

ZERO STABILITY IN HIERARCHICAL CO-SIMULATION

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ABSTRACT

This work presents investigations on zero-stability of hierarchical co-simulation methods with an arbitrary number of co-simulation levels. In comparison to traditional co-simulation, where all participating systems are coordinated by a single co-simulation, hierarchical co-simulation allows the introduction of further co-simulations on several levels beneath a top-level co-simulation. This way, individual macro step sizes and orchestration algorithms may be used on every level. In this paper, we investigate the implications of the introduction of such a hierarchy, which may extend to an arbitrarily chosen number of levels, on the important convergence property of zero stability.

1 INTRODUCTION

Co-simulation has become an important means to represent complex systems in a holistic way. It is understood (cf. Hafner and Popper (2017)) as the coupling of two or more simulations that differ in at least one of the following aspects:

- simulation tool
- solver algorithm
- step size

Thus, co-simulation allows every system part to be modeled with an individual, most suitable paradigm and to be simulated in an adequate simulation tool, while the coupling itself can be realized by varied approaches. Therein, the choice of a suitable macro step size presents a distinctive challenge, as it mostly comes down to a trade-off between accuracy and speed. In addition, stability issues are still a pressing problem in the area of co-simulation (Schweiger et al. 2019; Hafner and Popper 2021b). This work aims to diminish this gap for hierarchical co-simulation of continuous systems represented by differential algebraic equations (DAEs) and provide a basis for further research in this area.

The concept of hierarchical co-simulation is depicted in Figure 1. One of the most evident examples where it would be sensible to nest co-simulations this way is an application in which certain partial systems depend much more closely on values from one another than others and thus require more frequent data exchange. If this were handled by using a smaller time step for the original overall co-simulation, the whole simulation process would be slowed down unnecessarily. By the hierarchical approach, the more frequent synchronization between some subsystems can be achieved within the additional co-simulation. Still, all partial systems may be implemented as separate simulations and therefore subsystems themselves if required, for instance, due to highly differing modeling paradigms. For further clarification on these terms the reader is referred to Hafner and Popper (2017). Applications for the presented approach can be found in holistic simulations of production facilities including detailed machinery, room temperature, logistics and construction; multi-level realizations of urban systems including traffic and energy networks;

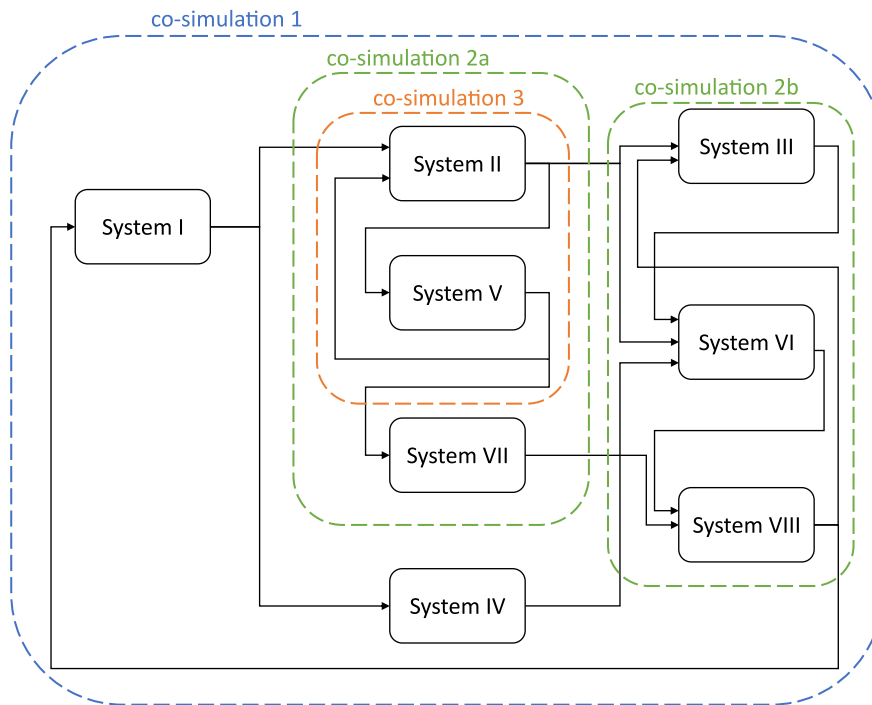


Figure 1: Schematic depiction of a hierarchical co-simulation approach. Coordination takes place on several levels by one top-level co-simulation that manages the communication between subsystems and further co-simulations. These may again coordinate subsystems and co-simulations on lower levels (Hafner and Popper 2021a).

or purely physical systems in one or more domains with differing time constants like the known benchmark for numerical stability investigations in co-simulation, a coupled multi-mass oscillator.

However, we have to investigate whether the introduction of further co-simulation levels influences convergence. The maintenance of consistency (i.e. the boundedness of the error of the numerical integration method per step, cf. Süli and Mayers (2003)) in comparison to traditional co-simulation has been shown in Hafner and Popper (2022), while improvement of numerical stability issues by the introduction of further co-simulation levels has been illustrated in Hafner and Popper (2021a), Hafner and Popper (2022). This work extends the investigations of Hafner and Popper (2021a), where we regard zero-stability properties for two levels of co-simulation, to arbitrary numbers of co-simulation levels.

2 FUNDAMENTALS

This section covers the numerical basis and fundamental work regarding stability in traditional co-simulation on which Section 3 builds. As this work concentrates on systems of differential algebraic equations, the approximate solution requires numerical integration methods as presented in detail in Süli and Mayers (2003), Hairer et al. (1993).

2.1 Zero-Stability of Numerical Integration Methods

For one-step integration methods (i.e. methods that use only the last calculated value for the approximation of the next one), consistency is sufficient for convergence of the numerical solution to the analytic solution. Multi-step methods (which take into account more than one previously calculated value) have to be investigated for zero-stability as well as consistency to ensure convergence (Süli and Mayers 2003, Theorem 12.5). Zero-stability means convergence of the method if the step size converges to zero. It

is equivalent to (Süli and Mayers 2003) and in some cases defined by (Hairer et al. 1993; Kübler and Schiehlen 2000) fulfillment of the root condition:

Lemma 1 (Zero-stability of multi-step methods) A multi-step method is zero-stable if the generating polynomial $\varrho(\zeta)$ (also called *first characteristic polynomial* (Süli and Mayers 2003)) satisfies the root condition, i.e.,

1. the roots of $\varrho(\zeta)$ lie on or within the unit circle and
2. the roots on the unit circle are simple.

For linear systems, zero-stability is independent of the simulation's initial values and therefore an important quality. Further background and additional information on basic terms of numerical integration methods may be found in Hairer et al. (1993), Süli and Mayers (2003).

2.2 Zero-Stability in Co-Simulation

For co-simulation methods, a definition of zero-stability can be found in Busch (2012):

Definition 1 (Zero-stability of co-simulation (Busch 2012)) "A coupling approach is called *zero-stable* if the co-simulation solution converges for an infinitesimal macro step size, i.e. $H \rightarrow 0$."

Therein, the macro step size H denotes the time between two synchronization references. In general, it is equal to the largest of the step sizes of the numerical integration methods used in the considered co-simulation's subsimulations.

In this work, we will focus on zero-stability based on the considerations from Kübler and Schiehlen (2000). They analyze zero-stability of loose-coupling co-simulation, which lays the groundwork for our further investigations regarding zero-stability of hierarchical co-simulation, and thus is explained in detail in the following. The mathematical description of coupled DAEs is given as follows:

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}^i(\mathbf{x}^i, \mathbf{u}^i, t), \quad \mathbf{x}^i(t_0) = \mathbf{x}_0^i \quad (1a)$$

$$\mathbf{y}^i(t) = \mathbf{g}^i(\mathbf{x}^i, \mathbf{u}^i, t) \quad (1b)$$

with $i = I, II, \dots, N$, $\mathbf{x}^i \in \mathbb{R}^{n_x^i}$, $\mathbf{u}^i \in \mathbb{R}^{n_u^i}$, $\mathbf{y}^i \in \mathbb{R}^{n_y^i}$ and

$$\mathbf{u}^i = \mathbf{L}^i \mathbf{y} = \begin{bmatrix} \mathbf{L}^{i,I} & \dots & \mathbf{L}^{i,i-1} & 0 & \mathbf{L}^{i,i+1} & \dots & \mathbf{L}^{i,N} \end{bmatrix} \begin{bmatrix} \mathbf{y}^I \\ \vdots \\ \mathbf{y}^{i-1} \\ \mathbf{y}^i \\ \mathbf{y}^{i+1} \\ \vdots \\ \mathbf{y}^N \end{bmatrix} \quad (1c)$$

with $\mathbf{L}^{i,j} \in \{0, 1\}^{n_u^i \times n_y^j} \quad \forall i, j \in \{I, \dots, N\}$ where $N \in \mathbb{N}$ denotes the number of coupled subsystems, each represented by a corresponding DAE system. Thereby, we assume a unique solution and Lipschitz continuous right-side functions \mathbf{f}^i in the first and second argument.

Remark 1 Notation with elements of $\mathbb{G} := \{I, II, \dots\}$ is used to avoid confusion with exponents and allow easy identification of subsystems. In arithmetic operations where elements of \mathbb{G} and \mathbb{N} are mingled, these are to be understood as operations between elements of \mathbb{N} by assigning every element of \mathbb{G} its image under the bijection that uniquely assigns the i -th element of \mathbb{G} the i -th element of \mathbb{N} , i.e. we identify a roman numeral from \mathbb{G} with the corresponding element from \mathbb{N} .

Definition 2 (Zero-stability of coupled integration (Kübler and Schiehlen 2000)) The coupled integration

$$\mathbf{x}_{k+1}^i = \Phi^i(\phi^i, m^i, \tilde{\mathbf{u}}^i) \quad (2a)$$

$$\mathbf{y}_{k+1}^i = \mathbf{g}^i(\mathbf{x}_{k+1}^i, \tilde{\mathbf{u}}_{k+1}^i, t_{k+1}) \quad (2b)$$

$$\mathbf{u}_k^i = \mathbf{L}^i \mathbf{y}_k \quad (2c)$$

of N subsystems is zero-stable if the discrete coupled system

$$\mathbf{x}_{k+1}^i = \Phi^i(\phi^i(h^i \rightarrow 0), m^i) \quad (3a)$$

$$\mathbf{y}_{k+1}^i = \mathbf{g}^i(\mathbf{x}_{k+1}^i, \mathbf{u}_k^i, t_{k+1}) \quad (3b)$$

$$\mathbf{u}_k^i = \mathbf{L}^i \mathbf{y}_k, \quad i = I, \dots, N \quad (3c)$$

is stable.

Here, $k+1 \in \mathbb{N}$ denotes the currently considered step of the coupled method (from time t_k to t_{k+1}), $\tilde{\mathbf{u}}^i$ the extrapolation of unknown inputs, m_i the multirate factor (ratio of the macro step size H to the micro (i.e. the subsystem's) step size h^i , constant per subsystem), ϕ^i the integration method including extrapolation, and $\Phi = [\Phi^I \dots \Phi^N]^T$ describes the global integration step.

Kübler and Schiehlen (2000) assume that

- one-step integration methods are used
- output equations are time-invariant
- output equations are linearly dependent on inputs.

Under these assumptions the outputs can be written as follows:

$$\mathbf{y}^i = \bar{\mathbf{g}}^i(\mathbf{x}^i) + \mathbf{D}^i(\mathbf{x}^i) \mathbf{u}^i \quad (4)$$

The discretized output equations yield

$$\mathbf{y}_{k+1}^i = \bar{\mathbf{g}}^i + \mathbf{D}^i \mathbf{u}_k^i \text{ with constant } \bar{\mathbf{g}}^i, \mathbf{D}^i. \quad (5)$$

Using this, it holds for the outputs of the global system

$$\mathbf{y}_{k+1} = \bar{\mathbf{g}} + \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{D}^I \mathbf{L}^{I,II} & \dots & \mathbf{D}^I \mathbf{L}^{I,N} \\ \mathbf{D}^{II} \mathbf{L}^{II,I} & \mathbf{0} & \dots & \mathbf{D}^{II} \mathbf{L}^{II,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}^N \mathbf{L}^{N,I} & \mathbf{D}^N \mathbf{L}^{N,II} & \dots & \mathbf{0} \end{bmatrix}}_{=\mathbf{D}} \mathbf{y}_k \quad (6)$$

that stability is guaranteed if the spectral radius ρ of \mathbf{D} is less than or equal to 1. In the special case of two participating systems, $\rho(\mathbf{D}) = 0$ if $\mathbf{D}^I = \mathbf{0} \vee \mathbf{D}^{II} = \mathbf{0}$, which means no feed-through in at least one of the systems (one of the outputs is not explicitly dependent on the inputs), thus no algebraic loop occurs. This, however, is a very strict restriction (as the requirement would be $\rho(\mathbf{D}) \leq 1$ and no algebraic loop means $\rho(\mathbf{D}) = 0$), so specific investigations of the systems in consideration are preferable if enough information on the participating systems is available.

Further details on the background, notation and results of this section would exceed the scope of this paper and may be found in Kübler and Schiehlen (2000).

3 ZERO-STABILITY IN HIERARCHICAL CO-SIMULATION

In the following, we will show that zero-stability can, depending on the corresponding one-level co-simulation, only be guaranteed for hierarchical decomposition in case of not only $\rho(\mathbf{D}) \leq 1$ but also $\|\mathbf{D}\|_\infty \leq 1$. In other cases (or those where we do not presume to know the stability properties of the corresponding single-level co-simulation), zero-stability has to be investigated separately for every co-simulation layer. To this aim, we start by considering hierarchical co-simulation consisting of two co-simulation layers, utilizing the notation in Section 2.2. This is generalized on arbitrary levels of nested co-simulations by the introduction of a specifically customized notation tree that enables the unique description required in the proof of the proposed zero-stability properties.

3.1 Zero-Stability in Hierarchical Co-Simulation with Two Levels

To begin our investigations on zero-stability of hierarchical co-simulation, we take the system given in (1), which is illustrated in Figure 2 and called CS_0 from now on. Therein, we introduce a second level of co-simulation: for an arbitrary, but fixed M with $1 < M < N$, w.l.o.g. systems M, \dots, N are wrapped up in a coupled system which serves as new M^{th} system on the upper level co-simulation. This is illustrated in Figure 3.

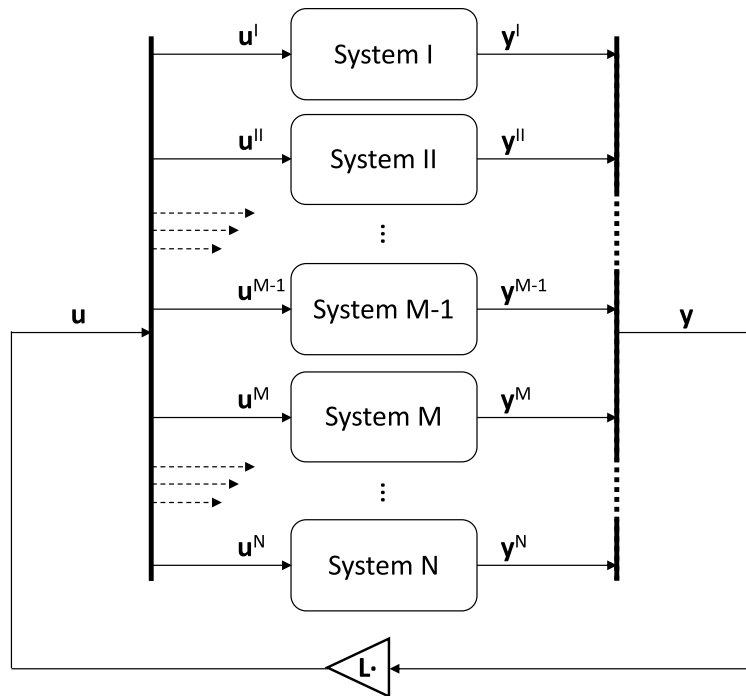


Figure 2: Illustration of the input-output relations in a traditional co-simulation approach for N coupled systems.

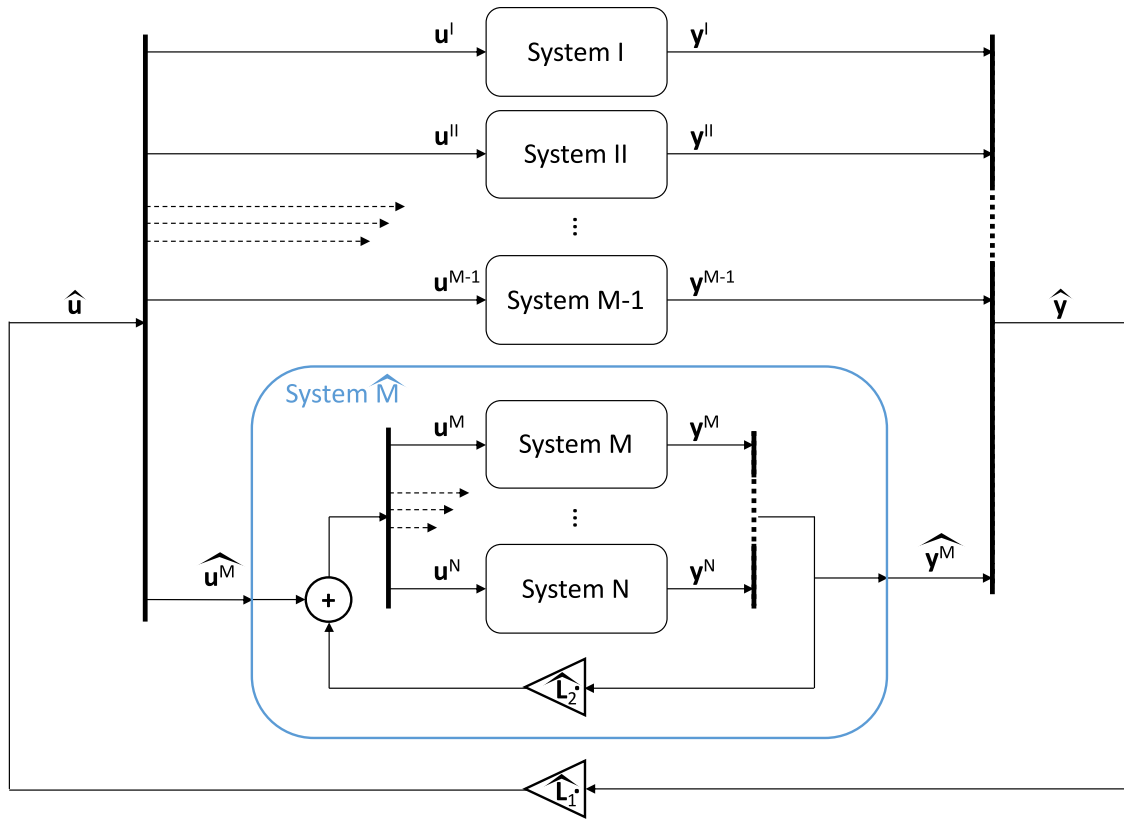


Figure 3: Illustration of the input-output relations in the hierarchical co-simulation of N systems on two levels (Hafner and Popper 2021a).

The coupling equation for the original coupled system is given (cf. (1c)) as

$$\begin{bmatrix} u^I \\ u^{II} \\ \vdots \\ u^{M-1} \\ u^M \\ \vdots \\ u^N \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & L^{I,II} & \dots & L^{I,M-1} & L^{I,M} & \dots & L^{I,N} \\ L^{II,I} & \mathbf{0} & \dots & L^{II,M-1} & L^{II,M} & \dots & L^{II,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L^{M-1,I} & L^{M-1,II} & \dots & \mathbf{0} & L^{M-1,M} & \dots & L^{M-1,N} \\ L^{M,I} & L^{M,II} & \dots & L^{M,M-1} & \mathbf{0} & \dots & L^{M,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L^{N,I} & L^{N,II} & \dots & L^{N,M-1} & L^{N,M} & \dots & \mathbf{0} \end{bmatrix}}_{=:L} \cdot \begin{bmatrix} y^I \\ y^{II} \\ \vdots \\ y^{M-1} \\ y^M \\ \vdots \\ y^N \end{bmatrix}.$$

For the hierarchical co-simulation illustrated in Figure 3, we obtain the coupling equations (7) for the upper co-simulation level (CS_1).

$$\begin{bmatrix} \mathbf{u}^I \\ \mathbf{u}^{II} \\ \vdots \\ \mathbf{u}^{M-1} \\ \widehat{\mathbf{u}}^M \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{L}^{I,II} & \dots & \mathbf{L}^{I,M-1} & \widehat{\mathbf{L}}^{I,M} \\ \mathbf{L}^{II,I} & \mathbf{0} & \dots & \mathbf{L}^{II,M-1} & \widehat{\mathbf{L}}^{II,M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{L}^{M-1,I} & \mathbf{L}^{M-1,II} & \dots & \mathbf{0} & \widehat{\mathbf{L}}^{M-1,M} \\ \widehat{\mathbf{L}}^{M,I} & \widehat{\mathbf{L}}^{M,II} & \dots & \widehat{\mathbf{L}}^{M,M-1} & \mathbf{0} \end{bmatrix}}_{=\widehat{\mathbf{L}}_1} \begin{bmatrix} \mathbf{y}^I \\ \mathbf{y}^{II} \\ \vdots \\ \mathbf{y}^{M-1} \\ \widehat{\mathbf{y}}^M \end{bmatrix} \quad (7)$$

with $\widehat{\mathbf{u}}^M$ as input to the new subsystem which replaces Systems M to N of CS_0 , $\widehat{\mathbf{y}}^M$ as its output and

$$\widehat{\mathbf{L}}^{i,M} = [\mathbf{L}^{i,M} \quad \mathbf{L}^{i,M+1} \quad \dots \quad \mathbf{L}^{i,N}], \quad i = I, \dots, M-1 \quad \text{and} \quad (8a)$$

$$\widehat{\mathbf{L}}^{M,i} = \begin{bmatrix} \mathbf{L}^{M,i} \\ \mathbf{L}^{M+1,i} \\ \vdots \\ \mathbf{L}^{N,i} \end{bmatrix}, \quad i = I, \dots, M-1. \quad (8b)$$

Given this, $\widehat{\mathbf{L}}_1$ can also be written as follows:

$$\widehat{\mathbf{L}}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{L}^{I,II} & \dots & \mathbf{L}^{I,M-1} & \mathbf{L}^{I,M} & \dots & \mathbf{L}^{I,N} \\ \mathbf{L}^{II,I} & \mathbf{0} & \dots & \mathbf{L}^{II,M-1} & \mathbf{L}^{II,M} & \dots & \mathbf{L}^{II,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}^{M-1,I} & \mathbf{L}^{M-1,II} & \dots & \mathbf{0} & \mathbf{L}^{M-1,M} & \dots & \mathbf{L}^{M-1,N} \\ \mathbf{L}^{M,I} & \mathbf{L}^{M,II} & \dots & \mathbf{L}^{M,M-1} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}^{N,I} & \mathbf{L}^{N,II} & \dots & \mathbf{L}^{N,M-1} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \quad (9)$$

Thus, the only difference between \mathbf{L} and $\widehat{\mathbf{L}}_1$ is the increased number of zero matrices in the lower right corner.

The discretized output equations of CS_1 are

$$\mathbf{y}_{k+1}^I = \bar{\mathbf{g}}^I + \mathbf{D}^I \mathbf{u}_k^I \quad (10a)$$

$$\mathbf{y}_{k+1}^{II} = \bar{\mathbf{g}}^{II} + \mathbf{D}^{II} \mathbf{u}_k^{II} \quad (10b)$$

$$\vdots \quad (10c)$$

$$\widehat{\mathbf{y}}_{k+1}^M = \widehat{\mathbf{g}}^M + \widehat{\mathbf{D}}^M \widehat{\mathbf{u}}_k^M. \quad (10d)$$

Note that while $\widehat{\mathbf{y}}^M$ in general corresponds to the stacked output vectors $\mathbf{y}^M, \dots, \mathbf{y}^N$ of CS_0 , the input vectors do not as the coupling with the outputs of systems M to N is considered within the new system \widehat{M} , cf. Figure 3 and (12).

The outputs of the global system can with (10a) be written as

$$\begin{bmatrix} \mathbf{y}_{k+1}^I \\ \vdots \\ \mathbf{y}_{k+1}^{M-1} \\ \widehat{\mathbf{y}}_{k+1}^M \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{g}}^I \\ \vdots \\ \bar{\mathbf{g}}^{M-1} \\ \widehat{\mathbf{g}}^M \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{D}^I & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & \mathbf{D}^{M-1} & \\ & & \widehat{\mathbf{D}}^M \end{bmatrix}}_{=\mathbf{D}_{CS_1}} \cdot \widehat{\mathbf{L}}_1 \cdot \begin{bmatrix} \mathbf{y}_k^I \\ \vdots \\ \mathbf{y}_k^{M-1} \\ \widehat{\mathbf{y}}_k^M \end{bmatrix}. \quad (11)$$

In analogy to the case of one co-simulation level, the co-simulation of the upper level is stable if $\rho(D_{CS_1}) \leq 1$. To find out whether this can be determined depending on the original coupled system, we have to find out the structure of D_{CS_1} . The only unknown in comparison to D of CS_0 is \widehat{D}^M , for which we have to take a look at System \widehat{M} , i.e. the second-level co-simulation CS_2 . The coupling equations within this co-simulation can be written (cf. Figure 3) as follows:

$$\begin{bmatrix} \mathbf{u}_k^M \\ \mathbf{u}_k^{M+1} \\ \vdots \\ \mathbf{u}_k^{N-1} \\ \mathbf{u}_k^N \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{L}^{M,M+1} & \dots & \mathbf{L}^{M,N-1} & \mathbf{L}^{M,N} \\ \mathbf{L}^{M+1,M} & \mathbf{0} & \dots & \mathbf{L}^{M+1,N-1} & \mathbf{L}^{M+1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{L}^{N-1,M} & \mathbf{L}^{N-1,M+1} & \dots & \mathbf{0} & \mathbf{L}^{N-1,N} \\ \mathbf{L}^{N,M} & \mathbf{L}^{N,M+1} & \dots & \mathbf{L}^{N,N-1} & \mathbf{0} \end{bmatrix}}_{=:\widehat{L}_2} \cdot \begin{bmatrix} \mathbf{y}_k^M \\ \mathbf{y}_k^{M+1} \\ \vdots \\ \mathbf{y}_k^{N-1} \\ \mathbf{y}_k^N \end{bmatrix} + \widehat{\mathbf{u}}_k^M \quad (12)$$

The discretized output equations are

$$\mathbf{y}_{k+1}^i = \bar{\mathbf{g}}^i + \mathbf{D}^i \mathbf{u}_k^i, \quad i = M, \dots, N. \quad (13)$$

Thus follows for the global output of CS_2

$$\widehat{\mathbf{y}}_{k+1}^M = \begin{bmatrix} \mathbf{y}_{k+1}^M \\ \vdots \\ \mathbf{y}_{k+1}^N \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{\mathbf{g}}^M \\ \vdots \\ \bar{\mathbf{g}}^N \end{bmatrix} + \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix} \widehat{L}_2 \begin{bmatrix} \mathbf{y}_k^M \\ \vdots \\ \mathbf{y}_k^N \end{bmatrix}}_{\widehat{\mathbf{g}}^M} + \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix} \widehat{\mathbf{u}}_k^M. \quad (14)$$

The part with \mathbf{y}_k^i , $i = M, \dots, N$ can be included in $\widehat{\mathbf{g}}^M$ as these are only internal states of CS_2 which are unknown in CS_1 . Hence (14) can be written as

$$\widehat{\mathbf{y}}_{k+1}^M = \widehat{\mathbf{g}}^M + \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix} \widehat{\mathbf{u}}_k^M \quad (15)$$

whence we obtain

$$\widehat{D}^M = \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix}, \quad (16)$$

which yields for D_{CS_1} from (11)

$$D_{CS_1} = \begin{bmatrix} \mathbf{0} & \mathbf{D}^I \mathbf{L}^{I,II} & \dots & \mathbf{D}^I \widehat{\mathbf{L}}^{I,M} \\ \mathbf{D}^{II} \mathbf{L}^{II,I} & \mathbf{0} & \dots & \mathbf{D}^{II} \widehat{\mathbf{L}}^{II,M} \\ \vdots & \ddots & \dots & \vdots \\ \mathbf{D}^{M-1} \mathbf{L}^{M-1,I} & \dots & \mathbf{0} & \mathbf{D}^{M-1} \widehat{\mathbf{L}}^{M-1,M} \\ \widehat{\mathbf{D}}^M \widehat{\mathbf{L}}^{M,I} & \dots & \widehat{\mathbf{D}}^M \widehat{\mathbf{L}}^{M,M-1} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{D}^I \mathbf{L}^{I,II} & \dots & \mathbf{D}^I \mathbf{L}^{I,M-1} & \mathbf{D}^I \mathbf{L}^{I,M} & \dots & \mathbf{D}^I \mathbf{L}^{I,N} \\ \mathbf{D}^{II} \mathbf{L}^{II,I} & \mathbf{0} & \dots & \mathbf{D}^{II} \mathbf{L}^{II,M-1} & \mathbf{D}^{II} \mathbf{L}^{II,M} & \dots & \mathbf{D}^{II} \mathbf{L}^{II,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}^{M-1} \mathbf{L}^{M-1,I} & \mathbf{D}^{M-1} \mathbf{L}^{M-1,II} & \dots & \mathbf{0} & \mathbf{D}^{M-1} \mathbf{L}^{M-1,M} & \dots & \mathbf{D}^{M-1} \mathbf{L}^{M-1,N} \\ \mathbf{D}^M \mathbf{L}^{M,I} & \mathbf{D}^M \mathbf{L}^{M,II} & \dots & \mathbf{D}^M \mathbf{L}^{M,M-1} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}^N \mathbf{L}^{N,I} & \mathbf{D}^N \mathbf{L}^{N,II} & \dots & \mathbf{D}^N \mathbf{L}^{N,M-1} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

due to

$$\widehat{\mathbf{D}}^M \widehat{\mathbf{L}}^{M,i} = \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix} \cdot \begin{bmatrix} \mathbf{L}^{M,i} \\ \mathbf{L}^{M+1,i} \\ \vdots \\ \mathbf{L}^{N,i} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^M \mathbf{L}^{M,i} \\ \mathbf{D}^M \mathbf{L}^{M+1,i} \\ \vdots \\ \mathbf{D}^N \mathbf{L}^{N,i} \end{bmatrix}, \quad i = I, \dots, M-1 \quad (17)$$

and

$$\begin{aligned} \mathbf{D}^i \widehat{\mathbf{L}}^{i,M} &= \mathbf{D}^i \cdot [\mathbf{L}^{i,M} \quad \mathbf{L}^{i,M+1} \quad \dots \quad \mathbf{L}^{i,N}] \\ &= [\mathbf{D}^i \mathbf{L}^{i,M} \quad \mathbf{D}^i \mathbf{L}^{i,M+1} \quad \dots \quad \mathbf{D}^i \mathbf{L}^{i,N}], \quad i = I, \dots, M-1. \end{aligned} \quad (18)$$

In comparison to matrix \mathbf{D} of co-simulation CS_0 , the only difference is the increased number of zero matrices in the lower right corner. In the following, we try to use this information to gain information on the properties of the spectral radius of \mathbf{D}_{CS_1} using knowledge on $\rho(\mathbf{D})$.

We know that for every matrix norm $\|\cdot\|$ and arbitrary matrix $\mathbf{A} = (a_{ij}); i = 1, \dots, m; j = 1, \dots, n; m, n \in \mathbb{N}$

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\| \quad (19)$$

holds (Horn and Johnson 2012, Thm. 5.6.9).

If we consider $\|\cdot\|_\infty$ given as

$$\|\mathbf{A}\|_\infty = \max_{i=1, \dots, m} \sum_{j=1}^n |a_{ij}| \quad (20)$$

we immediately see that $\|\mathbf{D}_{CS_1}\|_\infty \leq \|\mathbf{D}\|_\infty$. Unfortunately, this does *not* imply $\rho(\mathbf{D}_{CS_1}) \leq \rho(\mathbf{D})$, see e.g. Example 1.

Example 1 Let matrices \mathbf{A} and \mathbf{B} given as

$$\mathbf{A} = \begin{bmatrix} 0 & 0.1 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \\ 0.2 & 0 & 0 & -0.1 \\ 0 & 0.2 & -0.1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0.1 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \end{bmatrix}. \quad (21)$$

Here $\|\mathbf{A}\|_\infty = \|\mathbf{B}\|_\infty = 0.6$ but $\rho(\mathbf{A}) \approx 0.3317 < \rho(\mathbf{B}) \approx 0.3702$.

This means that in general, stability for hierarchical co-simulation has to be determined anew, even if the starting point is a stable co-simulation on one level. An exception is the case where not only $\rho(\mathbf{D}) \leq 1$ but also $\|\mathbf{D}\|_\infty \leq 1$, as from this follows further

$$\rho(\mathbf{D}_{CS_1}) \leq \|\mathbf{D}_{CS_1}\|_\infty \leq \|\mathbf{D}\|_\infty \leq 1 \quad (22)$$

which ensures zero-stability of the co-simulation on the upper level CS_1 .

For the stability properties of the coupling in CS_2 , we are interested in the input-output dependencies within the system only, thus we need to look at the spectral radius of \mathbf{D}_{CS_2} . \mathbf{D}_{CS_2} is found in (14):

$$\begin{aligned}
 \mathbf{D}_{CS_2} &= \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix} \widehat{\mathbf{L}}_2 \\
 &= \begin{bmatrix} \mathbf{D}^M & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{D}^N \end{bmatrix} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{L}^{M,M+1} & \dots & \mathbf{L}^{M,N-1} & \mathbf{L}^{M,N} \\ \mathbf{L}^{M+1,M} & \mathbf{0} & \dots & \mathbf{L}^{M+1,N-1} & \mathbf{L}^{M+1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{L}^{N-1,M} & \mathbf{L}^{N-1,M+1} & \dots & \mathbf{0} & \mathbf{L}^{N-1,N} \\ \mathbf{L}^{N,M} & \mathbf{L}^{N,M+1} & \dots & \mathbf{L}^{N,N-1} & \mathbf{0} \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{0} & \mathbf{D}^M \mathbf{L}^{M,M+1} & \dots & \mathbf{D}^M \mathbf{L}^{M,N-1} & \mathbf{D}^M \mathbf{L}^{M,N} \\ \mathbf{D}^{M+1} \mathbf{L}^{M+1,M} & \mathbf{0} & \dots & \mathbf{D}^{M+1} \mathbf{L}^{M+1,N-1} & \mathbf{D}^{M+1} \mathbf{L}^{M+1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{D}^{N-1} \mathbf{L}^{N-1,M} & \mathbf{D}^{N-1} \mathbf{L}^{N-1,M+1} & \dots & \mathbf{0} & \mathbf{D}^{N-1} \mathbf{L}^{N-1,N} \\ \mathbf{D}^N \mathbf{L}^{N,M} & \mathbf{D}^N \mathbf{L}^{N,M+1} & \dots & \mathbf{D}^N \mathbf{L}^{N,N-1} & \mathbf{0} \end{bmatrix}
 \end{aligned}$$

Since we see that \mathbf{D}_{CS_2} is a submatrix of \mathbf{D} , here again $\|\mathbf{D}_{CS_2}\|_\infty \leq \|\mathbf{D}\|_\infty$ holds, and thus $\rho(\mathbf{D}_{CS_2})$ has to be determined separately only if $\|\mathbf{D}\|_\infty > 1$.

To sum up, we can conclude that zero-stability of hierarchical co-simulation on two levels can be determined analogously to traditional single-level co-simulation. To this end, the matrices referring to the global system outputs on both co-simulation levels have to be examined – except for cases where the origin is a stable co-simulation with matrix \mathbf{D} fulfilling $\|\mathbf{D}\|_\infty \leq 1$, which is satisfied in particular for couplings where no feed-through occurs in at least one system, so $\|\mathbf{D}\|_\infty = \rho(\mathbf{D}) = 0$.

3.2 Zero-Stability in Hierarchical Co-Simulation with an Arbitrary Number of Levels

In this section, we extend these considerations to more than two levels of co-simulation:

Theorem 1 (Zero-stability of hierarchical co-simulation) Zero-stability of hierarchical co-simulation approaches can be determined by separately investigating zero-stability of the co-simulations on every level.

Proof. To begin with, we need to establish comprehensible notation of all considered systems and co-simulations. For this purpose, all participating simulations are depicted in a tree structure (introduced by the authors in Hafner and Popper (2022)), see Figure 4.

We will start from the topmost level, naming the overall co-simulation $S_{1,1}$. Beneath $S_{1,1}$, all further simulations enfold on J levels in total. On every level $j \in \{1, \dots, J+1\}$ all simulations – be they co-simulations themselves or “leaf” nodes without further branching beneath – are numbered from 1 to K_j . This means that on level j , we find simulations $S_{j,k}$ with $k = 1 \dots K_j$. While the ordering of these may be arbitrary, this notation is necessary to uniquely identify every co-simulation on every level in a fairly intelligible notation. Nevertheless, to clarify the belonging to the respective co-simulation, the sub-simulations of one node, i.e. all $N_{j,k}$ simulations coordinated by one co-simulation $S_{j,k}$ may be identified by $S_{j,k}^I, S_{j,k}^{II}, \dots, S_{j,k}^{N_{j,k}}$ in addition. This means that the i -th subsimulation of $S_{j,k}$, $i \in \{I, II, \dots, N_{j,k}\}$ may be called $S_{j,k}^i$ and equals, using the notation on the next level, $S_{j+1,l}$ for one $l \in \{1, \dots, K_{j+1}\}$:

$$S_{j,k}^i = S_{j+1,l} \quad \text{for } l = i + \sum_{m=1}^{k-1} N_{j,m} \quad (23)$$

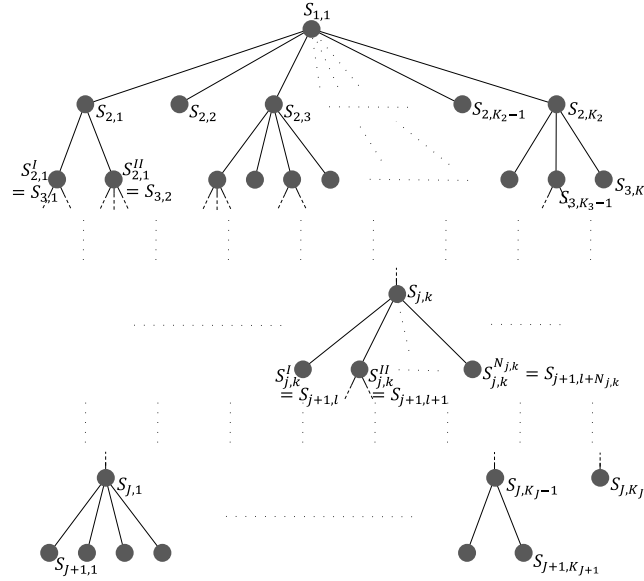


Figure 4: Illustration of the co-simulation hierarchy in a tree structure (Hafner and Popper 2022).

Note that naturally, the sum of all simulations that are co-simulated by simulations on level j equals the number of simulations on level $j + 1$:

$$\sum_{k=1}^{K_j} N_{j,k} = K_{j+1} \quad (24)$$

with the convention that for leaf nodes, $N_{j,k} := 0$. To emphasize that a regarded simulation $S_{j,k}$ is a co-simulation and not a leaf simulation, $S_{j,k}$ may also be denoted $CS_{j,k}$ (where $j \in \{1, \dots, J\}$).

With this notation in mind, we start from the top-most co-simulation $CS_{1,1}$, whose zero-stability is determined by the matrices $\mathbf{D}_{1,1}$ and $\mathbf{D}_{1,1}^i$, $i = I, II, \dots, N_{1,1}$. The zero-stability of every $CS_{1,1}^i$, on the other hand, depends – according to the investigations above – apart from $\mathbf{D}_{1,1}^i$, on $\mathbf{D}_{2,i}^r$, $r = I, II, \dots, N_{2,i}$. This can be continued to the co-simulations on the next-to-last level. Finally, zero-stability of every co-simulation $CS_{J,k}$ on level J is determined by the corresponding discretized output coefficient matrix denoted as $\mathbf{D}_{J,k}$. \square

Theorem 2 Consider a hierarchical co-simulation with, following above notation, matrix $\mathbf{D} := \mathbb{D}_{1,1}$ of the “flattened” overall co-simulation (i.e. the traditional, single-level co-simulation of all participating systems). Then

$$\|\mathbf{D}\|_{\infty} \leq 1 \Rightarrow \|\mathbf{D}_{j,k}\|_{\infty} \leq 1 \forall k \in \{1, \dots, K_j\}, j \in \{1, \dots, J\},$$

where J stands for the number of levels and K_j for the number of co-simulations per level.

Proof. Utilizing the notation introduced in the proof of Theorem 1, we climb back up the notation tree: co-simulation $CS_{J,k}$ is again the subsimulation of a co-simulation $CS_{J-1,l}$ on level $J - 1$ for one $l \in \{1, \dots, K_{J-1}\}$. Following the considerations above, we further regard the corresponding “flattened” co-simulation $\mathbb{C}S_{J-1,l}$, meaning a co-simulation in which the subsimulations of $CS_{J-1,k}^i$ would be coordinated directly, with corresponding matrix $\mathbb{D}_{J-1,l}$. Analogous to the example above follows that every $\mathbf{D}_{J-1,l}^i$ is a sub-matrix of $\mathbb{D}_{J-1,l}$, and $\mathbb{D}_{J-1,l}$ and $\mathbf{D}_{J-1,l}$ only differ by the increased number of zero matrices in $\mathbf{D}_{J-1,l}$, thus from $\|\mathbb{D}_{J-1,l}\|_{\infty} \leq 1$ follows $\|\mathbf{D}_{J-1,l}^i\|_{\infty} \leq 1$ for all $i \in \{I, II, \dots, N_{J-1,l}\}$. This can be continued for decreasing j until the topmost co-simulation. \square

Corollary 3 Given a hierarchical co-simulation whose flattened, one-level counterpart fulfills the assumption $\|\mathbf{D}\|_{\infty} \leq 1$, Theorem 1 implies zero-stability for every co-simulation on every level and therefore the hierarchical co-simulation approach in its entirety.

4 CONCLUSION AND OUTLOOK

In this paper, detailed investigations on the zero-stability of hierarchical co-simulation have been presented. Therein, an arbitrary number of systems and co-simulation levels is considered. The structure of these is uniquely denoted in a specifically designed notation tree that may prove valuable for further research in this area. Furthermore, we have been able to show that zero-stability in hierarchical co-simulation can be determined similarly to zero-stability of single-level co-simulation, albeit separately for every level of the hierarchical co-simulation. This completes the investigations on convergence of hierarchical co-simulation methods in combination with Hafner and Popper (2021a) and Hafner and Popper (2022), where consistency and numerical stability – the two other pillars of convergence of numerical integration methods – have been addressed. In the course of this, we have been able to demonstrate that accuracy and numerical stability may even be improved by the introduction of further co-simulation levels in comparison to traditional co-simulation, and no additional errors are to be expected due to the extra splitting. These investigations are essential to ensure that when the potential of hierarchical co-simulation is used, it is done prudently in order that no unanticipated – and, in the worst case, unnoticed – errors are introduced by the application of this approach. Nevertheless, the area of hierarchical co-simulation still leaves possibilities for further refinement, such as the utilization of sequential, iterative or adaptive orchestration algorithms, different extrapolation orders and higher order and/or multi-step subsystem solvers, which remain a topic for future investigations.

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