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# A COMPARISON BETWEEN STANDARDIZED TIME SERIES AND OVERLAPPING BATCHED MEANS

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## ABSTRACT

We study several methods for estimating the variance of the sample mean from a stationary stochastic process. Particular emphasis is placed on comparing the recently introduced methods of *standardized time series* and *overlapping batched means*. Comparison criteria include estimator bias and variance.

## 1. INTRODUCTION

An active area of research in simulation output analysis concerns the problem of confidence interval estimation for the mean of a stationary stochastic process. Over the last twenty years, several confidence interval estimation methodologies have been proposed: nonoverlapping batched means (NOBM), independent replications, ARMA time series modeling, spectral representation, regeneration, standardized time series (STS) [Schruben (1983)], and overlapping batched means (OBM) [Meketon and Schmeiser (1984)].

Approximate $100(1-\alpha)\%$ confidence interval estimators for the underlying process mean $\lambda$ are usually of the form:

$$\Pr\{ \lambda \in \bar{\bar{X}} \pm t_{1-\frac{1}{2}\alpha}\hat{V}^{\frac{1}{2}} \} \approx 1 - \alpha,$$

where $\bar{\bar{X}}$ is a point estimator for $\lambda$, $\hat{V}$ is an estimator for $\mathrm{Var}(\bar{\bar{X}})$, and $t_{1-\frac{1}{2}\alpha}$ is the appropriate quantile of a t-distribution. The accurate and precise estimation of $\mathrm{Var}(\bar{\bar{X}})$ is therefore of prime importance. In fact, the main differences among the confidence interval methodologies involve the estimation of $\mathrm{Var}(\bar{\bar{X}})$.

Various studies have been conducted in order to compare these methodologies against each other: Law and Kelton (1982) *empirically* investigate the NOBM, independent replications, time series modeling, spectral, and regeneration methodologies. Their results are based on a large number of simulation experiments with certain simple simulated processes; these experiments were all of very limited run length ($\leq$ 2560 observations). Goldsman and Schruben (1984) consider the NOBM and STS confidence interval estimators. With regard to several asymptotic performance characteristics of interest, they find that the STS estimator *strictly dominates* the NOBM estimator. Similarly, Meketon and Schmeiser (1984) show that the OBM estimator outperforms the NOBM estimator.

The current paper compares the variance estimators arising from a number of the above confidence interval methodologies. We pay special attention to the STS and OBM estimators. Our criteria for comparison among estimators include estimator bias and variance. Section 2 of this paper provides necessary background material. Results are presented in Section 3, and conclusions are drawn in Section 4.

## 2. BACKGROUND

We say that $\{Z(t)\colon t \geq 0\}$ is a continuous time *stationary increment stochastic process* (SISP) if

a. $Z(0) = 0$ and

b. for any positive integer n, $h \geq 0$, and $t_1 \leq t_2 \leq \cdots \leq t_n$,

$$(Z(t_1),\ Z(t_2)-Z(t_1),\ \ldots,\ Z(t_n)-Z(t_{n-1}))$$

and

$$(Z(t_1+h)-Z(h),\ Z(t_2+h)-Z(t_1+h),\ \ldots,\ Z(t_n+h)-Z(t_{n-1}+h))$$

have the same joint distribution.

(The analogous treatment for the discrete time case is straightforward.) As a simple example, suppose that $\{X(t)\colon t \geq 0\}$ is a stationary stochastic process. Then

$$Z(t) \equiv \int_0^t X(s)ds$$

is an SISP.

We shall be interested in the following:

a. the *mean rate of increase* of $Z(t)$, $\lambda \equiv E[Z(t)]/t$ ( $= E[X(t)]$).

b. the *variance time curve*, $V(t) \equiv \mathrm{Var}[Z(t)]$.

To motivate the variance time curve, again suppose that $\{X(t): 0 \le t \le T\}$ is a stationary process and $Z(t) = \int_0^t X(s)ds$. T is the run length of the simulation. Further, denote the autocovariance function of $\{X(t)\}$ as $R(u) \equiv Cov(X(s),X(s+u))$. Under mild conditions [cf. Goldsman and Meketon (1985)],

$$V(t) = 2 \int_0^t (t-u)R(u)du = \sigma^2 t + b + o(1/t)$$

for appropriate constants $\sigma^2$ and b. $\sigma^2$ is called the *process variance*; this quantity is useful in establishing confidence intervals for $\lambda$. Note that $V(t)/t = \sigma^2 + b/t$ plus an order term. So for large enough run length T, $V(T)/T \approx \sigma^2$.

We now review various well known estimators of $\sigma^2$:

### 2.1 Nonoverlapping Batched Means

Here, we fix $t = T/k$ and divide $\{Z(u): 0 \le u \le T\}$ into k contiguous, nonoverlapping batches, each of length t: Batch i consists of $\{Z(u): (i-1)t \le u \le it\}$, $i=1,\ldots,k$. Assuming that b/t is "small", we see that $\sigma^2 = V(t)/t = E[(Z(it)-Z((i-1)t)-\lambda t)^2]/t$, $\forall i$, by stationary increments. A reasonable estimator for $\lambda$ is clearly $\hat{\lambda}_T \equiv Z(T)/T$. Substituting $\hat{\lambda}_T$ for $\lambda$ and then averaging across batches, we arrive at the *nonoverlapping batched means* estimator for $\sigma^2$:

$$\hat{\sigma}^2_B \equiv \frac{1}{k} \sum_{i=1}^{k} [Z(it)-Z((i-1)t)-\hat{\lambda}_T t]^2/t.$$

This is the most popular process variance estimator used in simulation analysis.

### 2.2 Overlapping Batched Means

All batches of the form $\{Z(u): x \le u \le x+t\}$, $0 \le x \le T-t$, can be used to estimate $\sigma^2$. This leads to the *overlapping batched means* estimator:

$$\hat{\sigma}^2_O \equiv \frac{1}{T-t} \int_0^{T-t} [Z(x+t)-Z(x)-\hat{\lambda}_T t]^2/t \, dx.$$

Meketon and Schmeiser (1984) introduced the OBM estimator for simulation purposes. This estimator seems to be more efficient in its use of observations than the NOBM estimator.

### 2.3 Spectral Representation

The starting point for another variance estimator is the expression:

$$V(t) = 2 \int_0^t (t-u)R(u)du.$$

The autocovariance function R(u) is commonly estimated by the method of moments:

$$\hat{R}(u) \equiv \frac{1}{T-u} \int_0^{T-u} (X(s)-\hat{\lambda}_T)(X(s+u)-\hat{\lambda}_T)\, ds.$$

These facts yield the *spectral* estimator for $\sigma^2$:

$$\hat{\sigma}^2_S \equiv \frac{2}{t} \int_0^t (t-u)\hat{R}(u)du.$$

Other spectral-type estimators are described in Meketon (1980) and Goldsman and Meketon (1985).

### 2.4 Standardized Time Series

Using a functional central limit theorem, it is easy to show that as $T \to \infty$, the *standardized time series*, $B_T(s) \equiv (Z(sT)-sZ(T))/(\sigma\sqrt{T})$, converges to a Brownian bridge process, $B(s)$, $0 \le s \le 1$. [A Brownian bridge process is Brownian motion which is conditioned to start and stop at zero.] Schruben (1983) observes that:

$$E\left[ \left[ \int_0^1 B(s)ds \right]^2 \right] = 1/12,$$

where $\int_0^1 B(s)ds$ is the signed area below the Brownian bridge process. Under mild mixing and moment conditions, we can replace $B(s)$ by $B_T(s)$ in the above equation to get:

$$\lim_{T\to\infty} \frac{12}{T^3} E\left[ \left[ \int_0^T (Z(s) - \tfrac{s}{T}Z(T))ds \right]^2 \right] = \sigma^2.$$

Dividing the $\{Z(t)\}$ process into k contiguous, nonoverlapping batches, and appealing to stationary increments, Schruben derives the so-called *area* estimator for $\sigma^2$:

$$\hat{\sigma}^2_A \equiv \frac{12}{t^3 k} \sum_{i=1}^{k} A_i^2 ,$$

where

$$A_i \equiv \int_{(i-1)t}^{it} \left[ Z(u) - Z((i-1)t) - (u-(i-1)t) \frac{(Z(it)-Z((i-1)t))}{t} \right] du.$$

Schruben (1983) and Goldsman (1984) derive variance estimators based on other functionals of Brownian bridges.

## 3. RESULTS

We wish to compare the bias and variance of the variance estimators from the previous section. The bias of an estimator is the difference between its expected value and the parameter of interest; low bias is desirable. Goldsman and Meketon (1985) prove:

Theorem 3.1: Under mild conditions (including $t \to \infty$ and $T/t \to \infty$),

$$E[\hat{\sigma}_B^2] = E[\hat{\sigma}_O^2] = E[\hat{\sigma}_S^2] = \frac{V(t)}{t} + \frac{\sigma^2 t}{T} + o(\frac{1}{t})$$

and

$$E[\hat{\sigma}_A^2] = \frac{12}{t^3}\int_0^t yV(y)dy - \frac{3V(t)}{t}.$$

Since $V(t) = \sigma^2 t + b + o(1/t)$, we have:

$$\text{Bias}[\hat{\sigma}_B^2] = \text{Bias}[\hat{\sigma}_O^2] = \text{Bias}[\hat{\sigma}_S^2] = \frac{b}{t} + \frac{\sigma^2 t}{T} + o(1/t)$$

and

$$\text{Bias}[\hat{\sigma}_A^2] = \frac{3b}{t} + o(1/t).$$

Concerning the variance of the variance estimators, Goldsman and Meketon also prove:

Theorem 3-2:

$$\text{Var}(\hat{\sigma}_B^2) = \text{Var}(\hat{\sigma}_A^2) = 2\sigma^4 t/T$$

and

$$\text{Var}(\hat{\sigma}_O^2) = \text{Var}(\hat{\sigma}_S^2) = \frac{4}{3}\sigma^4 t/T.$$

## 4. CONCLUSIONS

In terms of the bias and variance criteria, the OBM and spectral variance estimators are superior to the NOBM estimator (since the former estimators have less variance). Further, the variance of the OBM estimator is less than that of the STS area estimator. Comparison of the OBM bias to the STS bias is not so straightforward; but if the quantity $T/t \to \infty$ sufficiently quickly, the OBM estimator clearly dominates. Further, Goldsman and Meketon (1985) show that in a number of respects, the OBM estimator has smaller mean squared error [variance plus squared bias] than the STS estimator.

By no means is all hope lost for the STS methodology. Schruben (1983) shows how to *combine* the NOBM and STS area estimators so as to produce another estimator for $\sigma^2$. This combined estimator has variance $\sigma^4 t/T$; however, the bias of the combined estimator is still not directly comparable to that of the OBM estimator.

It remains to be seen how the OBM and STS methods fare against each other in the small sample environment. This is a topic of current study.

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