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GENERATING NEGATIVELY CORRELATED GAMMA VARIATES USING THE BETA-GAMMA TRANSFORMATION

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The Beta-Gamma transform changes a Gamma distributed variate with shape parameter k into a Gamma variate with shape parameter q, where q is less than k. Using this transformation a scheme is given for generating negatively correlated Gamma variates, each with parameter k. The scheme can be implemented with nothing but a generator of Gamma deviates. It requires no inverse probability integral transform for Gamma variates.

1. ABSTRACT

In what follows $(X(k), Y(k))$ will denote a pair of possibly dependent random variates with correlation r and Gamma(β,k) marginal distributions. Also $G_1(k), G_2(k),\ldots$ will represent independent Gamma(β,k) variates and $B_1(m,n)$, $B_2(m,n),\ldots$ will represent independent Beta(m,n) variates. Although there are efficient schemes for generating Beta variates, they may also be generated as $B(m,n) = G_1(m)/\{G_1(m) + G_2(n)\}$.

The Beta-Gamma transformation (Lewis, 1982) is as follows. Let

$$X(q) = B(q,k-q)G(k)\ , \qquad q < k\ . \tag{1}$$

Then $X(q)$ is distributed as a Gamma(β,k) variate.

We now show how this transformation can be used to generate negatively correlated Gamma pairs $(X(k), Y(k))$. The emphasis here is on negatively correlated pairs of Gamma variates although the scheme can be adapted to obtain positive correlation. In general, however, positive correlation is easier to obtain than negative correlation, and particularly so with the Beta-Gamma transformation.

Let $(X_0(2k), Y_0(2k))$ have correlation r_0. Let a new pair $(X_1(k), Y_1(k))$ be generated as

$$X_1(k) = B_1(k,k)\,X_0(2k)\ ,$$

$$Y_1(k) = \{1 - B_1(k,k)\}\,Y_0(2k)\ . \tag{2}$$

Then the correlation between $X_1(k)$ and $Y_1(k)$ is given by

$$r_1 = \frac{k(r_0-1)}{1+2k} \qquad k > 0\ . \tag{3}$$

In particular if we start with an independent pair of Gamma variates, $X_0(2k)$ and $Y_0(2k)$, then

$$r_1 = \frac{-k}{1+2k}\ . \tag{4}$$

This has value $r_1 = -1/3$ if $k = 1$(exponential), $r_1 = -1/4$ if $k = 1/2$ and $r_1 \to -1/2$ as $k \to \infty$.

Of course the value of $r_1 = -1/3$ is only about half the maximum negative correlation of -0.645 attainable for antithetic exponential pairs.

Iteration of the process improves the attainable negative correlation. Start with independent pairs $X_0(4k)$ and $Y_0(4k)$ and obtain $X_1(2k)$ and $Y_1(2k)$ with correlation $r_1 = -2k/(4k+1)$. Then let

$$X_2(k) = B_2(k,k)X_1(2k)$$
$$= B_2(k,k)B_1(2k,2k)X_0(4k) ;$$
$$Y_2(k) = \{1 - B_2(k,k)\}\ Y_1(2k) \qquad (5)$$
$$= \{1 - B_2(k,k)\}\{1 - B_1(2k,2k)\}\ Y_0(4k) .$$

Then $r_2 = (-k)(6k+1)/\{(1+2k)(1+4k)\}$ which has value -0.4667 when $k = 1$ and $r_2 \to -3/4$ as $k \to \infty$. The next iteration produces a correlation $r_3 = -0.5259$ at $k = 1$ and $r_3 \to -7/8$ as $k \to \infty$.

The basic scheme (2) can be tuned by addition of one parameter to obtain correlations in the range from 0 to the value given by (3). Note too, that the scheme (2) can be generated purely from four Gamma variates:

$$X_1(k) = \frac{G_1(k)}{G_1(k) + G_2(k)} \cdot \{G_1(k) + G_2(k)\} = G_1(k) , \qquad (6)$$

$$Y_1(k) = \frac{G_2(k)}{G_1(k) + G_2(k)} \cdot \{G_3(k) + G_4(k)\} . \qquad (7)$$

The relationship (6) comes from the fact that in the ratio $G_1(k)/\{G_1(k) + G_2(k)\}$, the denominator is independent of the ratio. (This is a characterizing property of Gamma variates).

Further properties of these schemes for generating negatively correlated Gamma variates will be given elsewhere.

2. REFERENCES

Lewis, P. A. W. (1982). "Simple Multivariate Time Series for Simulations of Complex Systems". In Proceedings of 1981 Winter Simulation Conference, T. I. Oren, C. M. Delfosse, C. M. Shab (Eds.) p. 389-390.