STOCHASTIC SENSITIVITY ANALYSIS OF
LINEAR PROGRAMS BY SIMULATION
TECHNIQUES

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Introduction

Sensitivity analysis is the investigation of the stability of the optimal solution to a linear programming problem.

$$\begin{aligned} \max\ c'x &= z \\ \text{subj. to } Ax &= b \\ x &\geq 0 \end{aligned} \tag{1}$$

Suppose that A is an mxn matrix with rank equal to m and that an mxm submatrix, B, is the optimal basis. Let further S_B be the index set of the column vectors forming B. It is well known that the necessary and sufficient conditions for B to be optimal are the conditions of primal and dual feasibility, i.e.

$$\text{I.}\quad B^{-1}b \geq 0$$

$$\text{II.}\quad c_B'B^{-1}a_j \geq c_j,\ (j = 1, 2, \ldots, n) \tag{2a}$$

where c_B' is the row vector of objective function coefficients for $j \in S_B$ and a_j is the j-th column of A. As long as these conditions are maintained the basis is stable--that is, insensitive to variations in the coefficients. The ranges within which these variations may occur can be computed--ceteris paribus--for any particular component of c', b and a_j by well known formulae.[1,2]

Suppose now that a linear programming problem is to be solved where some or all of the components in the vectors c', b, and a_j are subject to random variations. This is the case of stochastic programming problems. Here the index set S_B of the optimal basis will likely depend upon the particular realization of a set of random variables and consequently it will not remain the same all the time. One can redefine the stability of the basis for this case, however, in terms of the probability that S_B remains unchanged. More precisely, if the optimal basis is unique or nondegenerate, the stability of a basis B can be measured by the probability:

$$\Pr(B^{-1}b \geq 0;\ c_B'BA - c' \geq 0) \tag{2b}$$

If this probability is equal to one, the optimal basis is considered stable, although the optimum of the objective function is still a random variable under the given assumptions.

Programming under the presence of random influence in the parameters is considered through different principles in the literature. Typical is the principle to minimize or maximize an expected value under certain conditions involving random variables and probabilities.[1,3,4] Though these methods solving the given problem are of high theoretical and practical value, it should be emphasized that their application is limited to such cases where the problem is repeated a great many times under similar circumstances because of the practical meaning of the expected value.

Methods have been developed to derive the probability distribution and expected value of the optimum of a stochastic linear programming problem under the assumption that the random variation in the parameters keeps a certain basis optimal with a high probability. There enters the problem of determination of the stability of a basis. Apparently, without a highly stable basis, the solution of the linear programming problem carried out with the expectations of the parameters may have questionable value for certain decisions.

The problems are considered under special assumptions, namely, in the cases where 1) b is a random vector, 2) c is a random vector, and 3) any one of the a_j's is a random vector. By duality, the problem of one random row in the linear programming problem goes back to the third case. Even in these relatively simple cases the numerical part of the problem implies serious difficulties, such as numerical integration of a function over a convex polyhedron. However, simulation techniques can provide a solution in many practical problems. With our particular assumptions, certain simplifications can be attained in simulating the random vectors and estimating the stability of any given basis very efficiently, as will be pointed out. Having an estimated value for the stability, the method suggested in [5] or [6] can be applied for the solution of the distribution problem, or the limitations of programming with expected values can be spelled out.

The approach outlined here also can be viewed as the extensions of the deterministic form of sensitivity analysis of linear programs to a stochastic form of sensitivity analysis. After the linear programming problem is solved, random vectors, exhibiting the supposed random variations in the coefficients are simulated and the effect of these variations on the optimum is evaluated without resolving the original problems. A major part of the paper deals with the techniques of performing these evaluations computationally most efficiently via simplified checks on transformed random vectors. The method provides complete sensitivity analysis if the random coefficients occur only in c' or b. If the elements of A are also random, the analysis can be applied only to selected random columns of A, one at a time, in the tradition of deterministic post-optimality methods.

Optimality Conditions with Transformed Random Vectors**

For the sake of simplicity we shall assume that every subset of the vectors in (1) is a system of linearly independent vectors. A subset of m vectors forms a matrix B which is a feasible basis if

$$B^{-1}b = x_B \geq 0 \tag{3}$$

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**This section utilizes some ideas laid out by A. Prokopa in an unpublished manuscript in 1965.

The complete set of vectors in (1), with $j = 1, \ldots, n$, can thus be represented in partitioned form: $A = [B:R]$.

According to the well known optimality condition of linear programs, a basis is optimal if (3) holds simultaneously with the following inequalities:

$$c_B'B^{-1}a_j \geq c_j, \text{ for all } j \notin S_B \tag{4}$$

where c_B' is an m-vector of objective function coefficients c_j; $j \in S_B$. In matrix form this expression corresponds to the vector inequality;

$$c_B'B^{-1}R \geq c_R' \tag{5}$$

where c_R' is a vector of c_j coefficients such that $j \notin S_B$.

We shall now consider random disturbances in b, c' and a_j by assuming that random vectors δ_b, δ_c and δ_j; with $E(\delta_b) = E(\delta_c) = E(\delta_j) = 0$ expectations and known covariance matrices $E(\delta_b\delta_b') = \Delta_b$, $E(\delta_c\delta_c') = \Delta_c$ and $E(\delta_j\delta_j') = \Delta_j$ are added to the respective parameter vectors under study. The subscripted matrix notations Δ_b, Δ_c, and Δ_j will be used to indicate the covariance matrices of the random vectors $(b+\delta_b)$, $(c+\delta_c)$ and $(a_j+\delta_j)$ thus formed, but notational distinction among the corresponding δ vectors will be applied only if necessary. The notation B_r will be used if $j \in S_B$ and $(a_j+\delta_j)$ is the rth column of the basis. If $c' = [c_B':c_R']$ then $\delta_c' = [\delta_B':\delta_R']$.

The stability of B will be investigated in terms of the following four probability measures:

a. $\Pr\ [B^{-1}(b+\delta_b) \geq 0]$

b. $\Pr\ [(c_B+\delta_B)'B^{-1}R \geq (c_R+\delta_R)']$

c. $\Pr\ [B_R^{-1}b \geq 0;\ c_B'B_r^{-1}R \geq c_R']$

d. $\Pr\ [c_B'B^{-1}(a_j+\delta_j) \geq c_j],\ j \notin S_B$

In each case the assumption will be made that the covariance matrix Δ sufficiently determines the random behavior of δ and it is known. It is clear from the above listed inequalities that the condition of basis stability depends on certain transformed values of the random vector δ. Consequently the purpose of the following sections is to find the appropriate transformations which will produce a new random vector η with $E(\eta) = 0$ and with covariance matrix Σ such that η will exhibit equivalent properties to those listed under a., b., c. and d. in satisfying the inequalities.

a. Condition a. is the condition of primal feasibility, and it can be rewritten as

$$\Pr\ [x_B + B^{-1}\delta_b \geq 0] \tag{6}$$

Let $\eta_b = B^{-1}\delta_b$. Then $E(\eta_b) = 0$ and

$$\Sigma_b = E(\eta_b \cdot \eta_b') = E(B^{-1}\delta_b\delta_b'B'^{-1}) = \\ = B^{-1}\Delta_b B'^{-1} \tag{7}$$

Consequently, the vector inequality in the bracket:

$$\Pr\ [x_B + \eta_b \geq 0] \tag{8}$$

will reproduce condition a. if the covariance matrix of η_b is Σ_b.

b. This is the condition of dual feasibility to random variation in the objective coefficient vector c'. The random vector $(c+\delta_c)'$ can be partitioned into basic and nonbasic parts in this case, i.e.

$$(c+\delta_c)' = [c_B':c_R'] + [\delta_B':\delta_R'] \tag{9}$$

Consequently, we will define $\Delta_c = E[(c+\delta_c)(c+\delta_c)']$ in partitioned form as

$$\Delta_c = \begin{bmatrix} \Delta_{BB} & \vdots & \Delta_{BR} \\ \cdots & \cdots & \cdots \\ \Delta_{RB} & \vdots & \Delta_{RR} \end{bmatrix} \tag{10}$$

Introducing $Q = B^{-1}R$ we may rewrite condition b. as

$$\Pr\ \{[c_B'Q - c_R'] + [\delta_B'Q - \delta_R'] \geq 0\} \tag{11}$$

and introduce the random vector

$$\eta_c' = [\delta_B'Q - \delta_R'] \tag{12}$$

which has zero expectation and covariance matrix:

$$\Sigma_c = E\{[Q'\delta_B - \delta_R][\delta_B'Q - \delta_R']\} \\ = Q'\Delta_{BB}Q - Q'\Delta_{BR} - \Delta_{RB}Q + \Delta_{RR} \tag{13}$$

Consequently, the expression

$$\Pr\ \{[c_B'Q - c_R'] + \eta_c' \geq 0\} \tag{14}$$

will satisfy the inequality condition in b., since $E(\eta_c) = 0$ with $E(\eta_c \cdot \eta_c') = \Sigma_c$

c. The inequality condition pertinent to this and the following case can be investigated through the concept of a basis change whereby the rth column of B, say b_r, is replaced by the random vector $(b_r+\delta_r)$. If we denote the rth row of B^{-1} by β_r', according to the well known transformation formula we obtain

$$B_r^{-1} = B^{-1} - \frac{B^{-1}\delta_r\beta_r'}{1 + \beta_r'\delta_r} \tag{15}$$

It is evident that a random change from B to B_r may effect both the primal and dual feasibility.

c.1. First we shall reformulate the first part of condition c. with the help of formula (15), and obtain the inequality

$$B^{-1}b - \frac{B^{-1}\delta_r\beta_r'b}{1 + \beta_r'\delta_r} \geq 0 \tag{16}$$

For feasible solutions we must have $x_B \geq 0$ and we shall start with the assumption that $x_r > 0$ where x_r is the rth component of x_B. From (16) it follows that

$$x_i - \frac{\beta_i'\delta_r}{1 + \beta_r'\delta_r} x_r \geq 0,\ i = 1,2,\ldots,m \tag{17}$$

must be satisfied and if $i \neq r$ and $x_r > 0$ (for the time being)

$$1/(1 + \beta_r'\delta_r) \geq 0\ , \tag{18}$$

that is

$$-\beta_r'\delta_r \leq 1 \tag{19}$$

Therefore, (17) and (19) are equivalent to the following expressions

(i) $\quad x_i + [\beta_r'x_i - \beta_i'x_r]\delta_r \geq 0$, for $i \neq r$

(ii) $1 + \beta_r'\delta_r \geq 0$, for $i = r$ (20)

Two new notations appear to simplify the computational aspects of expression (20). Let us define a matrix B_o^{-1} and a vector x_o as modification of B^{-1} and x_B such that the rth row i.e. β_r' and x_r are changed into

(i) $\beta_r' = [0,0,\ldots,0]$

(ii) $x_r = +1$ (21)

while all the other elements remain unchanged. With this substitution in (20) the feasibility condition subject to random variations in the rth column of the basis can be summarized as follows:

$$x_o + [x_o\beta_r' - x_r B_o^{-1}]\delta_r \geq 0 \tag{22}$$

The expression in the bracket is an (mxm) matrix of known constants which we shall denote by P and which transforms the random vector δ_r into another vector η_P, i.e.

$$\eta_P = P\delta_r \tag{23}$$

where $E(\eta_P) = 0$ and $E(\eta_P \cdot \eta_P') = \Sigma_P = P\Delta_r P'$. With this result expression (16) is simplified into the following equivalent form:

$$x_o + \eta_P \geq 0 \tag{24}$$

We must keep in mind here that (24) is only the condition of primal feasibility and for the case under study the optimality condition implies that both this and condition c.2. are simultaneously satisfied.

c.2. The condition of dual feasibility with the random vector b_r in the basis can be given as

$$c_B'B^{-1}a_j - \frac{c_B'B^{-1}\delta_r\beta_r'a_j}{1 + \beta_r'\delta_r} \geq c_j \;,\; j \notin S_B \tag{25}$$

We shall introduce the notations $w' = c_B'B^{-1}$ and $z_j = w'a_j$ and conclude that

$$(z_j - c_j) - \frac{\beta_r'a_j w'\delta_r}{1 + \beta_r'\delta_r} \geq 0 \tag{26}$$

is an equivalent inequality which can be transformed by utilizing (18) in the following forms

(i) $1 + [B_r' - \frac{\beta_r'a_j}{(z_j - c_j)} w']\delta_r \geq 0$;

$j \notin S_B$, $(z_j - c_j) > 0$

(ii) $-\beta_r'a_j w'\delta_r \geq 0$; $j \notin S_B$, $(z_j - c_j) = 0$ (27)

Consider now vectors u and v such that

(i) $u_j = 1$, $v_j = \beta_r'a_j/(z_j - c_j)$

for all $j \notin S_B$ if $(z_j - c_j) > 0$

(ii) $u_j = 0$, $v_j = \beta_r'a_j$

for all $j \notin S_B$ if $(z_j - c_j) = 0$

With these notations (27) can be written as

$$u + [u\beta_r' - vw']\delta_r \geq 0 \tag{28}$$

Denoting the (n-m) x m matrix in the bracket by T, we may define the random vector η_T as

$$\eta_T = T\delta_r$$

with $E(\eta_T) = 0$ and $E(\eta_T \cdot \eta_T') = \Sigma_T = T\Delta_r \cdot T'$. Consequently, the expression

$$u + \eta_T \geq 0 \tag{29}$$

stating the condition of dual feasibility is equivalent to (25). Naturally, both sets of inequalities--the ones in (24) and in (29)--must be satisfied simultaneously at optimality. Hence condition c. is equivalent to the following transformed expression

$$\Pr\ (x_o + \eta_P \geq 0 \;;\; u + \eta_T \geq 0) \tag{30}$$

The connection between the two formulae becomes more specific if we allow for degenerate solution and will permit $x_r = 0$ with condition c.1. In this case the sign of $1/(1 + \beta_r'\delta_r$ does not effect primal feasibility but it enters in expression (26) and will require the converse inequality to hold. This implies that whenever $x_r = 0$ and the rth inequality in (22) is violated, the converse of inequality (28) should be satisfied in order to maintain optimality.

d. The effect of introducing a random vector $(a_j + \delta_j)$ in place of a_j for $j \notin S_B$, can be investigated by the following inequality condition:

$$(z_j - c_j) + w'\delta_j \geq 0 \tag{31}$$

Here δ_j is a random vector with 0 expectation and Δ_j covariance matrix; however, $w'\delta_j$ is no longer a random vector but only a random variable denoted by η_j with $E(\eta_j) = 0$ and variance $\sigma_j^2 = w'\Delta_j w$. Consequently, for any nonbasic vector $(a_j + \delta_j)$ the following inequality preserves the basis:

$$(z_j - c_j) + \eta_j \geq 0 \tag{32}$$

The above transformations not only simplify the form of inequality relative to the random vectors but also modify the dimensions of the η vectors to our advantage as compared with the dimensions of the random parameter vectors in the cases of b. and d. Assuming that A is an (mxn) matrix with m < n, in the first case the n dimensional $(c + \delta_c)$ vector is replaced by the (n-m) dimensional η_c vector, while in the second case the m dimensional $(a_j + \delta_j)$ vector is replaced by a single random variable.

Monte Carlo Estimation of Basis Stability

The techniques of estimating the probability that a given set of inequalities (as listed under conditions a., b., c. and d. in the previous section) will be satisfied with a given basis is essentially the sampling techniques of estimating the probability of binary events. Each independent realization of a random parameter vector can be regarded as a sample with two possible outcomes: the basis either remains optimal or not. The previous section has shown that this question can be answered in terms of checking the pertinent inequalities directly against certain transformed random vectors. Hence the problem of estimation is reduced to two technical questions: (1) How many samples are needed and (2) How to generate random vectors. Both questions are covered in various places of the literature of simulation techniques; therefore, only a short summary of the necessary steps will be given here.

(1) There is a close relation between the sample size, n, and the precision of the estimated measure of

probability for binary events. If y is the number of events when optimality was maintained out of n samples generated, the estimated measure of stability is

$$\hat{p} = y/n \tag{33}$$

with the standard deviation

$$\sigma_{\hat{p}} = (p(1-p)/n)^{1/2} \tag{34}$$

If the relative precision is defined as

$$\alpha = (y-np)/np \tag{35}$$

and it is to be attained with P_k probability within the $\pm k\sigma_{\hat{p}}$ confidence interval, the sample size which meets the specifications is

$$n = (k/\alpha)^2(1-p)/p \tag{36}$$

The usual values of P_k and k are .95 and 1.96 respectively while α may be selected to be equal to .01 or smaller especially if high value of $\hat{p}$ is expected. The given figures already correspond to a sample size of several hundred and as a rule each additional decimal point in precision will require a hundredfold increase in sample size. This property emphasizes the necessity for simplifying the computations of generating random vectors as much as possible.

(2) It is observable in the derivation of the η vectors in the previous section that even in cases where Δ is diagonal, Σ is going to contain off-diagonal, i.e. covariance terms. For this reason and in order to cover the more general--and realistic--cases when the parameter variations are already correlated we shall assume that the η vectors are random vectors from a multivariate normal population. The generation of such vectors by computer involves three essential steps. First, the generation of pseudo-random numbers on the (0,1) interval. This procedure is well known and documented in the literature in several versions out of which a combination of the multiplicative and mixed congruential methods can be recommended.[7] Second, the generation of independent normal variables (with zero mean, unit variance) which can be performed again in several known ways according to the literature.[7,8] Certain subsets of these independent normal variables may form conceptually random vectors, denoted here by ε with zero expectation and variance-covariance matrix, $E(\varepsilon\cdot\varepsilon') = I$.

The third step is to transform the generated ε vectors into η vectors. This is done with the help of so-called "square root" method.[9] We shall assume that

$$\eta = C\varepsilon \tag{37}$$

and therefore we require that

$$\Sigma = CC' \tag{38}$$

If C is chosen to be a lower triangular matrix, it can be uniquely determined from Σ. Let σ_{ij} denote the general element of Σ, then we have the following recursive formuli:

$$c_{i1} = \sigma_{i1}/(\sigma_{11})^{1/2} \qquad 1 \le i \le n$$

$$c_{i1} = (\sigma_{ii} - \Sigma_{k=1}^{i-k} c_{ik}^2)^{1/2} \qquad 1 < i \le n$$

$$c_{ij} = (\sigma_{ij} - \Sigma_{k-1}^{j-1} c_{ik}\cdot c_{jk})/c_{jj} \qquad 1 < j < i \le n$$

$$c_{jj} = 0 \qquad 1 < j \le n \tag{39}$$

Once the elements of C are determined--and this is necessary only once per simulation run--the components η_i are generated from ε_i as

$$\eta_i = \Sigma_{j=1}^{i} c_{ij}\varepsilon_j \qquad i = 1, \ldots, r \tag{40}$$

where r is the dimension of the random vector η. This computation must be repeated n times to arrive at the desired probability estimate of basis stability.

Conclusions

The techniques described in the previous sections represent the only computational method available for the estimation of basis stability for stochastic linear programs to date. The method utilizes the concept of random vector transformations in order to obtain maximum efficiency in the computer program and consequently, it is limited to the simulation of multivariate normal vectors. It is assumed, in other words, that the covariance matrix--or at least the main diagonal variances--of the random coefficient vectors are known.

One very likely use of this technique is a testing process whereby the stability of a given solution is estimated in order to see whether confidence intervals can or cannot be calculated according to [5] or [6]. Another utilization of the discussed methods may give rise to the introduction of routine stochastic sensitivity analysis software as an optional feature of regular linear programming codes.

References

(1) Dantzig, G. B. Linear Programming and Extensions. Princeton Press, 1966.

(2) Barnett, S. "Stability of the Solution to a Linear Programming Problem." Operational Research Quarterly, Vol. 13, No. 3, pp. 219-228, 1962.

(3) Tintner, G. "A Note on Stochastic Linear Programming." Econometrica, Vol. 28, pp. 490-495, 1960.

(4) Charnes, A. and Cooper, W. W. "Chance-Constrained Programming." Management Science, Vol. 6, pp. 73-79, 1959.

(5) Prokopa, A. "On the Probability Distribution of the Optimum of a Random Linear Program." J. SIAM Control, Vol. 4, pp. 211-222, 1966.

(6) Bracken, J. and Soland, R. M. "Statistical Decision Analysis of Stochastic Linear Programming Problems." Naval Research Logistic Quarterly, Vol. 13, pp. 205-226, 1966.

(7) Naylor, T. H., Balintfy, J. L., Burdick, D. and Chu, K. Computer Simulation Techniques. John Wiley & Sons, 1966.

(8) Muller, Mervin. "A Comparison of Methods for Generating Normal Deviates on a Computer." Journal of A.C.M., Vol. 6, pp. 376-383, 1959.

(9) Scheuer, M. E. and Stoller, D. C. "On the Generation of Normal Random Vectors." Technometrics, Vol. 4, pp. 278-281, 1962.