[(S_T-K)^+] = \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\left(\frac{\partial S_T}{\partial\theta}\right)\cdot\left(\int_{-\infty}^{S_T-K}e^{-\frac{u^2}{4\varepsilon}}du + (S_T-K)e^{-\frac{(S_T-K)^2}{4\varepsilon}}\right)\right].$$

Then

$$\text{DF-IPA} := \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi\varepsilon}}\left(\frac{\partial S_T}{\partial\theta}\right)\cdot\left(\int_{-\infty}^{S_T-K}e^{-\frac{u^2}{4\varepsilon}}du + (S_T-K)e^{-\frac{(S_T-K)^2}{4\varepsilon}}\right). \tag{10}$$

θ represents different parameters in the model corresponding to different Greeks. For example, in (10), `delta` corresponds to $\theta = S_0$, and `vega` corresponds to $\theta = \sigma$.

We can similarly derive the DF-IPA formula for the Greeks of Asian digital options, and detailed derivations are available upon request.

3 NUMERICAL EXPERIMENTS

For comparison purposes and in order to distinguish the notations of the classical method from ours, we denote the IPA estimator based on our Dirac Delta family method by “DF-IPA”, and denote the classical IPA and LRM by “IPA” and “LRM” respectively. In this section, each result is conducted through Monte Carlo simulations and all the estimators are estimated through 100 independent runs.

3.1 Greeks of European Options under Variance Gamma Process

In this section, we consider the same example of a Variance Gamma (VG) model as presented in (Glasserman and Liu 2011), which proposed the LRM using the characteristic functions or Laplace transforms. Assume that under the risk-neutral measure, the asset dynamic follows $S_T = S_0\exp((r+b)T + X_T)$, where $b = \log(1-\theta\nu-\sigma^2\nu/2)/\nu$. Here θ, ν and σ are the parameters of a VG model, which is defined by the following: Let W_t be a standard Brownian motion, $\gamma_t^{(\nu)}$ be the gamma process with drift $\mu = 1$ and variance parameter ν, then the VG model X_t can be represented as a Brownian motion subordinated to a gamma subordinator, i.e.,

$$X_t = B_{\gamma_t^\nu}^{(\theta,\sigma)} = \theta\gamma_t^{(\nu)} + \sigma W_{\gamma_t^{(\nu)}},$$

whose Laplace transform is given by

$$L(u) = \mathbb{E}\left[e^{-uX_t}\right] = \left(\frac{1}{1+\theta\nu u - \rho^2\nu u^2/2}\right)^{t/\nu},$$

where $B^{(\theta,\sigma)}$ is a Brownian motion with drift θ and volatility σ and W is a standard Brownian motion. For the European call option, denote $p(X) := e^{-rT}\mathbb{E}[(S_T - K)^+]$, we focus on the Greeks `delta` ($\partial p/\partial S_0$) and `vega` ($\partial p/\partial \sigma$).

We replicate the parameters in (Glasserman and Liu 2011), where $S_0 = 100, r = 0.05, T = 1, \sigma = 0.2, \theta = -0.15, K = 100$, $\nu = \{1, 0.5\}$. They also provide the values of Greeks (delta, vega) as the benchmark, see Table 1 in (Glasserman and Liu 2011). The root mean square error (RMSE) is a commonly used metric for evaluating the performance of a model. It measures the average magnitude of the error between the results of our method and the benchmark. The RMSE is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y})^2},$$

where n is the number of independent runs, y_i is the result of the i-th run and $\hat{y}$ is the benchmark. In Table 3, we report RMSE of five estimators with different sample sizes and 100 independent runs, where "LRM" denotes the proposed method in (Glasserman and Liu 2011), employing the parameters: truncation parameter $T_p = 100$ and grid spacing $\delta = 0.05$. It is shown that DF-IPA significantly outperforms the remaining two estimators in all cases, even with $M = 10^4$, the performance of DF-IPA is better than that of IPA and LRM at $M = 10^6$.

Table 3: RMSE for Greeks of European call options under the VG model.

		$M = 10^4$		$M = 10^5$		$M = 10^6$	
	RMSE	`delta`	`vega`	`delta`	`vega`	`delta`	`vega`
	DF-IPA	0.0055	0.7731	0.0016	0.2741	5.9421E-04	0.0745
	IPA	0.0371	1.3557	0.0375	1.2911	0.0373	1.1810
$\nu = 1$	LRM	0.7726	6.9233	0.5390	4.5282	0.3767	3.7516
	DF-IPA	0.0056	0.7696	0.0020	0.2654	4.9188E-04	0.0875
	IPA	0.0359	1.6572	0.0356	1.4967	0.0355	1.4561
$\nu = 0.5$	LRM	0.6765	6.0926	0.5051	4.0755	0.3271	3.5637

3.2 Greeks of Asian Digital Option under OU Process

In this section, we consider an example of Asian digital option, where the underlying asset is assumed to follow an Ornstein-Uhlenbeck (OU) process that satisfies the following stochastic differential equation (SDE):

$$dS_t = \kappa(\mu - S_t)dt + \sigma dW_t, \quad t \geq 0, \tag{11}$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion and the solution of (11) can be written as

$$S_t = \mu + (S_0 - \mu)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-u)}dW_u.$$

The payoff of the Asian digital option is given by $\mathbb{1}_{\{\bar{S}\geq K\}}$ with $\bar{S} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$ and $t_i = i \cdot T/n$, which is considered in (Liu and Hong 2011). We aim to estimate `delta`, `vega` and `theta` for $A(\theta) = e^{-rT}\mathbb{E}[\mathbb{1}_{\{\bar{S}\geq K\}}]$, which are defined by $\partial A/\partial S_0, \partial A/\partial \sigma$ and $-\partial A/\partial T$, respectively. In our setting, $g(S) = 1$, $h(S) = \bar{S} - K$. By (8), we obtain the following estimators using the Delta family method:

$$\text{DF-IPA} = e^{-rT} \cdot \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi\varepsilon}} \cdot \frac{\partial \bar{S}}{\partial \theta} e^{-\frac{(\bar{S}-K)^2}{4\varepsilon}},$$

The classical IPA and LRM estimators can be also derived, which are illustrated in (Liu and Hong 2011). We replicate the parameters specified as $r = 0.05, \sigma = 0.3, \kappa = 0.2, \mu = 98, S_0 = 100, K = 100$ and $T = 1$ in (Liu and Hong 2011), and they also provide the exact values with different numbers of discrete monitoring, which are regarded as the benchmark. We report the variance of the three estimators with varying sample sizes for fixed $\varepsilon = 1/300$, which is found to be sufficiently accurate from pilot numerical experiments. In Figure 1, it is obvious that DF-IPA outperforms the other two estimators significantly in terms of variance. Moreover, the variance of DF-IPA remains quite small even with a small sample size, indicating that its performance is more stable than the classical IPA and LRM. We conduct a comparison among these three estimators for `delta`, `vega` and `theta` in terms of the root mean squared error (RMSE). The results are presented in Table 4, where N denotes the number of discrete monitoring time points. One can observe that the DF-IPA exhibits the lowest root mean square error (RMSE) in all cases. The results of DF-IPA show minimal variation with different numbers of discrete monitoring N, while the results of IPA and LRM fluctuate with different N.

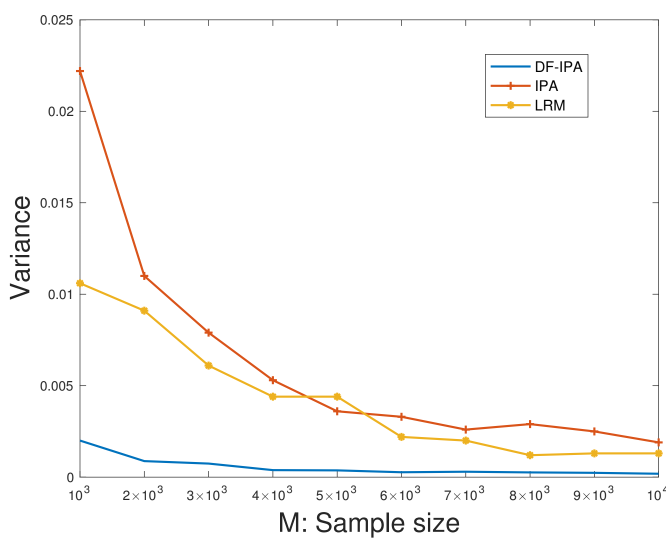

Figure 1: Variance of our method with respect to sample size M for delta under the OU process.

Table 4: RMSE for Greeks of Asian digital option under the OU process.

	N=10			N=20			N=50		
	`delta`	`vega`	`theta`	`delta`	`vega`	`theta`	`delta`	`vega`	`theta`
DF-IPA	0.0292	0.0124	0.0020	0.0156	0.0155	0.0062	0.0209	0.0132	0.0019
IPA	0.0600	0.0344	0.0120	0.1169	0.0412	0.0300	0.1049	0.0781	0.0594
LRM	0.0718	0.0228	0.0172	0.0707	0.0207	0.0094	0.0399	0.0256	0.0081

4 CONCLUSION

In this paper, drawing on the Dirac Delta family method, we devise a new stochastic derivative estimator, the DF-IPA estimator. Explicit error bounds are established for the newly proposed estimator and we manage to bypass the hard-to-verify assumption of intechangeability of limit and differentiation in the literature on IPA estimators. We illustrate the new class of estimators through numerical examples and demonstrate the improved efficiency as compared to IPA estimator, as measured by a smaller RMSE.

It is of interest to extend the DF-IPA estimator to cases of computing sensitivities of quantiles (Hong 2009; Cui and Ding 2022) and distortion risk measures (Glynn et al. 2021). It is also interesting to compare the performance of our method with the generalized likelihood ratio method in (Peng et al. 2019) and other variants. We leave these topics to future research.

A PROOFS

A.1 Proof of Lemma 1

First, we can compute that for any $\delta > 0$,

$$\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\right] = \mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq \delta\}}\right] + \mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\mathbb{1}_{\{h(S)<\delta\}}\right],$$

and

$$\begin{aligned}&\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right]\\ &= \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)\geq\delta\}}\right] + \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)<\delta\}}\right].\end{aligned}$$

This implies that

$$\begin{aligned}&\left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\right] - \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right]\right|\\ &\leq \left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq\delta\}}\right] - \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)\geq\delta\}}\right]\right|\\ &\quad + \left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\mathbb{1}_{\{h(S)<\delta\}}\right]\right| + \left|\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)<\delta\}}\right]\right|.\end{aligned}$$

We can compute that

$$\begin{aligned}&\left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq\delta\}}\right] - \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)\geq\delta\}}\right]\right|\\ &= \left|\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{h(S)}^{\infty} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)\geq\delta\}}\right]\right|\\ &\leq \frac{1}{2\sqrt{\pi\varepsilon}}\int_{\delta}^{\infty} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{E}\left[|\partial_\theta g(S)|\right]\\ &= \mathbb{P}(X_\varepsilon\geq\delta)\mathbb{E}\left[|\partial_\theta g(S)|\right],\end{aligned}$$

where $X_\varepsilon \sim \mathscr{N}(0, 2\varepsilon)$. For any $\theta > 0$,

$$\mathbb{P}(X_\varepsilon\geq\delta) \leq \mathbb{E}[e^{\theta X_\varepsilon}]e^{-\theta\delta} = e^{\theta^2\varepsilon}e^{-\theta\delta},$$

so that $\mathbb{P}(X_\varepsilon \geq \delta) \leq e^{-\frac{\delta^2}{4\varepsilon}}$ by choosing $\theta = \frac{\delta}{2\varepsilon}$. Therefore, we obtain

$$\begin{aligned}&\left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq\delta\}}\right] - \frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)\geq\delta\}}\right]\right|\\ &\leq e^{-\frac{\delta^2}{4\varepsilon}}\mathbb{E}\left[|\partial_\theta g(S)|\right].\end{aligned}$$

Next, we can compute that

$$\left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\mathbb{1}_{\{h(S)<\delta\}}\right]\right| \leq \left(\mathbb{E}|\partial_\theta g(S)|\right)^{1/2}\left(\mathbb{P}(0\leq h(S)\leq\delta)\right)^{1/2},$$

where we have applied the Cauchy-Schwarz inequality and moreover,

$$\begin{aligned}&\left|\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)<\delta\}}\right]\right|\\ &\leq\left|\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)<\delta\}}\mathbb{1}_{\{h(S)>-\delta\}}\right]\right|\\ &\qquad\qquad+\left|\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\cdot\mathbb{1}_{\{h(S)\leq-\delta\}}\right]\right|\\ &\leq\mathbb{E}\left[|\partial_\theta g(S)|\cdot\mathbb{1}_{\{h(S)<\delta\}}\mathbb{1}_{\{h(S)>-\delta\}}\right]+\mathbb{P}(X_\varepsilon\leq-\delta)\mathbb{E}\left[|\partial_\theta g(S)|\right]\\ &\leq\left(\mathbb{E}|\partial_\theta g(S)|\right)^{1/2}\left(\mathbb{P}(-\delta\leq h(S)\leq\delta)\right)^{1/2}+e^{-\frac{\delta^2}{4\varepsilon}}\mathbb{E}\left[|\partial_\theta g(S)|\right].\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}&\left|\mathbb{E}\left[\partial_\theta g(S)\mathbb{1}_{\{h(S)\geq 0\}}\right]-\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[\partial_\theta g(S)\int_{-\infty}^{h(S)} e^{-\frac{u^2}{4\varepsilon}}du\right]\right|\\ &\leq 2\left(\mathbb{E}|\partial_\theta g(S)|\right)^{1/2}\left(\mathbb{P}(-\delta\leq h(S)\leq\delta)\right)^{1/2}+2e^{-\frac{\delta^2}{4\varepsilon}}\mathbb{E}\left[|\partial_\theta g(S)|\right].\end{aligned}$$

This completes the proof.

A.2 Proof of Lemma 2

First, we can compute that

$$\begin{aligned}&-\partial_y\mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0}\\ &=\lim_{\eta\to 0}\frac{1}{\eta}\left(\mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq 0\}}\right]-\mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq\eta\}}\right]\right)\\ &=\lim_{\eta\to 0}\frac{1}{\eta}\int_{h^{-1}(0)}^{h^{-1}(\eta)} g(s)\partial_\theta h(s)f(s)ds,\end{aligned}$$

where we assume that f is the probability density function of S and moreover we assume that h is a monotonic increasing function such that its inverse function h^{-1} exists.

Let us assume that

$$\lim_{\eta\to 0}\frac{h^{-1}(\eta)-h^{-1}(0)}{\eta}=c,$$

for some $c\in[0,\infty]$. We discuss c in three cases in the sequel.

(i) If $c = 0$, then, it is easy to see that

$$-\partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0} = 0.$$

This is a trivial case, and in this case, we do not need to introduce an estimator of $-\partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0}$ since we know this quantity equals to 0.

(ii) If $0 < c < \infty$, then

$$-\partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0} = c \cdot g(h^{-1}(0))\partial_\theta h(h^{-1}(0))f(h^{-1}(0)).$$

Note that if $g(h^{-1}(0))\partial_\theta h(h^{-1}(0))f(h^{-1}(0)) = 0$, then we do not need to introduce an estimator of $-\partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0}$ since we know this quantity equals to 0.

(iii) If $c = \infty$, then $-\partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0}$ diverges and since it is a trivial case, we do not discuss this case.

More generally, if $\int_{h^{-1}(0)}^{h^{-1}(\eta)} g(s)\partial_\theta h(s)f(s)ds$ does not go to 0 as $\eta \to 0$, then

$$-\partial_y \mathbb{E}\left[g(S)\partial_\theta h(S)\mathbb{1}_{\{h(S)\geq y\}}\right]\Big|_{y=0} = \infty,$$

and we do not need to introduce an estimator of it. If $\lim_{\eta\to 0}\int_{h^{-1}(0)}^{h^{-1}(\eta)} g(s)\partial_\theta h(s)f(s)ds = 0$, then by L'Hôpital's rule,

$$\begin{aligned}\lim_{\eta\to 0}\frac{1}{\eta}\int_{h^{-1}(0)}^{h^{-1}(\eta)} g(s)\partial_\theta h(s)f(s)ds &= \lim_{\eta\to 0}\frac{g(h^{-1}(\eta))\partial_\theta h(h^{-1}(\eta))f(h^{-1}(\eta))}{h'(h^{-1}(\eta))}\\ &= \lim_{s\to 0}\frac{g(s)\partial_\theta h(s)f(s)}{h'(s)}.\end{aligned}$$

On the other hand,

$$\frac{1}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[g(S)e^{-\frac{(h(S))^2}{4\varepsilon}}\partial_\theta h(S)\right] = \frac{1}{2\sqrt{\pi\varepsilon}}\int_{-\infty}^{\infty} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds.$$

We assumed that h is monotonically increasing, and let us assume that s_* is the unique value such that $h(s_*) = 0$. We also assume that $g(s_*)\partial_\theta h(s_*)f(s_*) \neq 0$.

For any $M > 0$,

$$\begin{aligned}&\frac{1}{2\sqrt{\pi\varepsilon}}\int_{-\infty}^{\infty} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds\\ &= \frac{1}{2\sqrt{\pi\varepsilon}}\int_{h^{-1}(-M)}^{h^{-1}(M)} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds\\ &\qquad + \frac{1}{2\sqrt{\pi\varepsilon}}\int_{s\notin[h^{-1}(-M),h^{-1}(M)]} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds,\end{aligned}$$

where it is easy to see that

$$\left|\frac{1}{2\sqrt{\pi\varepsilon}}\int_{s\notin[h^{-1}(-M),h^{-1}(M)]} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds\right| \leq \frac{e^{-\frac{M^2}{4\varepsilon}}}{2\sqrt{\pi\varepsilon}}\mathbb{E}\left[|g(S)\partial_\theta h(S)|\right].$$

We consider case (ii), that is,

$$\lim_{\eta\to 0}\frac{h^{-1}(\eta)-h^{-1}(0)}{\eta}=c\in(0,\infty).$$

This is equivalent to

$$h'(s_*)=\frac{1}{c}\in(0,\infty).$$

By Laplace's method, one has

$$\int_{h^{-1}(-M)}^{h^{-1}(M)} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds\sim g(s_*)\partial_\theta h(s_*)f(s_*)\sqrt{\frac{2\pi\varepsilon}{\frac{\partial^2}{\partial s^2}\frac{(h(s))^2}{4}|_{s=s_*}}},$$

as $\varepsilon\to 0$, where $f\sim g$ means $f/g\to 1$. One can compute that

$$\frac{\partial^2}{\partial s^2}\frac{(h(s))^2}{4}\bigg|_{s=s_*}=\frac{1}{2}\left((h'(s))^2+h(s)h''(s)\right)\bigg|_{s=s_*}=\frac{1}{2c^2}.$$

Hence, we conclude that

$$\frac{1}{2\sqrt{\pi\varepsilon}}\int_{-\infty}^{\infty} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds\sim c\cdot g(s_*)\partial_\theta h(s_*)f(s_*),$$

as $\varepsilon\to 0$. Indeed, by applying Proposition 3 in (Aristoff and Zhu 2018), one can show that there exists some $C_0>0$ which can be computed out explicitly, such that for any sufficiently small ε,

$$\left|\frac{1}{2\sqrt{\pi\varepsilon}}\int_{-\infty}^{\infty} g(s)e^{-\frac{(h(s))^2}{4\varepsilon}}\partial_\theta h(s)f(s)ds-c\cdot g(s_*)\partial_\theta h(s_*)f(s_*)\right|\le C_0\varepsilon.$$

The proof is complete.

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