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A MONTE CARLO SAMPLING PLAN BASED ON PRODUCT FORM ESTIMATION

George S. Fishman

Department of Operations Research
University of North Carolina at Chapel Hill

ABSTRACT

This paper derives an improved bound on the time required to estimate the volume of a convex body in m–dimensional euclidean space with a specified relative accuracy.

1 INTRODUCTION

Dyer et al. (1989) and Lovasz and Simonovits (1989) derive expressions for bounding the sample size required to estimate the volume of a convex body in m-dimensional euclidean space with a specified relative accuracy. The purpose of this paper is to present an alternative bound. Let $\underline{R}$ denote a region with unknown volume $\lambda(\underline{R})$ in the m-dimensional unit hypercube. If one generates a random point $\mathbf{X}$ uniformly in $[0,1]^m$ and sets

$$\phi(\mathbf{X}) = 1 \quad \text{if } \mathbf{X} \in \underline{R}$$
$$= 0 \quad \text{otherwise}$$

then $E\phi(\mathbf{X}) = \lambda$ and $\text{var}\ \phi(\mathbf{X}) = \lambda(1 - \lambda)$ with corresponding coefficient of variation $\gamma(\phi(\mathbf{X})) = [(1 - \lambda)/\lambda]^{1/2}$. If, for example, $\underline{R}$ is the m–dimensional hypersphere centered at $(1/2,\ldots,1/2)$, then $\gamma(\phi(\mathbf{X})) = O([(2m + 4)/\pi e]^{m/4})$ as $m \to \infty$, demonstrating a serious limitation for standard Monte Carlo sampling. An alternative approach, suggested in Dyer et al. (1989), eliminates the potential for exponential growth.

Define $\underline{R}$ as a bounded open convex region in $\mathbb{R}^m$ and assume that we are given a hypersphere of radius ω containing $\underline{R}$, a hypersphere of radius $s\ (>0)$ contained in $\underline{R}$ and a procedure which can determine whether or not any point $\mathbf{x}$ is in $\underline{R}$ or not. These properties enable one to find a nonsingular, affine transformation which, when applied to $\underline{R}$, results in the transformed body containing the hypersphere $\underline{A}(1)$ of unit radius centered at $\mathbf{0}$ and being contained in a concentric hypersphere $\underline{A}(r)$ of radius $r = m^{1/2}(m+1)$ (Grotschel, et al. 1988). Moreover, finding this transformation takes time polynomial in m. The transformed body is said to be *well rounded.* For present purposes, assume that $\underline{R}$ is the transformed body so that $\underline{A}(1) \subseteq \underline{R} \subseteq \underline{A}(r)$.

Let

$$\rho = 1 - 1/m$$

$$t = t(m) = \lceil \log_{1/\rho} r \rceil$$

$$\rho_i = \max(\rho^i r, 1) \qquad 0 \leq i \leq t.$$

Let $\underline{R}(r)$ denote $\underline{R}$ scaled up by r. Since $\underline{A}(1) \subset \underline{R}$, $\underline{A}(r) \subset \underline{R}(r)$. Let

$$\underline{K}_i = \underline{K}(\rho_i) = \underline{R}(\rho_i) \cap \underline{A}(r)$$

$$1 \leq i \leq t \qquad (1)$$

and observe that $\underline{K}(\rho_i) \supseteq \underline{K}(\rho\rho_i)$ so that $\lambda(\underline{K}_i) \geq \rho^m \lambda(\underline{K}_{i-1})$. This inequality is essential to part iii of our theorem.

Algorithm CONVOL estimates the volume ratios $\mu_i = \lambda(\underline{K}_i)/\lambda(\underline{K}_{i-1})$, $1 \leq i \leq t$, and combines them to produce an estimate of $\lambda(\underline{R})$. Figure 1 illustrates the steps for $i = 1$. The algorithm follows similarly to a procedure in Dyer et al. (1989) that specifies a particular method for generating $\mathbf{X}$. The rationale for the estimating approach follows from:

Algorithm CONVOL

Purpose: To estimate $\lambda(\underline{R})$.

Input: $\underline{K}_1, \ldots, \underline{K}_t$, $\mathbf{n} = (n_1, \ldots, n_t)$.

Output: $\overline{\lambda}_{\mathbf{n}}(\underline{R})$.

Method:

$i \leftarrow 1$.

While $i \leq t$:

$T_i \leftarrow 0$ and $j \leftarrow 1$.

While $j \leq n_i$:

Generate $\mathbf{X}$ uniformly distributed in $\mathscr{K}_{i-1}$.

If $\mathbf{X} \in \underline{K}_i$, $T_i \leftarrow T_i + 1$.

$j \leftarrow j + 1$.

$i \leftarrow i + 1$.

$\overline{\lambda}_{\mathbf{n}}(\underline{R}) \leftarrow \lambda(\underline{A}(r)) \prod_{i=1}^{t} (T_i/n_i)$.

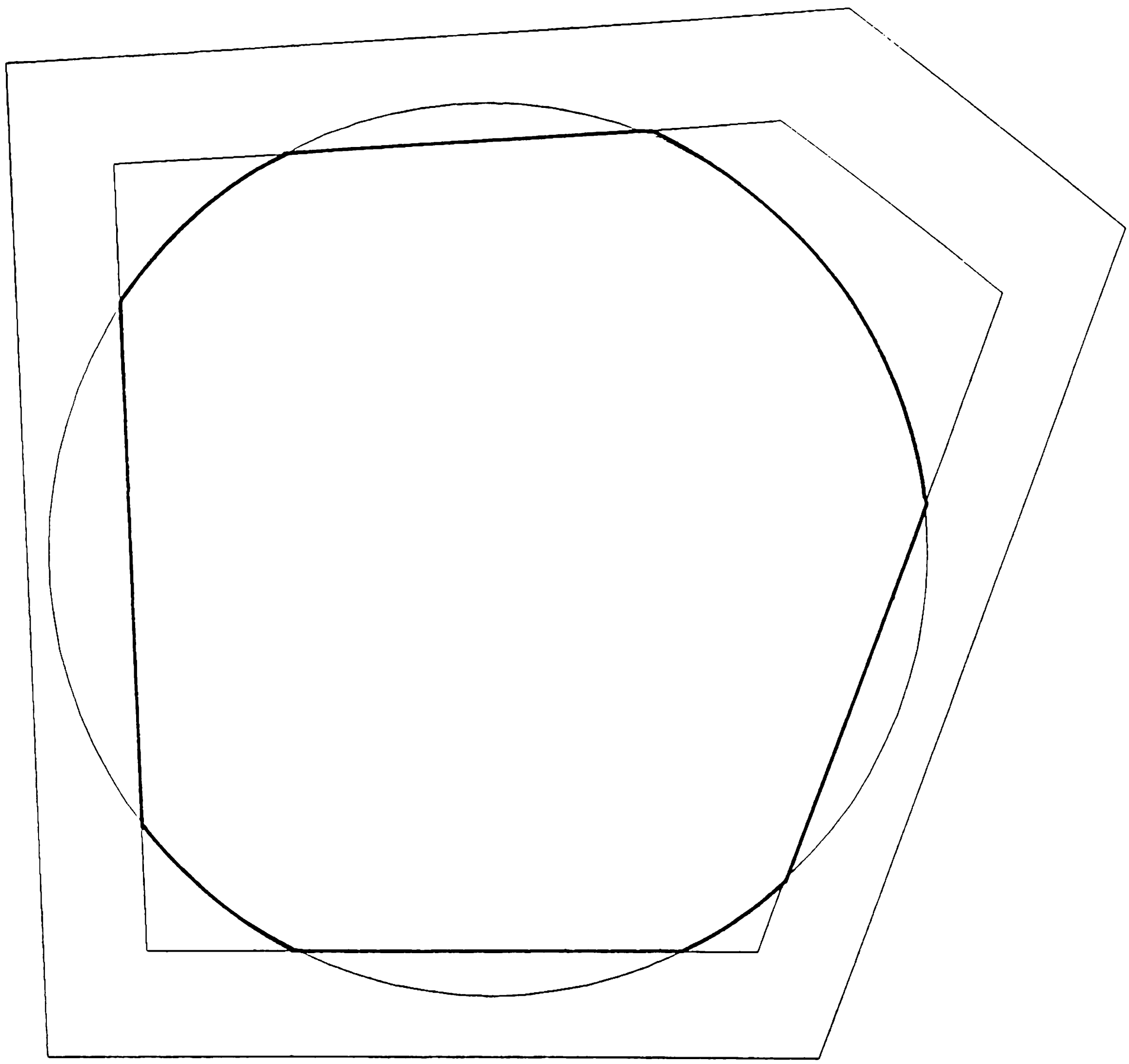

Figure 1: Estimating the Volume of a Convex Body

Theorem. The quantity $\overline{\lambda}_{\mathbf{n}}(\underline{R})$ has

i. $E\overline{\lambda}_{\mathbf{n}}(\underline{R}) = \lambda(\underline{R})$

ii. $\operatorname{var} \overline{\lambda}_{\mathbf{n}}(\underline{R}) = \lambda^2(\underline{R})\left[\prod_{i=1}^{t}\left[1 + \frac{1-\mu_i}{\mu_i n_i}\right] - 1\right]$

where

$$\mu_i = \lambda(\underline{K}_i)/\lambda(\underline{K}_{i-1}) \quad 1 \le i \le t$$

and

iii. for $n_1 = \ldots = n_t = n$

$$\gamma(\overline{\lambda}_{\mathbf{n}}(\underline{R})) \le [(1+\frac{3}{n})^t - 1]^{1/2}$$

iv. If $\lim_{m \to \infty} \frac{t}{n} = 0$, the bound in iii

converges.

v. A sample size

$$n(\lambda(\underline{R})\epsilon_r,\delta) = \left\lceil \frac{3}{(1+\delta\epsilon_r^2)^{1/t} - 1} \right\rceil$$

$$\le \left\lceil \frac{m \ \ln[m^{1/2}(m+1)]+1}{\ln(1+\delta\epsilon_r^2)} \right\rceil \qquad (2)$$

guarantees the $(\lambda(\underline{R})\epsilon_r,\delta)$ relative accuracy criterion

$$\operatorname{pr}[|\overline{\lambda}_n(\underline{R}) - \lambda(\underline{R})| < \lambda(\underline{R})\epsilon_r] \ge 1 - \delta$$

and the bound in (2) sharp.

Proof. Since $\underline{K}_i \subset \underline{K}_{i-1}$, $1 \le i \le t$,

$$\operatorname{pr}(\mathbf{X} \in \underline{K}_i | \mathbf{X} \in \underline{K}_{i-1}) = \lambda(\underline{K}_i)/\lambda(\underline{K}_{i-1}) \le 1.$$

Therefore,

$$E \prod_{i=1}^{t} (T_i/n_i) = \prod_{i=1}^{t} E(T_i/n_i)$$

$$= \lambda(\underline{K}_t)/\lambda(\underline{K}_0).$$

Since $\lambda(\underline{K}_0) = \lambda(\underline{A}(r))$ and $\underline{K}_t = \underline{R}$, part i follows.

Since $E(T_i/n_i)^2 = \mu_i^2\left[1+\frac{1-\mu_i}{\mu_i n_i}\right]$, part ii follows by independence.

The squared coefficient of variation has the form

$$\gamma^2(\overline{\lambda}_{\mathbf{n}}(\underline{R})) = \prod_{i=1}^{t}\left[1 + \frac{1-\mu_i}{\mu_i n}\right] - 1.$$

Since $\mu_i \geq \rho^m \geq 1/4$, $1 \leq i \leq t$,

$$\gamma^2(\bar{\lambda}_n(\underline{R})) \leq \left[1 + \frac{1-\rho^m}{\rho^m n}\right]^t - 1$$

$$\leq \left[1 + \frac{3}{n}\right]^t - 1,$$

which establishes iii. Part iv is obvious.

Part v follows directly from Chebyshev's inequality using the worst–case variance $\lambda^2(\underline{R})[(1 + 3/n)^t - 1]$ and applying the inequality $-\ln(1 - x) > x$, $x < 1$. These lead to

$$t = \frac{\ln[m^{1/2}(m+1)]}{-\ln \rho}$$

$$+ \theta < m \ln[m^{1/2}(m+1)] + \theta$$

$$0 \leq \theta < 1.$$

It should also be noted that for any upper bound $\mu_* < 1$ on $\mu_1,\ldots,\mu_m$, $(1+\frac{1-\mu^*}{\mu^* n})^t$ converges only if $\lim_{m\to\infty} \frac{t}{n} = 0$, implying that the $O(m \ln m)$ bound on sample size is sharp. ■

The successful implementation of Algorithm CONVOL rests on the existence of algorithms whose times are bounded by polynomial functions in m for:

a. determining a hypersphere contained in $\underline{R}$

b. determining a concentric hypersphere that contains $\underline{R}$

c. determining whether or not a point $\mathbf{x}$ is in $\underline{R}$

d. generating a random $\mathbf{X}$ uniformly distributed on each region $\underline{K}_0, \underline{K}_1, \ldots, \underline{K}_{t-1}$.

Such algorithms exist for all four tasks. In particular, Dyer et al. give a polynomial-time algorithm for generating an $\mathbf{X}$ that is approximating uniformly distributed on $\underline{K}_i$.

Items a and b need be executed only once at the beginning of the sampling experiment whereas times c and d need to be executed on each replication. As formulated in the Theorem, the specified relatively accuracy obtains with $tn(\lambda(\underline{R})\epsilon_r, \delta) = O(m^2(\ln m)^2)$ determinations of set membership (item c) and sample generations (item d). As originally formulated in Dyer et al., this bound was $O(m^4(\ln m)^5)$ and, as formulated in Lovász and Simonovits it was $O(m^3(\ln m)^4)$. These alternative approaches relied on Hoeffding's inequality and Chernoff's bound. For example, see Hoeffding (1963).

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AUTHOR'S BIOGRAPHY

GEORGE FISHMAN is professor of Operations Research at the University of North Carolina at Chapel Hill. His principal interest is the development of statistical methodology applicable to the analysis of output from discrete event digital simulation models. He is the author of *Concepts and Methods in Discrete Event Digital Simulation* published by Wiley in 1973 and of *Principles of Discrete Event Simulation* published by Wiley in 1978. He is a frequent contributor to the operations research and statistical literature on this topic. At present, he is working on variance reducing methods for applying the Monte Carlo method to counting problems and on designing Markov transition matrices to accelerate convergence in Metropolis and Gibbs sampling. Professor Fishman has been simulation departmental editor for *Management Science* and is a member of the Operations Research Society of America, the Institute of Management Science and the American Statistical Association.