APPLICATION OF MONTE CARLO SIMULATION
TO A CIRCLE PACKING PROBLEM

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OBJECTIVE

Frequently, there is a requirement to pack cylindrical objects of varying radii in a larger cylindrical container. Several methods of packing the objects have been investigated in [1] and [2]. This paper describes a method to pack the circles using Monte Carlo simulaiton and then analyzes the resulting data to determine the relationship among several selected variables.

A basic assumption is that the cylindrical objects are of equal length. Under this assumption, the problem reduces to an investigation of the corss section of the cylindrical container, or equivalently to an investigation of small circles packed within a larger circle. We specifically investigate how packing density (ratio of the total area of the inner circles to the area of the outer circle) varies as the number and radii of the inner circles vary.

METHOD

As the initial step in the Monte Carlo process, ten pairs of uniformly distributed random numbers between 0 and 1 were generated. The pairs of random numbers were converted to polar coordinates with the center of the outer circle acting as the center of the coordinate system. The first number of a pair, when multiplied by 100 units (i.e., the radius of the outer circle), became the magnitude; the second number, when multiplied by 2π became the angle. The polar coordinates, themselves, designated points within the outer circle which would be centers for ten inner circles. Once the centers were established, the polar coordinates were converted to rectangular coordinates to facilitate the checking of constraints.

The first constraint is that the inner circles cannot overlap the outer circle. This constraint is satisfied if

$$r_i \le a_i = 100 - \sqrt{x_i^2 + y_i^2} \quad \text{for } i = 1, 2, \ldots, 10,$$

where r_i is the radius of the i^{th} circle, x_i and y_i are the rectangular coordinates of the center of the i^{th} circle, and 100 is the radius of the outer circle.

The second constraint is the requirement that the inner circles do not overlap each other. This constraint is satisfied if r_i

$$r_i + r_j \le b_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

for all $i \ne j$.

In other words, the sum of the radii of any two inner circles must be less than or equal to the distance between their centers.

In order to preclude the clustering of circles in one sector of the outer circle, a restriction was placed on the acceptable values of a_i and b_{ij}. If for any i and j the constraints a_i or b_{ij} assumed values less than 25 units, the corresponding centers were rejected and new centers were generated. If necessary, the rejection and generation process was repeated five times. If ten points could not be found to conform to the clustering restriction, the minimum value of 25 units was cut in half and the process was repeated. If necessary, the routine would have helped the minimum value again; but in practice only one reduction in the constraint was necessary.

Once ten acceptable centers were established and the constraints on the radii were derived, linear programming was utilized to determine the radii. The linear program may be stated as:

$$\text{maximize} \sum_{i=1}^{10} r_i$$

$$\text{subject to} \quad r_i \le a_i$$

$$\text{and} \quad r_i + r_j \le b_{ij} \text{ for all } i \ne j.$$

Once the radii were determined for the first set of ten centers, the procedure was repeated for a total of 20 times. Then the number of inner circles was increased to 15 and the procedure was repeated 20 times. The same was performed for 20, 25, and 30 inner circles although for 30 circles only 5 sets of centers were generated because of the excessive amount of time required to find 30 centers which met the constraints. The results for the 85 solution sets are listed under Figure 1.

ANALYSIS OF DATA

The primary tool in the analysis was the Statistical Analysis System (SAS), developed by the Department of Statistics, North Carolina State University, and described in [3]. As input to the regression procedure of SAS we chose as independent variables the number of inner circles, the mean radius of the circles, and the variance of the radii. The packing density was chosen to be the dependent variable. For the 85 sets of data the regression procedure produced the following expression:

$$D = .0938 - .0172N - .0178R + .0008R^2 + .0026N*R + .0100N*VAR$$

where D = predicted packing density
N = number of inner circles
R = mean radius
VAR = variance of the radii

In order to determine which variables had the greatest effect on packing density, the stepwise procedure of SAS was used. This procedure finds the first single-variable model which produces the largest R-square (square of the multiple correlation coefficient) statistic. In our case the product N*R produced the largest value of .1767. By adding the variable N to the model, the R-square statistic rose to .5423. With the addition of the product N*VAR and the variable R^2, the R-square value approached .9952. The effect of the variable R was barely perceptible.

In order to ascertain how packing density varies with the number of circles, we evaluated the partial derivative of D with respect to N, or

$$\frac{\partial D}{\partial N} = -.0172 + .0026R + .0100VAR$$

Since the minimum value of R is 11.39 and the minimum value of VAR is 5.48, the minimum value of the derivative is .067. Since this derivative assumes positive values over the domain of the variables, we conclude that the packing density is an increasing function with respect to the number of circles.

Differentating the same expression with respect to R, we get

$$\frac{\partial D}{\partial R} = -.0178 + .0016R + .0026N$$

Since the minimum value for N is 10, the minimum value for this derivative is .0026. Consequently, the packing density is also an increasing function with respect to the mean radius.

By observation one can see that the packing density increases as the variance of the radii increases.

CONCLUSIONS

Randomly generating the centers of the inner circles and maximizing the sum of their radii yield results which are statistically significant. In particular, one discovers that packing density is affected the most by the product N*R, followed by the variable N, the product N*R, followed by the variable N, the product N*VAR, and the variable R^2. Furthermore, the packing density is an increasing function of the variables N, R, and VAR when each is treated separately.

FIGURE 1

OBS NUMBER	N	($x10^{-2}$) AVG (R)	($x10^{-2}$) VAR
1	10	2204	9753
2	10	2272	2091
3	10	1975	2912
4	10	2275	3672
5	10	2119	16854
6	10	2054	3228
7	10	2058	9622
8	10	2113	8126
9	10	2262	11075
10	10	2075	7164
11	10	2223	2292
12	10	2164	6557
13	10	2054	4277
14	10	1910	16860
15	10	2277	1667
16	10	2107	5117
17	10	2086	7650
18	10	2277	548
19	10	2183	9831
20	10	1893	7072
21	15	1592	7320
22	15	1698	7956
23	15	1661	8933
24	15	1460	9109
25	15	1589	7430
26	15	1607	7257
27	15	1781	6729
28	15	1719	6955
29	15	1775	4353
30	15	1585	9385
31	15	1670	5704
32	15	1688	6521
33	15	1634	9000
34	15	1736	6122
35	15	1660	5866
36	15	1666	6037
37	15	1864	6165
38	15	1547	11383
39	15	1745	7672
40	15	1701	6676
41	20	1442	4341
42	20	1566	3439
43	20	1519	5380
44	20	1522	5831
45	20	1557	3490
46	20	1550	4960
47	20	1565	3206
48	20	1577	6500
49	20	1535	3409
50	20	1386	4795
51	20	1582	5831
52	20	1472	4141
53	20	1533	3261
54	20	1501	4053
55	20	1470	4136
56	20	1545	2381
57	20	1496	3970
58	20	1515	2642
59	20	1490	4867
60	20	1557	5301
61	25	1353	3928
62	25	1329	4617
63	25	1417	2573
64	25	1373	4004
65	25	1293	3756
66	25	1363	5997
67	25	1332	5960
68	25	1265	5050
69	25	1325	4264
70	25	1317	4747
71	25	1347	4900
72	25	1354	4857
73	25	1323	4031
74	25	1283	4475
75	25	1378	4315
76	25	1422	3061
77	25	1298	4272
78	25	1312	3631
79	25	1409	4850
80	25	1226	4468
81	30	1177	3861
82	30	1139	4006
83	30	1207	3271
84	30	1219	3206
85	30	1145	3886

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