ESTIMATION OF THE BEST ALTERNATIVE

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ABSTRACT

In many simulation studies there are available several alternatives and the experimenter is simulating in order to choose which one is best with regard to certain specified criteria. Once an alternative has been selected, people may raise the question: "How good is it?" For this problem one naturally tries to provide a confidence interval for the parameter of the best alternative.

Estimation of the best alternative has been studied in the last decade, and most of the works are in the category of k-population problems under one factor. Below we summarize these results and extend them to experiments in factorial designs - one-way and two-way classifications.

I. BACKGROUND AND INTRODUCTION

For quite a long time statistics has dealt with observations that came from one population or two populations while many other experimental problems that do not, in fact, fall into the category of one- or two-population were put under this frame. In the classical k-population problems the goal of the experiment is usually to test the hypothesis that the k($\geq$ 2) populations are homogeneous in terms of means or other characteristics. However, if the goal is not a test of homogeneity, the traditional analysis of variance technique is inadequate. If the goal is to select the best alternative among k of them in a factorial experiment, Bechhofer (1), a pioneering worker, built a ranking and selection procedure to choose such a best alternative with regard to certain criteria of "bestness" rather than testing the hypothesis that k populations are really only one. However, in his procedure only selection was considered. Once the best alternative has been selected with certain probability of correct selection, one may raise the question: "How good is the best one?" To answer this question one naturally tries to construct a point or an interval estimate for the value of the parameter of the best alternative. In recent years estimation of ordered parameters has been extensively studied on k population problems under the consideration of one-factor experiment. In section II, we will introduce interval estimation of the largest normal mean under various situations in one-factor factorial design using the traditional assumptions (normality and independence of errors). In section III, this work is considered for a two-factor factorial design with no interaction. Appropriate tables are selected for applications.

II. ONE-WAY CLASSIFICATION

Let's consider an experiment with a single factor at k($\geq$ 2) treatment levels. In this design of experiment it is assumed that the following linear statistical model holds:

$$X_{ij} = \mu_i + \varepsilon_{ij},\ i=1, \ldots, k,\ j=1,2,\ldots \qquad (1)$$

where X_{ij} is the j^{th} observation on the i^{th} treatment level, μ_i is the i^{th} treatment mean which is unknown, and ε_{ij} are independent and normally distributed with mean zero and variance σ_i^2 for i=1, ...,k and j=1,2,.... Our goal is an interval estimate on the mean of the best alternative (or treatment), $\mu_{[k]} = \max(\mu_1,\ldots,\mu_k)$, among k treatments when the best alternative has been selected according to ranking and selection procedure. Various cases are possible: variances σ_i^2's known or unknown, equal or unequal, and sample sizes equal or unequal.

Case 1. Variances σ_i^2 Known. When variances $\sigma_1^2, \ldots, \sigma_k^2$ are known (equal or unequal) Chen and Dudewicz (3) proposed a procedure to obtain a confidence interval for the largest mean $\mu_{[k]}$ (the mean of the best alternative) as follows: In the one-factor factorial model of expression (1), n_i independent observations $X_{i1}, X_{i2}, \ldots, X_{in_i}$ are taken from the i^{th} treatment level for i=1,...,k. These observations are assumed to obey a normal probability distribution with unknown mean μ_i and known variance σ_i^2. (This assumption is equivalent to that of the error term ε_{ij}.) Then, the usual estimate of μ_i is given by the arithmetic sample mean

$$\bar{X}_i = (X_{i1} + X_{i2} + \ldots + X_{in_i})/n_i$$

for i=1,2,...,k. These sample data may be summarized as in Table 1. By normal theory, the sample mean $\bar{X}_i$ will obey a normal probability distribution

TABLE 1

One-Factor Experiment

	treatment level		
	1	2 . . .	k
Observations	X_{11} X_{12} . . . X_{1n_1}	X_{21} X_{22} : X_{2n_2} . . .	X_{k1} X_{k2} . . : X_{kn_k}
Sample size	n_1	n_2 . . .	n_k
Sample mean	$\bar{X}_1$	$\bar{X}_2$ . . .	$\bar{X}_k$

with mean μ_i and variance σ_i^2/n_i. Now, a 100γ% (95% if γ = .95) confidence interval for the largest mean $\mu_{[k]}$ is

$$I_1 = (\max_i[\bar{X}_i - (L-d)\sigma_i/\sqrt{n_i}],\ \max_i[\bar{X}_i + d\sigma_i/\sqrt{n_i}]) \quad (2)$$

where L and d are tabulated in Table 2 (reproduced from Chen and Dudewicz (3)) for various confidence coefficients γ. For practical computation, one finds L and d respectively from appropriate entry of Table 2, computes the quantity $\bar{X}_i - (L - d)\sigma_i/\sqrt{n_i}$ for i=1, ..., k, and then finds the maximum of these k values, $\bar{X}_1 - (L - d)\sigma_1/\sqrt{n_1}$, ..., and $\bar{X}_k - (L - d)\sigma_k/\sqrt{n_k}$. This maximum value will be the lower limit of confidence interval of expression (2). Similarly, one can find the upper limit of interval of expression (2) by taking maximum of $(\bar{X}_1 + d\sigma_1/\sqrt{n_1}, \ldots, \bar{X}_k + d\sigma_k/\sqrt{n_k})$.

In the special case where variances are equal and known, i.e., $\sigma_1^2 = \ldots = \sigma_k^2 = \sigma^2$, say and the numbers of observations from each treatment level are also equal, i.e., $n_1 = \ldots = n_k = n$, say interval I_1 in expression (2) reduces to

$$I_1' = (\bar{X}_{[k]} - (L - d)\sigma/\sqrt{n},\ \bar{X}_{[k]} + d\sigma/\sqrt{n})$$

where $\bar{X}_{[k]}$ is the maximum of $\bar{X}_1, \ldots, \bar{X}_k$, which was considered by Dudewicz and Tong (7).
Case 2. Variances Equal but Unknown. When variances $\sigma_1^2, \ldots, \sigma_k^2$ are equal (= σ^2, say) but unknown, Chen and Dudewicz (3) proposed a similar (to that of Case 1) procedure to obtain a confidence interval for the largest mean $\mu_{[k]}$ except that the

TABLE 2

Values of L (upper entry) and d (lower entry)

k \ γ	0.80	0.90	0.95	0.99
2	2.563103 1.281552	3.289707 1.644854	3.919928 1.959964	5.151658 2.575829
3	2.599909 1.130417	3.329066 1.509463	3.959107 1.837609	5.187286 2.475334
4	2.660401 1.054289	3.389878 1.446547	4.017437 1.783419	5.238657 2.432835
5	2.719553 1.009469	3.447301 1.410819	4.071764 1.753084	5.286179 2.409284
6	2.772974 0.980189	3.498359 1.387838	4.119849 1.733679	5.328189 2.394288
7	2.820580 0.959610	3.543520 1.371811	4.162299 1.720188	5.365282 2.383895
8	2.863120 0.944361	3.583710 1.359992	4.200039 1.710261	5.398276 2.376264
9	2.901394 0.932608	3.619772 1.350914	4.233886 1.702648	5.427888 2.370422
10	2.936084 0.923272	3.652399 1.343721	4.264500 1.696625	5.454695 2.365804
12	2.996829 0.909373	3.709431 1.333046	4.318010 1.687697	5.501605 2.358972
14	3.048641 0.899521	3.758001 1.325501	4.363587 1.681397	5.541625 2.354158
16	3.093691 0.892172	3.800196 1.319886	4.403193 1.676713	5.576453 2.350582
18	3.133469 0.886479	3.837436 1.315543	4.438159 1.673094	5.607243 2.347821
20	3.169033 0.881939	3.870721 1.312084	4.469424 1.670213	5.634810 2.345625
22	3.201158 0.878234	3.900784 1.309265	4.497674 1.667866	5.659745 2.343837
25	3.244132 0.873799	3.940999 1.305893	4.535481 1.665060	5.693161 2.341701

pooled sample variance estimate,

$$S^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/(N - k)$$

where $N = n_1 + n_2 + \ldots + n_k$ with N - k d.f., takes the place of σ_i^2. (Data are as in Table 1.) Hence a 100γ% confidence interval for $\mu_{[k]}$ is given by

$$I_2 = (\max_i[\bar{X}_i - (L-d)S/\sqrt{n_i}],\ \max_i[\bar{X}_i + dS/\sqrt{n_i}]) \quad (3)$$

where d = L/2 when k=2, d can be found from the usual Student's-t table at γ confidence coefficient, and when k > 2, L and d can be found from Table 3 for γ = .95 (reproduced from Chen and Dudewicz (4)). The computational procedure of I_2 in (3) is similar

TABLE 3

Values of L_0 (left entry) and d_0 (right entry) for Interval Estimation of the Largest Normal Mean with Common Unknown Variance, $\gamma = 0.95$.

ν \ k	3	4	5	6	7	8	9	10
4	5.64 2.55	5.78 2.44	5.92 2.38	6.05 2.34	6.17 2.31	6.27 2.29	6.36 2.28	6.44 2.27
5	5.22 2.37	5.34 2.27	5.46 2.22	5.56 2.19	5.66 2.16	5.75 2.15	5.82 2.13	5.89 2.12
6	4.96 2.26	5.07 2.17	5.18 2.13	5.27 2.10	5.36 2.08	5.43 2.06	5.50 2.05	5.57 2.04
7	4.79 2.19	4.89 2.11	4.99 2.06	5.08 2.04	5.16 2.02	5.23 2.00	5.29 1.99	5.35 1.98
8	4.67 2.14	4.77 2.06	4.86 2.02	4.94 1.99	5.01 1.97	5.08 1.96	5.14 1.95	5.19 1.94
9	4.58 2.10	4.67 2.03	4.76 1.99	4.83 1.96	4.90 1.94	4.96 1.93	5.02 1.92	5.07 1.91
10	4.51 2.07	4.60 2.00	4.68 1.96	4.75 1.93	4.82 1.92	4.88 1.90	4.93 1.89	4.98 1.89
11	4.45 2.05	4.54 1.98	4.62 1.94	4.69 1.91	4.75 1.90	4.81 1.88	4.86 1.87	4.91 1.87
12	4.41 2.03	4.49 1.96	4.57 1.92	4.64 1.89	4.70 1.88	4.75 1.87	4.80 1.86	4.85 1.85
13	4.37 2.01	4.45 1.94	4.53 1.91	4.59 1.88	4.65 1.87	4.71 1.86	4.76 1.85	4.80 1.84
14	4.34 2.00	4.42 1.93	4.49 1.90	4.56 1.87	4.61 1.86	4.67 1.84	4.72 1.83	4.76 1.83
15	4.31 1.99	4.39 1.92	4.46 1.89	4.53 1.86	4.58 1.85	4.63 1.83	4.68 1.82	4.72 1.82
16	4.29 1.98	4.36 1.91	4.43 1.88	4.50 1.85	4.55 1.84	4.60 1.82	4.65 1.82	4.69 1.81
17	4.27 1.97	4.34 1.90	4.41 1.87	4.47 1.85	4.53 1.83	4.58 1.82	4.62 1.81	4.66 1.80
18	4.25 1.96	4.32 1.90	4.39 1.86	4.45 1.84	4.51 1.82	4.56 1.81	4.60 1.80	4.64 1.80
19	4.23 1.95	4.30 1.89	4.37 1.86	4.43 1.83	4.49 1.82	4.54 1.81	4.58 1.80	4.62 1.79
20	4.22 1.95	4.29 1.89	4.36 1.85	4.42 1.83	4.47 1.81	4.52 1.80	4.56 1.79	4.60 1.79
25	4.16 1.93	4.23 1.86	4.30 1.83	4.36 1.81	4.41 1.79	4.45 1.78	4.49 1.77	4.53 1.77
30	4.13 1.91	4.19 1.85	4.26 1.82	4.31 1.80	4.36 1.78	4.41 1.77	4.45 1.76	4.48 1.76
35	4.10 1.90	4.17 1.84	4.23 1.81	4.29 1.79	4.33 1.77	4.38 1.76	4.42 1.75	4.45 1.75
40	4.08 1.89	4.15 1.83	4.21 1.80	4.26 1.78	4.31 1.76	4.35 1.75	4.39 1.75	4.43 1.74
50	4.06 1.88	4.12 1.82	4.18 1.79	4.23 1.77	4.28 1.75	4.32 1.74	4.36 1.74	4.39 1.73
60	4.04 1.87	4.10 1.81	4.16 1.78	4.21 1.76	4.26 1.75	4.30 1.74	4.34 1.73	4.37 1.73
80	4.02 1.86	4.08 1.80	4.14 1.77	4.19 1.75	4.24 1.74	4.27 1.73	4.31 1.72	4.34 1.72
∞	3.96 1.84	4.02 1.78	4.07 1.75	4.12 1.73	4.16 1.72	4.20 1.71	4.23 1.70	4.26 1.70

to that of I_1 in (2) except using the pooled sample standard deviation S instead of σ_i. In the special case when the sample sizes are all equal (equal to n, say), the interval I_2 reduces to

$$I_2' = (\bar{X}_{[k]} - (L - d)S/\sqrt{n},\ \bar{X}_{[k]} + dS/\sqrt{n})$$

which was considered by Saxena (8).

Case 3. Variances Unequal Unknown. When variances $\sigma_1^2, \ldots, \sigma_k^2$ are unequal and unknown no single sampling procedure exists. Hence, Chen and Dudewicz (5) proposed a two-stage fixed-width confidence interval for the largest mean $\mu_{[k]}$. In the one-factor model of (1), we proceed our sampling procedure as follows: Take an initial random sample $X_{i1}, X_{i2}, \ldots, X_{in_0}$ of size $n_0 (\geq 2)$ from treatment level i $(1 \leq i \leq k)$, and define

$$\bar{X}_i = \sum_{j=1}^{n_0} X_{ij}/n_0,$$

$$S_i^2 = \sum_{j=1}^{n_0} (X_{ij} - \bar{X}_i)^2/(n_0 - 1),$$

and

$$n_i = \max\{n_0 + 1, [(S_i/h)^2]\}$$

where h is fixed >0 and [y] denotes the smallest integer $\geq$ y. Take $n_i - n_0$ additional observations $X_{i,n_0+1}, \ldots, X_{i,n_i}$ from level i and define

$$\tilde{X}_i = \sum_{j=1}^{n_i} a_{ij}X_{ij}.$$

Here, the a_{ij}'s $(j = 1, \ldots, n_i;\ i = 1, \ldots, k)$ are as chosen by Dudewicz (6): $a_{i1} = \ldots = a_{in_0} = b_1/n_0$ and $a_{i,n_0+1} = \ldots = a_{i,n_i} = b_2/(n_i - n_0)$ where

$$b_1 = \frac{n_0}{n_i}\left(1 + \sqrt{1 - \frac{n_i}{n_0}\left(1 - \left(\frac{h}{S_i}\right)^2 (n_i - n_0)\right)}\right)$$

and $b_2 = 1 - b_1$. Then $\tilde{X}_i = b_1\bar{X}_i + b_2\bar{Y}_i$ with $\bar{Y}_i = \Sigma_{j=n_0+1}^{n_i} X_{ij}/(n_i - n_0)$ for i=1, ..., k. Let $\tilde{X}_{[k]}$ denote the largest of $\tilde{X}_1, \ldots, \tilde{X}_k$. Then for a given fixed-length L (>0), a 100γ% confidence interval for the largest mean $\mu_{[k]}$ is

$$I_3 = (\tilde{X}_{[k]} - c,\ \tilde{X}_{[k]} + L - c) \qquad (4)$$

where c together with h (a value that determines n_i and hence $\tilde{X}_i$) are tabulated in Table 4 with k = 2(1) 5, 7, 10, 15, 20, and 25; and $\nu = n_0 - 1 = 1(1)30(5)$

TABLE 4

Values of (h, c) for Two-Stage Confidence Intervals of $\mu_{[k]}$ when $\gamma = 0.95$.

ν \ k	2	3	4	5	7	10	15	20	25
1	.0271 .58	.0212 .63	.0176 .66	.0152 .68	.0120 .72	.0092 .76	.0067 .79	.0053 .81	.0044 .83
2	.0958 .56	.0849 .59	.0777 .61	.0724 .63	.0648 .66	.0574 .69	.0497 .71	.0448 .73	.0412 .75
3	.1363 .55	.1251 .58	.1176 .60	.1120 .61	.1040 .63	.0959 .65	.0873 .68	.0815 .69	.0773 .70
4	.1597 .55	.1489 .57	.1416 .59	.1362 .60	.1284 .62	.1205 .64	.1120 .66	.1064 .67	.1021 .68
5	.1746 .55	.1641 .57	.1571 .59	.1519 .60	.1444 .61	.1368 .63	.1287 .65	.1232 .66	.1191 .67
6	.1838 .54	.1746 .57	.1678 .58	.1628 .59	.1556 .61	.1483 .62	.1405 .64	.1352 .65	.1313 .66
7	.1923 .54	.1823 .56	.1757 .58	.1708 .59	.1638 .61	.1567 .62	.1492 .64	.1441 .65	.1403 .66
8	.1979 .54	.1881 .56	.1816 .58	.1769 .59	.1700 .60	.1632 .62	.1558 .63	.1509 .64	.1472 .65
9	.2023 .54	.1927 .56	.1863 .58	.1816 .59	.1749 .60	.1682 .62	.1611 .63	.1563 .64	.1527 .65
10	.2058 .54	.1963 .56	.1901 .58	.1855 .59	.1789 .60	.1723 .62	.1653 .63	.1607 .64	.1571 .65
11	.2087 .54	.1994 .56	.1932 .58	.1886 .58	.1822 .60	.1757 .61	.1688 .63	.1642 .64	.1608 .65
12	.2112 .54	.2019 .56	.1958 .57	.1913 .58	.1849 .60	.1785 .61	.1718 .63	.1673 .64	.1639 .64
13	.2132 .54	.2040 .56	.1980 .57	.1935 .58	.1872 .60	.1809 .61	.1743 .63	.1698 .63	.1665 .64
14	.2150 .54	.2058 .56	.1999 .57	.1055 .58	.1892 .60	.1830 .61	.1764 .62	.1720 .63	.1688 .64
15	.2165 .54	.2074 .56	.2015 .57	.1971 .58	.1909 .60	.1848 .61	.1783 .62	.1739 .63	.1707 .64
16	.2179 .54	.2088 .56	.2029 .57	.1986 .58	.1925 .60	.1864 .61	.1799 .62	.1756 .63	.1724 .64
17	.2191 .54	.2101 .56	.2042 .57	.1999 .58	.1938 .60	.1878 .61	.1814 .62	.1771 .63	.1740 .64
18	.2201 .54	.2112 .56	.2053 .57	.2011 .58	.1950 .59	.1890 .61	.1827 .62	.1784 .63	.1753 .64
19	.2211 .54	.2122 .56	.2063 .57	.2021 .58	.1961 .59	.1901 .61	.1838 .62	.1796 .63	.1765 .64
20	.2219 .54	.2130 .56	.2073 .57	.2030 .58	.1970 .59	.1911 .61	.1849 .62	.1807 .63	.1776 .64
21	.2227 .54	.2138 .56	.2081 .57	.2039 .58	.1979 .59	.1920 .61	.1858 .62	.1817 .63	.1786 .64
22	.2234 .54	.2146 .56	.2088 .57	.2046 .58	.1987 .59	.1928 .61	.1867 .62	.1826 .63	.1795 .63
23	.2241 .54	.2152 .56	.2095 .57	.2053 .58	.1994 .59	.1936 .61	.1875 .62	.1834 .63	.1803 .63
24	.2246 .54	.2158 .56	.2101 .57	.2060 .58	.2001 .59	.1943 .61	.1882 .62	.1841 .63	.1811 .63
25	.2252 .54	.2164 .56	.2107 .57	.2066 .58	.2007 .59	.1949 .61	.1888 .62	.1848 .63	.1818 .63
26	.2256 .54	.2169 .56	.2112 .57	.2071 .58	.2013 .59	.1955 .61	.1894 .62	.1854 .63	.1824 .63
27	.2261 .54	.2174 .56	.2117 .57	.2076 .58	.2018 .59	.1961 .61	.1900 .62	.1860 .63	.1830 .63
28	.2266 .54	.2179 .56	.2122 .57	.2081 .58	.2023 .59	.1966 .60	.1905 .62	.1865 .63	.1836 .63
29	.2270 .54	.2183 .56	.2126 .57	.2085 .58	.2027 .59	.1970 .60	.1910 .62	.1870 .63	.1841 .63
30	.2274 .54	.2187 .56	.2130 .57	.2089 .58	.2032 .59	.1975 .60	.1915 .62	.1875 .63	.1846 .63
35	.2289 .54	.2203 .56	.2147 .57	.2106 .58	.2049 .59	.1993 .60	.1934 .62	.1895 .63	.1866 .63
40	.2301 .54	.2215 .56	.2159 .57	.2119 .58	.2062 .59	.2007 .60	.1948 .62	.1909 .62	.1881 .63
45	.2310 .54	.2224 .56	.2169 .57	.2129 .58	.2073 .59	.2017 .60	.1959 .62	.1921 .62	.1892 .63
50	.2317 .54	.2232 .56	.2177 .57	.2137 .58	.2081 .59	.2026 .60	.1968 .61	.1930 .62	.1902 .63
55	.2323 .54	.2238 .56	.2183 .57	.2143 .58	.2088 .59	.2033 .60	.1975 .61	.1938 .62	.1910 .63
60	.2328 .54	.2243 .56	.2189 .57	.2149 .58	.2093 .59	.2039 .60	.1981 .61	.1944 .62	.1916 .63
80	.2341 .54	.2257 .56	.2203 .57	.2164 .58	.2109 .59	.2055 .60	.1998 .61	.1961 .62	.1934 .63
120	.2355 .54	.2271 .56	.2218 .57	.2179 .58	.2124 .59	.2071 .60	.2015 .61	.1978 .62	.1951 .63

60, 80, 120. (Selected from Chen (2).)

If our goal is to estimate the smallest mean by an interval, the procedures given above are the same except that one changes the signs of all observations by multiplying a negative one.

III. TWO-WAY CLASSIFICATION

In this section we consider a factorial experiment with two factors having no interaction. The first factor, let's call factor A, has I (≥ 2) treatment levels and the second factor, factor B, has J (≥ 2) treatment levels and hence there are $I \times J$ treatment combinations. Then in this design of experiment it is assumed that the following well-known linear statistical model holds:

$$X_{ij\ell} = \mu + \alpha_i + \beta_j + \varepsilon_{ij\ell}, \qquad (4)$$

$$i = 1, \ldots I; \; j = 1, \ldots, J; \; \ell = 1, 2, \ldots, n$$

where $X_{ij\ell}$ is the ℓ^{th} observation at the i^{th} level of factor A and the j^{th} level of factor B, μ is the general mean, α_i is the main effect of factor A at level i and β_j is the main effect of factor B at level j with the restrictions $\sum_i \alpha_i = 0$ and $\sum_j \beta_j = 0$. It is also assumed that the errors $\varepsilon_{ij\ell}$ are independent and normally distributed with mean 0 and variance σ_{ij}^2 for $\ell = 1,2,\ldots$. Let $\mu_{i.} = \mu + \alpha_i$ denote the true average with factor A at level i, $i=1,\ldots, I$; let $\mu_{.j} = \mu + \beta_j$ denote the true average with factor B at level j, $j=1,\ldots,J$; and let $\mu_{ij} = \mu + \alpha_i + \beta_j$ denote the true average with factor A at level i and with factor B at level j. Let $\mu_{[I.]}$ denote the largest one of $\mu_{1.}, \ldots, \mu_{I.}$, let $\mu_{[.J]}$ denote the largest one of $\mu_{.1}, \ldots, \mu_{.J}$, and let $\mu_{[IJ]}$ denote the largest one of $\mu_{11}, \ldots, \mu_{I1}, \ldots, \mu_{IJ}$.

The goals for two-factor factorial experiment may be:

(1) Estimate by an interval the mean of the best treatment level, $\mu_{[I.]}$, of factor A regardless of factor B.

(2) Estimate by an interval the mean of the best treatment level, $\mu_{[.J]}$, of factor B regardless of factor A.

(3) Estimate by an interval the mean of the best treatment combination, $\mu_{[IJ]}$, among all treatment combinations.

For the goals listed above, independent observations are obtained according to the model (4) of factorial experiment. These sample data are summarized in Table 5 where $\bar{X}_{i.}$ and $n_{i.}$ are the i^{th} row mean and sample size, respectively, of factor A, $\bar{X}_{.j}$ and $n_{.j}$ are the j^{th} column mean and sample size, respectively, of factor B, and $\bar{X}_{ij.}$ and n are the $(i,j)^{th}$ treatment combination mean and size respectively.

TABLE 5

Two-Factor Experiment

		Factor B				Sample Size	Sample Mean
		1	2	...	J	(row)	(row)
Factor A	1	X_{111}	X_{121}		X_{1J1}		
		X_{112}	X_{122}		X_{1J2}	$n_{1.}$	$\bar{X}_{1.}$
		⋮	⋮		⋮		
		X_{11n}	X_{12n}		X_{1Jn}		
	2	X_{211}	X_{221}		X_{2J1}		
		X_{212}	X_{222}		X_{2J2}	$n_{2.}$	$\bar{X}_{2.}$
		⋮	⋮		⋮		
		X_{21n}	X_{22n}		X_{2Jn}		
	⋮	⋮	⋮	...	⋮	⋮	⋮
	I	X_{I11}	X_{I21}		X_{IJ1}		
		X_{I12}	X_{I22}		X_{IJ2}	$n_{I.}$	$\bar{X}_{I.}$
		⋮	⋮		⋮		
		X_{I1n}	X_{I2n}		X_{IJn}		
Sample size (Column)		$n_{.1}$	$n_{.2}$	...	$n_{.J}$		
Sample mean (Column)		$\bar{X}_{.1}$	$\bar{X}_{.2}$	...	$\bar{X}_{.J}$		

For goal (1), when $\sigma_{ij}^2 = \sigma^2$ with σ^2 known the procedure is the same as in case 1 of section II except that we replace $\bar{X}_i$ by $\bar{X}_{i.}$, n_i by $n_{i.}$, and σ_i by σ. When $\sigma_{ij} = \sigma$ with σ unknown the procedure is the same as in case 2 of section II with ((I-1)(J-1) degrees of freedom)

$$S^2 = \sum_i \sum_j n(\bar{X}_{ij.} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X})^2/(I-1)(J-1)$$

where $\bar{X}$ is the grand total sample mean. For the goal (2), the procedures of cases (1) and (2) of section II can also be applied just like what we do for goal (1) except using $\bar{X}_{.j}$, $n_{.j}$ instead of $\bar{X}_{i.}$, $n_{i.}$. Finally for goal (3) when σ_{ij}^2's are known, the procedure is the same as in case 1 of section II except using $\bar{X}_{ij\cdot}$, n, and σ_{ij} instead of $\bar{X}_i$, n_i, and σ_i respectively. When $\sigma_{ij}^2 = \sigma^2$ with σ unknown the procedure in case 2 can be applied using S^2 as an estimate of σ^2. In the final situation when σ_{ij}^2's are unequal and unknown the two-stage procedure described in case 3 can be applied for each treatment combination.

IV. AN ILLUSTRATIVE EXAMPLE

In this section we consider the estimation of the largest and the smallest mean traffic fatality rates in the southeastern United States including Alabama, Florida, Georgia, North Carolina, South Carolina and Tennessee. The motor-vehicle fatality rate for each state is published each year by the National Safety Council in the annual editions of Accident Facts. Basically, we consider the fatalities which occur within one year as a result of an accident involving a motor-vehicle on a trafficway. The death is attributed to the state where the accident occurs. These fatality rates are given in Table 6.

TABLE 6

SELECTED MOTOR-VEHICLE TRAFFIC FATALITIES PER YEAR PER 100 MILLION VEHICLE MILES IN THE SOUTHEASTERN U.S.

	STATE					
YEAR	AL	FL	GA	NC	SC	TN
60	7.0	5.7	6.3	6.8	7.8	5.6
61	7.0	5.4	6.0	6.5	7.9	4.8
62	6.9	5.6	6.2	6.7	7.8	4.9
63	7.5	6.0	6.8	7.0	7.7	6.4
64	7.0	5.9	6.5	7.6	8.0	6.9
65	7.4	6.1	6.5	7.3	7.3	6.8
66	7.0	6.0	7.0	7.4	7.9	7.4
67	7.1	5.4	6.7	7.1	7.0	7.0
68	7.0	6.1	6.9	7.2	7.0	6.4
69	7.0	5.7	6.4	6.5	6.4	7.1
70	6.4	5.2	6.2	6.0	6.2	6.6
71	6.8	5.0	5.7	5.9	5.8	5.5
72	6.0	4.5	4.7	5.8		5.1
73	6.2	4.5	5.3	5.3		4.9
74	4.1	3.7				

Source: These rates are selected directly from the paper entitled, "An application of nonparametric selection procedures to an analysis of motor-vehicle traffic fatality rates" by Gary C. McDonald, Proceedings of 1977 Winter Simulation Conference. For the purpose of illustration some latest and estimated rates are not included in the table in order to have unequal sample sizes.

In this example, a state is a population; the characteristic is the fatality rate recorded each year for each state. We are interested in the state which has the largest mean (motor-vehicle fatality) rate or the smallest mean rate and the values of the mean rate themselves. Here we assume that data do not violate the assumptions of independence, normality and homogeneity of variances. (Actually, the fatality rates recorded for each state are independent of those for others; normality and homogeneity of variances have been checked by goodness-of-fit test and the Bartlett's Chi-quare test, respectively. We find that these basic assumptions are satisfied.) In the present example we have k = 6 states (populations). The sample sizes are $n_1 = 15$, $n_2 = 15$, $n_3 = 14$, $n_4 = 14$, $n_5 = 12$, $n_6 = 14$; the sample means are $\bar{X}_1 = 6.69$, $\bar{X}_2 = 5.39$, $\bar{X}_3 = 6.23$, $\bar{X}_4 = 6.65$, $\bar{X}_5 = 7.23$, $\bar{X}_6 = 6.10$; the sample variances are $s_1^2 = 0.68$, $s_2^2 = 0.49$, $s_3^2 = 0.41$, $s_4^2 = 0.47$, $s_5^2 = 0.57$, $s_6^2 = 0.87$; and the pooled sample variance with degrees of freedom $\nu = 84 - 6 = 78$ is found to be

$$s^2 = \sum_{i=1}^{k} (n_i - 1)s_i^2 / \left(\sum_{i=1}^{k} n_i - k\right)$$

$$= \frac{14(.68)+14(.49)+13(.41)+13(.47)+11(.57)+13(.87)}{78}$$

$$= 0.58.$$

From this calculation we can see that South Carolina has the largest sample mean rate ($\bar{X}_5 = 7.23$) and Florida has the smallest sample mean rate ($\bar{X}_2 = 5.39$).

If we define the "best" state to be the one with the smallest true mean rate $\mu_{[1]}$ (which is unknown) and the "worst" state with the largest true mean rate $\mu_{[6]}$ (unknown), then, according to ranking and selection procedures, South Carolina (with the largest sample mean rate) will be identified as the worst state and Florida (with the smallest sample mean rate) the best state. However, the true mean rate of the worst or the best state has not yet been estimated. Hence, if we want to know how bad the worst state is, we would like to provide a confidence interval for the mean fatality rate of the worst state. Now if we wish to determine a 95% confidence interval for the true mean rate associated with the worst state, that is, the largest mean $\mu_{[6]}$, then we need to compute $\bar{X}_i - (L - d)S/\sqrt{n_i}$ and $\bar{X}_i + dS/\sqrt{n_i}$ for each sample by using the interval of expression (3) in case 2 of section II. For k = 6, $\nu = 78$, and $\gamma = .95$, we find L = 4.19, and d = 1.75, respectively, from Table 3. The computations needed to construct the confidence interval are shown in Table 7.

Expression (3) indicates that the two end points of the interval I_2 are the largest numbers in the corresponding two last columns in Table 7. The largest in the left column is 6.70, and in the right is 7.61. Then the 95% confidence interval for $\mu_{[6]}$ is

$I_2 = (6.70, 7.61)$

TABLE 7

COMPUTATIONS FOR CONSTRUCTING A CONFIDENCE INTERVAL I_2

Sample i	$\bar{X}_i - (L - d)S/\sqrt{n_i}$	$\bar{X}_i + dS/\sqrt{n_i}$
1	$6.69 - \frac{(4.19 - 1.75)(.76)}{\sqrt{15}} = 6.21$	$6.69 + \frac{(1.75)(.76)}{\sqrt{15}} = 7.03$
2	$5.39 - \frac{(4.19 - 1.75)(.76)}{\sqrt{15}} = 4.91$	$5.39 + \frac{(1.75)(.76)}{\sqrt{15}} = 5.73$
3	$6.23 - \frac{(4.19 - 1.75)(.76)}{\sqrt{14}} = 5.73$	$6.23 + \frac{(1.75)(.76)}{\sqrt{14}} = 6.59$
4	$6.65 - \frac{(4.19 - 1.75)(.76)}{\sqrt{14}} = 6.15$	$6.65 + \frac{(1.75)(.76)}{\sqrt{14}} = 7.01$
5	$7.23 - \frac{(4.19 - 1.75)(.76)}{\sqrt{12}} = 6.70$	$7.23 + \frac{(1.75)(.76)}{\sqrt{12}} = 7.61$
6	$6.10 - \frac{(4.19 - 1.75)(.76)}{\sqrt{14}} = 5.60$	$6.10 + \frac{(1.75)(.76)}{\sqrt{14}} = 6.46$

and we are 95 percent sure that the interval (6.70, 7.61) will contain the largest true mean fatality rate. Since only one of the six sample means is contained in this interval, in this case, $\bar{X}_5$, it is tempting to also assert that the state South Carolina is the worst one with confidence level $\gamma = .95$. (This happens to have the same conclusion as that by ranking and selection procedures. In fact, the interval may contain at least one of the sample means.)

Moreover, if the goal is to tell how good the best state is, we will, by a similar manner, construct a 95% confidence interval for the smallest mean rate associated with the best state. The computational procedure is the same except that one replaces the value of $\bar{X}_i$ by its negative counterpart and the estimating procedure is the same as in case 2 of section II. After the interval for $-\mu_{[1]}$ (the largest one now) has been calculated, we multiply a -1 to the interval to obtain what we desire. In our example, the desired 95% confidence interval for the smallest mean rate $\mu_{[1]}$ is (5.05, 5.87). Since only one of the six sample means is contained in this interval, in this case $\bar{X}_2$, it is tempting to also assert, with confidence level $\gamma = .95$, that the state Florida is the best state which has the smallest mean fatality rate in the southeastern United States.

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