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GENERALIZING THE GENERALIZED LIKELIHOOD RATIO METHOD THROUGH A PUSH-OUT LEIBNIZ INTEGRATION APPROACH

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ABSTRACT

We extend the generalized likelihood ratio (GLR) method using a novel push-out Leibniz integration approach. Extending the conventional push-out likelihood ratio (LR) method, our approach allows parameter-dependent sample spaces after the change of variables, introducing a surface integral in addition to the standard LR estimator, which may require extra simulation. Our approach also applies to cases with only "local" change of variables, includes existing GLR estimators as special cases, and covers a broader range of discontinuous sample performances.

1 INTRODUCTION

Consider an output sample performance $\varphi(g(X,\theta))$, where $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ is a bounded measurable function. The function $g(\cdot,\cdot) : \mathbb{R}^n \times \Theta \mapsto \mathbb{R}^n$ is invertible with respect to x and twice continuously differentiable in both arguments. Here, $\Theta \subset \mathbb{R}$ is a bounded open interval, and X is an n-dimensional random vector with bounded support $\Omega \subset \mathbb{R}^n$, having a density function $f(\cdot,\cdot) : \Omega \times \Theta \mapsto \mathbb{R}$ that is continuously differentiable in both arguments.

We aim to estimate the derivative of $\mathbb{E}(\varphi(g(X,\theta)))$ with respect to θ. Since φ is not assumed continuous, infinitesimal perturbation analysis (IPA), which requires continuity of the sample path, is not applicable. If a change of variables $Y = g(X,\theta)$ is feasible, we can push the parameter θ from $\varphi(g(X,\theta))$ into the density f, avoiding differentiation of a discontinuous function (Rubinstein 1992). The likelihood ratio (LR) method can then be applied, although it typically requires the support of Y to be independent of θ. In this study, we extend the push-out LR method using the Leibniz integral rule, allowing it to handle cases where the support of Y depends on θ. The new estimator also applies to situations where the change of variables only "locally" exists, and includes existing GLR estimators (Peng et al. 2018) as special cases. The same estimator is proposed under different regularity conditions by Puchhammer and L'Ecuyer (2022), which focuses on density estimation. We demonstrate its effectiveness with a simple simulation example.

2 METHODOLOGY

Step 1: Change of variables. Making the change of variables $y = g(x,\theta)$, we can write

$$\mathbb{E}(\varphi(g(X,\theta))) = \int_{g(\Omega,\theta)} \varphi(y) f(g^{-1}(y,\theta),\theta) |\det(J_{g^{-1}}(y,\theta))| dy, \tag{1}$$

where $g(\Omega,\theta)$ is the image of Ω under map g, and $J_{g^{-1}}(y,\theta)$ is the Jacobian matrix of g^{-1} w.r.t. y.
Step 2: Function approximation. The Leibniz integral rule requires the integrand to be differentiable (Flanders 1973). Since φ is bounded, we can approximate φ with a sequence of smooth functions $\{\varphi_n\}$ that converges to φ in L^1 as $n \to \infty$ (Peng et al. 2018).
Step 3: Differentiation by the Leibniz integral rule. Let $\vec{v}(y) := \partial_\theta g(x,\theta)|_{x=g^{-1}(y,\theta)}$. Substituting φ_n in (1) and applying the Leibniz integral rule (Flanders 1973), we can express $\frac{d}{d\theta}\mathbb{E}(\varphi_n(g(X,\theta)))$ as:

$$\int_{g(\Omega,\theta)} \varphi_n(y) \frac{d}{d\theta}(f(g^{-1}(y,\theta),\theta)|\det(J_{g^{-1}}(y,\theta))|) + \operatorname{div}(\varphi_n(y) f(g^{-1}(y,\theta),\theta)|\det(J_{g^{-1}}(y,\theta))|\vec{v}(y)) dy.$$

Step 4: Reverse the change of variables. To extend our method when g is only locally invertible (i.e., when only J_g is invertible), we reverse the change of variables by substituting $x = g^{-1}(y,\theta)$:

$$\frac{d}{d\theta}\mathbb{E}(\varphi_n(g(X,\theta))) = \int_\Omega \varphi_n(g(x,\theta))(d(x,\theta)+l(x,\theta))f(x,\theta)dx + \int_{\partial\Omega} \varphi_n(g(x,\theta))s(x,\theta)^T\vec{n}(x)f(x,\theta)ds. \quad (2)$$

where $d(x,\theta) = \mathrm{div}(-f(x,\theta)s(x,\theta))/f(x,\theta)$, $l(x,\theta) = \partial_\theta \log f(x,\theta)$, $s(x,\theta) = J_g^{-1}(x,\theta)\partial_\theta g(x,\theta)$, and $\vec{n}(x)$ is the normal vector to the surface $\partial\Omega$.

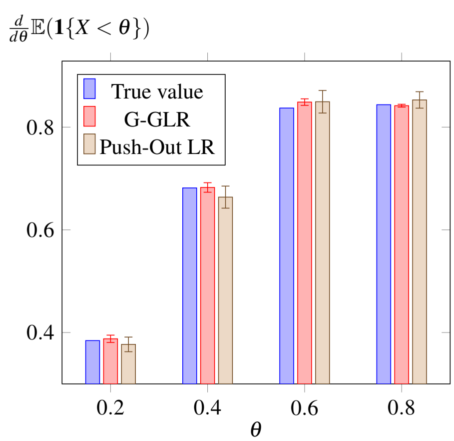
Step 5: Take the limit $n \to \infty$. Under suitable conditions, (2) holds with φ_n replaced by φ. The first term on the right side has an unbiased estimator $\varphi(g(X,\theta))(d(X,\theta)+l(X,\theta))$.

Step 6: Compute the surface integral for hyperrectangle support. Let $f_{X|X_i}$ be the conditional density of X given X_i. If $\Omega = [a_1,b_1]\times\cdots\times[a_n,b_n]$, an unbiased estimator for the surface integral term is:

$$\sum_{i=1}^{n}(\varphi(g(X,\theta))f_{X_i}(b_i)s(X,\theta)^T e_i\big|_{X\sim f_{X|X_i=b_i}} - \varphi(g(X,\theta))f_{X_i}(a_i)s(X,\theta)^T e_i\big|_{X\sim f_{X|X_i=a_i}}).$$

3 SIMULATION EXAMPLE

We compare the proposed generalized GLR (G-GLR) method with the push-out LR method using a toy example: $\frac{d}{d\theta}\mathbb{E}(\mathbf{1}\{X<\theta\})$, where X follows an exponential distribution with parameter $\theta > 0$. The G-GLR method has lower variance but may require extra simulation for multidimensional inputs.



θ	True value	G-GLR	Push-Out LR
0.2	0.384	0.388 ± 0.018	0.377 ± 0.038
0.4	0.682	0.683 ± 0.014	0.664 ± 0.032
0.6	0.837	0.849 ± 0.008	0.850 ± 0.026
0.8	0.844	0.842 ± 0.003	0.853 ± 0.019

Figure 1: Simulation results: Point estimates and standard errors for $\frac{d}{d\theta}\mathbb{E}(\mathbf{1}\{X<\theta\})$.

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